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Research paper

Well-posedness and inverse problems for semilinear nonlocal wave equations

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ABSTRACT

This article is devoted to forward and inverse problems associated with time-independent semilinear nonlocal wave equations. We first establish comprehensive well-posedness results for some semilinear nonlocal wave equations. The main challenge is due to the low regularity of the solutions of linear nonlocal wave equations. We then turn to an inverse problem of recovering the nonlinearity of the equation. More precisely, we show that the exterior Dirichlet-to-Neumann map uniquely determines homogeneous nonlinearities of the form $f(x,u)$ under certain growth conditions. On the other hand, we also prove that initial data can be determined by using passive measurements under certain nonlinearity conditions. The main tools used for the inverse problem are the unique continuation principle of the fractional Laplacian and a Runge approximation property. The results hold for any spatial dimension $n \in \mathbb{N}$.

1. Introduction

1.1. Mathematical model and main results

In this paper, we study forward and inverse problems for *semilinear nonlocal wave equations*. We prove that an a priori unknown nonlinearity $f(x,\tau)$, belonging to a certain subclass of Carathéodory functions, can be uniquely recovered from the *Dirichlet-to-Neumann* (DN) map related to the semilinear nonlocal wave equation

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here and throughout this article $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain for $n \in \mathbb{N}$ with exterior $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$, $T > 0$ a finite time horizon and $s > 0$ a non-integer. Moreover, we set $A_t := A \times (0,t)$, for any subset $A \subseteq \mathbb{R}^n$ and $t > 0$. In our study the magnitude of the time horizon does not play any role. The nonlocal wave equation appears, for instance, as a special case in the study of *peridynamics*, which is a nonlocal elasticity theory particularly used to study material dynamics with discontinuities, see *e.g.*, [1]. For the time being let us assume the well-posedness of (1.1) in a suitable function space. Then, for any two measurement sets $W_1, W_2 \subset \Omega_e$, let

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us formally introduce the DN map $\Lambda^{f}_{u_0,u_1}$ of (1.1) via

$$\Lambda^{f}_{u_0,u_1}\varphi = (-\Delta)^s u_\varphi\Big|_{(W_2)_T},$$

for any $\varphi \in C_c^\infty((W_1)_T)$, where $u_\varphi : \mathbb{R}^n_T \to \mathbb{R}$ denotes the unique solution of (1.1). The DN map will be rigorously defined in Section 3.2. In our study we want to answer:

Question 1. *Can one determine under suitable assumptions the nonlinearity f or the initial conditions u_0, u_1?*

In the special case of f being homogeneous and satisfying the conditions in Assumption 3.4, we are able to establish the following positive results.

Theorem 1.1 (*Recovery of the Nonlinearity*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $T > 0$ and $s > 0$ a non-integer. Let $W_1, W_2 \subset \Omega_e$ be open sets. Suppose the nonlinearities f_j satisfy the conditions in Assumption 3.4 with $F(x,\tau) \geq 0$. Suppose also that $f_j(x,\tau)$ are $(r+1)$-homogeneous with respect to τ with r as in Assumption 3.4 and $0 < r \leq 1$. Let $\Lambda_j := \Lambda^{f_j}_{0,0}$ be the DN maps of*

$$\begin{cases} \partial_t^2 u_j + (-\Delta)^s u_j + f_j(x,u_j) = 0 & \text{in } \Omega_T,\\ u_j = \varphi & \text{in } (\Omega_e)_T,\\ u_j(0) = \partial_t u_j(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

for $j = 1,2$, satisfying

$$\Lambda_1(\varphi)\Big|_{(W_2)_T} = \Lambda_2(\varphi)\Big|_{(W_2)_T} \tag{1.3}$$

for any $\varphi \in C_c^\infty((W_1)_T)$. Then there holds $f_1(x,\tau) = f_2(x,\tau)$ for almost all $x \in \Omega$ and $\tau \in \mathbb{R}$.

Moreover, with the nonlocality at hand, we are able to determine the initial data without any knowledge of the nonlinearities.

Theorem 1.2 (*Recovery of the Initial Values by Passive Measurements*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $T > 0$ and $s > 0$ a non-integer. Suppose the nonlinearities f_j satisfy the conditions in Assumption 3.4 and $(u_{0,j}, u_{1,j}) \in \widetilde{H}^s(\Omega) \times L^2(\Omega)$. Let $\Lambda_j := \Lambda^{f_j}_{u_{0,j},u_{1,j}}$ be the DN maps of*

$$\begin{cases} \partial_t^2 u_j + (-\Delta)^s u_j + f_j(x,u_j) = 0 & \text{in } \Omega_T,\\ u_j = \varphi & \text{in } (\Omega_e)_T,\\ u_j(0) = u_{0,j}, \quad \partial_t u_j(0) = u_{1,j} & \text{in } \Omega, \end{cases} \tag{1.4}$$

for $j = 1,2$. Suppose that $\Lambda_1(0)\Big|_{(W_2)_T} = \Lambda_2(0)\Big|_{(W_2)_T}$ holds, then one has

$$u_{0,1} = u_{0,2}, \quad u_{1,1} = u_{1,2} \text{ in } \Omega.$$

Remark 1.3. Let us emphasize the following observations.

(i) As the exterior data $\varphi = 0$ in $(\Omega_e)_T$, this measurement is usually regarded as the *passive measurement*,[1] which forms a *single measurement* result.

(ii) The preceding theorem states that even though we might not be able to determine the unknown nonlinearity in general, we can still recover the initial data. Analogous results for local wave equations were studied in the work [2]. However, in order to determine the initial data in the local case, one needs some extra tools, such as observability estimate for linear wave equations. Nevertheless, the nonlocality helps us to recover the initial condition without any observability estimates.

In addition, when the coefficient $f(x,u) = a(x)u$, for sufficiently regular coefficient $a(x)$, we are able to determine both initial data and $a(x)$ simultaneously.

Corollary 1.4 (*Simultaneous Recovery of Both Initial Data and Coefficients*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $T > 0$ and $s > 0$ a non-integer. Suppose either $0 \leq a_j \in L^p(\Omega)$ for some p satisfying (3.9), or $a_j \in L^\infty(\Omega)$, and $(u_{0,j}, u_{1,j}) \in \widetilde{H}^s(\Omega) \times L^2(\Omega)$ for $j = 1,2$. Let $\Lambda_j := \Lambda^{a_j}_{u_{0,j},u_{1,j}}$ be the DN maps of the linear nonlocal wave equation*

$$\begin{cases} \partial_t^2 u_j + (-\Delta)^s u_j + a_j(x) u_j = 0 & \text{in } \Omega_T,\\ u_j = \varphi & \text{in } (\Omega_e)_T,\\ u_j(0) = u_{0,j}, \quad \partial_t u_j(0) = u_{1,j} & \text{in } \Omega, \end{cases}$$

for $j = 1,2$. Suppose that

$$\Lambda_1(\varphi)\Big|_{(W_2)_T} = \Lambda_2(\varphi)\Big|_{(W_2)_T}$$

[1] Here the passive measurements are generated by unknown sources, without injecting new inputs or affecting the existing one.

for any $\varphi \in C_c^\infty((W_1)_T)$, then there holds

$$u_{0,1} = u_{0,2}, \quad u_{1,1} = u_{1,2} \quad \text{and} \quad a_1 = a_2 \text{ in } \Omega.$$

1.2. Earlier literature

Inverse problems for nonlocal partial differential equations (PDEs) have been widely studied in the recent years. In the pioneering work [3] the authors determined unknown potential q in the fractional Schrödinger equation $(-\Delta)^s u + qu = 0$ in a bounded open set by using its exterior DN map. The main tools in their proof are the *unique continuation property* (UCP) for the fractional Laplacian, and the *Runge approximation property*. These two remarkable ingredients help us to study several inverse problems that are widely open in the local setting. For example, both a surrounding potential and an unknown inclusion can be determined simultaneously as shown in [4]. If-and-only-if -monotonicity relations have been derived in [5–7], and a determination of both drift and potentials result was proved in [8]. The aforementioned results are either open, or not true for their local counterparts. In short, we regard the nonlocality as a tool in solving related inverse problems.

So far, most of the existing works in this area are focused on the determination or reconstruction of lower order coefficients via their DN maps (for example, see [9–11]). Meanwhile, some nonlinear nonlocal inverse problems have been addressed in [12–15]. Very recently, the recovery of leading order coefficients in nonlocal operators has been investigated in several works, such as [16,17], where the authors found a novel reduction formula from the nonlocal to the local case via the Caffarelli–Silvestre type extension property. In [18], the authors determined the metric on closed Riemannian manifolds from the local source-to-solution map related to the fractional Laplace–Beltrami equation. In addition, Calderón-type inverse problems for the fractional conductivity operator are considered in [19–21], which converges to the usual conductivity operator in the classical limit $s \to 1^-$. Last but not least, there are even uniqueness results for leading order coefficients in nonlinear nonlocal equations such as equations of porous medium type [22] or of p-Laplace type [23,24].

Inverse problems for nonlinear (local) hyperbolic equations have been widely investigated. In fact, it is now known that the nonlinear interaction of waves can generate new waves, which can be taken advantage of when studying related inverse problems. This field started from [25], where it was proved that the local measurements can determine global topology and differentiable structure uniquely for a semilinear wave equation with a quadratic nonlinearity. In further, inverse problems were studied for general semilinear wave equations on Lorentzian manifolds [26], and related inverse problems were investigated for the Einstein–Maxwell equation in [27]. We also refer to [2,28–33], and the many fruitful references therein for more inverse problems for hyperbolic PDEs.

Next, let us mention that there are also a few results on inverse problems for nonlocal wave equations. On the one hand in [34] it has been shown that from the DN map related to

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega \end{cases}$$

one can uniquely determine the potential $q \in L^\infty(\Omega)$, when $0 < s < 1$. On the other hand in [35], the same unique determination question has been studied for the so called *(nonlinear) nonlocal viscous wave equation*

$$\begin{cases} \partial_t^2 u + (-\Delta)^s \partial_t u + (-\Delta)^s u + f(u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases} \tag{1.5}$$

In this article it has been established that unique determination of f holds if

(1) $f(u) = qu$ with $q \in L^\infty(0,T;L^p(\Omega))$, where p satisfies (3.9), and q has a certain continuity property in the time variable
(2) or f satisfying the condition (i) in Assumption 3.4, being $r+1$ homogeneous and $0 < r \leq 2$.

Let us point out that a main difference between the problems (1.1) and (1.5) is that in the later case one can establish a Runge approximation theorem in $L^2(0,T;\widetilde{H}^s(\Omega))$ (see [35, Proposition 4.2]) instead of $L^2(\Omega_T)$. This in turn allows to handle larger r values. Finally, let us recall that the difference of these approximation results rest on the fact that solutions to Eq. (1.5) are much more regular as for (1.1) and in fact the latter can be obtained by a certain approximation process of the first one (see [36] Claim 4.2 below). Actually, the loss term $(-\Delta)^s\partial_t$ regularizes the solution u to (1.5) such that $\partial_t u$ belongs to $L^2(0,T;H^s(\mathbb{R}^n))$, whereas the solution v of the nonlocal wave Eq. (1.1) only satisfies $\partial_t v \in L^2(\mathbb{R}^n_T)$.

1.3. Organization of the paper

We start in Section 2 by defining the used function spaces, the fractional Laplacian, and by recalling their main properties. Section 3 concerns the well-posedness of the nonlinear nonlocal wave equation. Under certain decay assumptions we show the existence, uniqueness and energy estimates for the solutions. We also define the exterior DN map, which will be the measurement data used for inverse problems later on. Finally, in Section 4 we study the continuity of the nonlinear terms, prove a Runge approximation theorem under arbitrary initial data, and finally establish determination of the nonlinear terms and initial data, given the DN map.

2. Preliminaries

Throughout this article the space dimension n is a fixed positive integer and $\Omega \subset \mathbb{R}^n$ is an open set. In this section, we introduce fundamental properties of function spaces and operators which will be used in our study.

2.1. Fractional Sobolev spaces and fractional Laplacian

We denote by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ Schwartz functions and tempered distributions respectively. We define the Fourier transform by

$$\mathcal{F}u(\xi) := \int_{\mathbb{R}^n} u(x)e^{-\mathrm{i}x\cdot\xi}\,dx,$$

which is occasionally also denoted by $\hat{u}$, where $\mathrm{i} = \sqrt{-1}$. By duality the Fourier transform can be extended to the space of tempered distributions where it will again be denoted by $\mathcal{F}u = \hat{u}$, where $u \in \mathscr{S}'(\mathbb{R}^n)$. We denote the inverse Fourier transform by $\mathcal{F}^{-1}$.

Given $s \in \mathbb{R}$, the L^2-based fractional Sobolev space $H^s(\mathbb{R}^n)$ is the set of all tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{H^s(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)} < \infty,$$

where $\langle D\rangle^s$ is the Bessel potential operator of order s with Fourier symbol

$$\left(1 + |\xi|^2\right)^{s/2}.$$

The fractional Laplacian of order $s \geq 0$ can be defined as the Fourier multiplier

$$(-\Delta)^s u = \mathcal{F}^{-1}\left(|\xi|^{2s}\,\hat{u}(\xi)\right),$$

for $u \in \mathscr{S}'(\mathbb{R}^n)$ whenever the right-hand side of the above identity is well-defined. In addition, it is also known that for $s \geq 0$, an equivalent norm on $H^s(\mathbb{R}^n)$ is given by

$$\|u\|^*_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}, \tag{2.1}$$

and the fractional Laplacian $(-\Delta)^s : H^t(\mathbb{R}^n) \to H^{t-2s}(\mathbb{R}^n)$ is a bounded linear operator for all $s \geq 0$ and $t \in \mathbb{R}$. The next results assert the unique continuation property (UCP) and a suitable Poincaré inequality for the fractional Laplacian on bounded domains $\Omega \subset \mathbb{R}^n$.

Proposition 2.1 (*UCP for Fractional Laplacians*). *Let $s > 0$ be a non-integer and $t \in \mathbb{R}$. If $u \in H^t(\mathbb{R}^n)$ satisfies $u = (-\Delta)^s u = 0$ in a nonempty open subset $V \subset \mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n$.*

The preceding proposition was first shown in [3, Theorem 1.2] for the case $s \in (0,1)$, in which case the fractional Laplacian $(-\Delta)^s$ can be equivalently computed as the singular integral (up to a constant depending on $n \in \mathbb{N}$ and $s \in (0,1)$)

$$(-\Delta)^s u(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\,dy$$

for sufficiently nice functions u, where p.v. stands for the Cauchy principal value. For the higher order case $s > 1$, one can apply the standard Laplacian to the equation, then the classical UCP for the Laplacian yields iteratively the desired result. With UCP at hand, one may derive a remarkable Runge approximation for nonlocal equations, which was first observed by [3, Theorem 1.3]. In addition, we will prove Runge approximation for a nonlocal wave equation in Proposition 4.1.

Proposition 2.2 (*Poincaré Inequality (cf. [37, Lemma 5.4])*). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For any $s \geq 0$, there exists $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\Omega)$.

Next we introduce some local variants of the above fractional Sobolev spaces. If $\Omega \subset \mathbb{R}^n$ is an open set, $F \subset \mathbb{R}^n$ a closed set and $s \in \mathbb{R}$, then we set

$$\begin{aligned} H^s(\Omega) &:= \left\{u|_\Omega : u \in H^s(\mathbb{R}^n)\right\}, \\ \widetilde{H}^s(\Omega) &:= \text{closure of } C_c^\infty(\Omega) \text{ in } H^s(\mathbb{R}^n), \\ H^s_F &:= \{u \in H^s(\mathbb{R}^n)\,;\, \mathrm{supp}(u) \subset F\}. \end{aligned}$$

Meanwhile, $H^s(\Omega)$ is a Banach space with respect to the quotient norm

$$\|u\|_{H^s(\Omega)} := \inf\left\{\|U\|_{H^s(\mathbb{R}^n)} : U \in H^s(\mathbb{R}^n) \text{ and } U|_\Omega = u\right\}.$$

Hence, using the fact that (2.1) is an equivalent norm on $\widetilde{H}^s(\Omega)$, Propositions 2.2 and the density of $C_c^\infty(\Omega)$ in $\widetilde{H}^s(\Omega)$, we have:

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $s \geq 0$. Then an equivalent norm on $\widetilde{H}^s(\Omega)$ is given by*

$$\|u\|_{\widetilde{H}^s(\Omega)} = \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}.$$

The observation of Lemma 2.3 will be of constant use in the well-posedness theorem below.

2.2. Bochner spaces

Next, we introduce some standard function spaces for time-dependent PDEs adapted to the nonlocal setting considered in this article. First if X is a given Banach space and $(a,b)\subset\mathbb{R}$, then we let $C^k([a,b];X)$, $L^p(a,b;X)$ ($k\in\mathbb{N}, 1\leq p<\infty$) stand for the space of k-times continuously differentiable functions and the space of measurable functions $u:(a,b)\to X$ such that $t\mapsto\|u(t)\|_X\in L^p([a,b])$ (with the usual modification for $p=\infty$). These spaces carry the norms

$$\begin{aligned}\|u\|_{L^p(a,b;X)} &:= \left(\int_a^b \|u(t)\|_X^p\,dt\right)^{1/p}<\infty,\\ \|u\|_{C^k([a,b];X)} &:= \sup_{0\leq\ell\leq k}\left\|\partial_t^\ell u\right\|_{L^\infty(a,b;X)}.\end{aligned} \tag{2.2}$$

Additionally, whenever $u\in L^1_{\mathrm{loc}}(a,b;X)$ with X being a space of functions over a subset of some euclidean space, such as $L^2(\Omega)$ or $H^s(\mathbb{R}^n)$, then u is identified with a function $u(x,t)$ and $u(t)$ denotes the function $x\mapsto u(x,t)$ for almost all t. This is justified by the fact, that any $u\in L^q(a,b;L^p(\Omega))$ with $1\leq q,p<\infty$ can be seen as a measurable function $u:\Omega\times(a,b)\to\mathbb{R}$ such that the norm $\|u\|_{L^q(a,b;L^p(\Omega))}$, as defined in (2.2), is finite. In particular, one has $L^p(0,T;L^p(\Omega))=L^p(\Omega_T)$ for $1\leq p<\infty$. Recall that we denote $A_T=A\times(0,T)$ for any set $A\subset\mathbb{R}^n$ and $T>0$. Clearly, a similar statement holds for the spaces $L^q(a,b;H^s(\mathbb{R}^n))$ and their local versions. Furthermore, the distributional derivative $\frac{du}{dt}\in\mathscr{D}'((a,b);X)$ is identified with the derivative $\partial_t u\in\mathscr{D}'(\Omega\times(a,b))$ as long as it is well-defined. Here $\mathscr{D}'((a,b);X)$ stands for all continuous linear operators from $C_c^\infty((a,b))$ to X.

3. The forward problem of nonlinear nonlocal wave equations

The main purpose of this section is to establish the well-posedness of the semilinear nonlocal wave equation for certain nonlinearities $f(u)$. As a preliminary step we first show the well-posedness of the linear nonlocal wave equation with a potential q only belonging to some L^p space.

3.1. Well-posedness

Let us start by stating the well-posedness theorem for the linear nonlocal wave equation.

Theorem 3.1 (*Well-Posedness of Linear Nonlocal Wave Equation*). *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $T>0$ and $s>0$ a non-integer. Suppose that $q\in L^p(\Omega)$, where p satisfies*

$$\begin{cases} n/s\leq p\leq\infty, & \text{if } 2s<n,\\ 2<p\leq\infty, & \text{if } 2s=n,\\ 2\leq p\leq\infty, & \text{if } 2s\geq n.\end{cases}$$

Then for any pair $(u_0,u_1)\in\widetilde{H}^s(\Omega)\times L^2(\Omega)$ and $F\in L^2(\Omega_T)$ there exists a unique solution $u\in C([0,T];\widetilde{H}^s(\Omega))$ with $\partial_t u\in C([0,T];L^2(\Omega))$ of

$$\begin{cases}\left(\partial_t^2+(-\Delta)^s+q\right)u=F & \text{in } \Omega_T,\\ u=0 & \text{in } (\Omega_e)_T,\\ u(0)=u_0,\quad \partial_t u(0)=u_1 & \text{in } \Omega.\end{cases} \tag{3.1}$$

More concretely, this means that there holds

$$\frac{d}{dt}\langle\partial_t u,v\rangle_{L^2(\Omega)}+\left\langle(-\Delta)^{s/2}u,(-\Delta)^{s/2}v\right\rangle_{L^2(\mathbb{R}^n)}+\langle qu,v\rangle_{L^2(\Omega)}=\langle F,v\rangle_{L^2(\Omega)}, \tag{3.2}$$

for all $v\in\widetilde{H}^s(\Omega)$ in the sense of $\mathscr{D}'((0,T))$. Furthermore, u satisfies the following energy identity

$$\begin{aligned}&\|\partial_t u(t)\|^2_{L^2(\Omega)}+\left\|(-\Delta)^{s/2}u(t)\right\|^2_{L^2(\mathbb{R}^n)}+2\langle qu,\partial_t u\rangle_{L^2(\Omega_t)}\\ &=\|u_1\|^2_{L^2(\Omega)}+\left\|(-\Delta)^{s/2}u_0\right\|^2_{L^2(\mathbb{R}^n)}+2\langle F,\partial_t u\rangle_{L^2(\Omega_t)}\end{aligned} \tag{3.3}$$

for all $t\in[0,T]$. Moreover, if $(u_{0,j},u_{1,j})\in\widetilde{H}^s(\Omega)\times L^2(\Omega)$, $F_j\in L^2(\Omega_T)$ and u_j denotes the related unique solution to (3.2) for $j=1,2$, then the following continuity estimate holds

$$\begin{aligned}&\|u_1-u_2\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))}+\|\partial_t u_1-\partial_t u_2\|_{L^\infty(0,T;L^2(\Omega))}\\ &\leq C\left(\|u_{0,1}-u_{0,2}\|_{\widetilde{H}^s(\Omega)}+\|u_{1,1}-u_{1,2}\|_{L^2(\Omega)}+\|F_1-F_2\|_{L^2(\Omega_T)}\right),\end{aligned} \tag{3.4}$$

for some $C>0$ depending on $T>0$.

Remark 3.2. Notice that the previous well-posedness result can easily be used to construct a unique solution $u \in C([0,T];H^s(\mathbb{R}^n))$ with $\partial_t u \in C([0,T];L^2(\mathbb{R}^n))$ to the problem

$$\begin{cases}\left(\partial_t^2 + (-\Delta)^s + q\right) u = f & \text{in } \Omega_T,\\ u = \varphi & \text{in } (\Omega_e)_T,\\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega,\end{cases}$$

whenever the initial conditions, force term, potential are as above and the exterior Dirichlet data φ is sufficiently regular, say $\varphi \in C^2([0,T];H^{2s}(\mathbb{R}^n))$. Furthermore, (3.4) and the mapping properties of the fractional Laplacian directly imply the continuity estimate

$$\begin{aligned}&\|u_1 - u_2\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \left\|\partial_t u_1 - \partial_t u_2\right\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\\ \leq C &\left(\left\|u_{0,1} - u_{0,2}\right\|_{H^s(\mathbb{R}^n)} + \left\|u_{1,1} - u_{1,2}\right\|_{L^2(\mathbb{R}^n)} + \left\|F_1 - F_2\right\|_{L^2(\Omega_T)}\right.\\ &\left. + \left\|\varphi_1 - \varphi_2\right\|_{C^2([0,T];H^{2s}(\mathbb{R}^n))} \right).\end{aligned}$$

Here, $(u_{0,j}, u_{1,j}) \in H^s(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $F_j \in L^2(\Omega)$, $\varphi_j \in C^2([0,T];\widetilde{H}^{2s}(\Omega_e))$ with $u_{0,j} - \varphi_j(0) \in \widetilde{H}^s(\Omega), u_{1,j} - \partial_t\varphi_j(0) \in L^2(\Omega)$ and u_j denotes the related unique solution to

$$\begin{cases}\left(\partial_t^2 + (-\Delta)^s + q\right) u = F_j & \text{in } \Omega_T,\\ u = \varphi_j & \text{in } (\Omega_e)_T,\\ u(0) = u_{0,j}, \quad \partial_t u(0) = u_{1,j} & \text{in } \Omega,\end{cases}$$

for $j = 1, 2$.

Proof of Theorem 3.1. Let us define

$$V = \widetilde{H}^s(\Omega), \quad H = L^2(\Omega), \quad V' = \widetilde{H}^s(\Omega)', \tag{3.5}$$

where V is endowed with the equivalent norm from Lemma 2.3, and

$$a_0(u,v) = \left\langle (-\Delta)^{s/2}u, (-\Delta)^{s/2}v\right\rangle_{L^2(\mathbb{R}^n)}, \quad a_1(u,v) = \langle qu, v\rangle_{L^2(\Omega)}. \tag{3.6}$$

for $u, v \in V$. Here $\widetilde{H}^s(\Omega)'$ denotes the antidual of $\widetilde{H}^s(\Omega)$. Observe that a_0 and a_1 are continuous sesquilinear forms on $\widetilde{H}^s(\Omega)$ and furthermore there holds

$$\left|\langle qu, v\rangle_{L^2(\Omega)}\right| \leq C\|q\|_{L^p(\Omega)}\|u\|_{\widetilde{H}^s(\Omega)}\|v\|_{L^2(\Omega)}$$

for some $C > 0$. The case $p = \infty$ is clear. In the case $\frac{n}{s} \leq p < \infty$ with $2s < n$ one can use Hölder's inequality with

$$\frac{1}{2} = \frac{n-2s}{2n} + \frac{s}{n},$$

$L^{r_2}(\Omega) \hookrightarrow L^{r_1}(\Omega)$ for $r_1 \leq r_2$ as $\Omega \subset \mathbb{R}^n$ is bounded and Sobolev's inequality to obtain

$$\begin{aligned}\left|\langle qu, v\rangle_{L^2(\Omega)}\right| &\leq \|qu\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\ &\leq \|q\|_{L^{n/s}(\Omega)}\|u\|_{L^{\frac{2n}{n-2s}}(\Omega)}\|v\|_{L^2(\Omega)}\\ &\leq C\|q\|_{L^{n/s}(\Omega)}\|u\|_{L^{\frac{2n}{n-2s}}(\Omega)}\|v\|_{L^2(\Omega)}\\ &\leq C\|q\|_{L^p(\Omega)}\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\Omega)}\end{aligned} \tag{3.7}$$

for all $u, v \in \widetilde{H}^s(\Omega)$.

In the case $2s > n$ one can use the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ together with Lemma 2.3 and the boundedness of Ω to see that the estimate (3.7) holds. In the case $n = 2s$ one can use the boundedness of the embedding $\widetilde{H}^s(\Omega) \hookrightarrow L^{\bar{p}}(\Omega)$ for all $2 \leq \bar{p} < \infty$, Hölder's inequality and the boundedness of Ω to get the estimate (3.7). In fact, the aforementioned embedding in the critical case follows by [38] and the Poincaré inequality. Next observe that the spaces V, H, V' and sesquilinear forms a_0, a_1 as given in (3.5), (3.6), respectively, satisfy the assumptions in [39, Chapter XVIII, §5,Sections 1.1,1.2,5.2]. Therefore, we can apply [39, Chapter XVIII, §5, Theorem 3 and 4] to deduce that there exists a unique solution u of (3.1) in the sense that (3.2) holds. Note that a priori this solution can be complex, but since we always take real-valued data in our problems we know that u is real-valued as well. Finally, the energy identity is nothing else than a rephrasing of [39, Chapter XVIII, §5, Lemma 7].

The continuity estimate (3.4) follows from the energy identity, since in this case we have

$$\begin{aligned}&\left\|\partial_t u_1(t) - \partial_t u_2(t)\right\|^2_{L^2(\Omega)} + \left\|u_1(t) - u_2(t)\right\|^2_{\widetilde{H}^s(\Omega)}\\ &+ 2\left\langle q(u_1 - u_2), \partial_t(u_1 - u_2)\right\rangle_{L^2(\Omega_t)}\\ = &\left\|u_{1,1} - u_{1,2}\right\|^2_{L^2(\Omega)} + \left\|u_{0,1} - u_{0,2}\right\|^2_{\widetilde{H}^s(\Omega)}\\ &+ 2\left\langle F_1 - F_2, \partial_t(u_1 - u_2)\right\rangle_{L^2(\Omega_t)}.\end{aligned}$$

Using Cauchy–Schwarz, Young's inequality, and the estimate (3.7) we obtain

$$\begin{aligned}&\left\|\partial_t u_1(t)-\partial_t u_2(t)\right\|^2_{L^2(\Omega)}+\left\|u_1(t)-u_2(t)\right\|^2_{\widetilde{H}^s(\Omega)}\\ \leq C&\left(\left\|u_{1,1}-u_{1,2}\right\|^2_{L^2(\Omega)}+\left\|u_{0,1}-u_{0,2}\right\|^2_{\widetilde{H}^s(\Omega)}\right.\\ &\left.+\left\|F_1-F_2\right\|^2_{L^2(\Omega_T)}+\int_0^t\left\|\partial_t u_1(\rho)-\partial_t u_2(\rho)\right\|^2_{L^2(\Omega)}\,d\rho\right).\end{aligned}$$

Finally, Grönwall's inequality and taking supremum over $[0,T]$ yield the claimed continuity property. Thus we can conclude the proof. □

Next, we move on to the nonlinear problem. We consider here slightly more general setting than in Eq. (1.1), containing also a nonlinear term $g(x,\partial_t u)$. For this purpose let us specify more precisely the assumptions we make on the nonlinearities f and g. We start by recalling the notion of a Carathéodory function.

Definition 3.3. Let $U\subset\mathbb{R}^n$ be an open set. We say that $f:U\times\mathbb{R}\to\mathbb{R}$ is a *Carathódory function*, if it has the following properties:

(i) $\tau\mapsto f(x,\tau)$ is continuous for a.e. $x\in U$,
(ii) $x\mapsto f(x,\tau)$ is measurable for all $\tau\in\mathbb{R}$.

Assumption 3.4. Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Omega\times\mathbb{R}\to\mathbb{R}$ be two Carathéodory functions satisfying the following conditions:

(i) f has partial derivative $\partial_\tau f$, which is a Carathéodory function, and there exists $a\in L^p(\Omega)$ such that

$$\left|\partial_\tau f(x,\tau)\right|\lesssim a(x)+|\tau|^r \tag{3.8}$$

for all $\tau\in\mathbb{R}$ and a.e. $x\in\Omega$.[2] Here the exponents p and r satisfy the restrictions

$$\begin{cases} n/s\leq p\leq\infty, & \text{if } 2s<n,\\ 2<p\leq\infty, & \text{if } 2s=n,\\ 2\leq p\leq\infty, & \text{if } 2s\geq n,\end{cases} \tag{3.9}$$

and

$$\begin{cases} 0\leq r<\infty, & \text{if } 2s\geq n,\\ 0\leq r\leq\frac{2s}{n-2s}, & \text{if } 2s<n,\end{cases} \tag{3.10}$$

respectively. Moreover, f fulfills the integrability condition $f(\cdot,0)\in L^2(\Omega)$.

(ii) There is a constant $C_1>0$ such that the function $F:\Omega\times\mathbb{R}\to\mathbb{R}$ defined via

$$F(x,\tau)=\int_0^\tau f(x,\rho)\,d\rho$$

satisfies $F(x,\tau)\geq -C_1$ for all $\tau\in\mathbb{R}$ and $x\in\Omega$.

(iii) g is uniformly Lipschitz continuous in $x\in\Omega$ with $g(\cdot,0)\in L^2(\Omega)$.

Remark 3.5. Let $q\in L^\infty(\Omega)$ be a non-negative function. An example of a nonlinearity f, which satisfies the conditions in Assumption 3.4 is given by a fractional power type nonlinearity $f(x,\tau)=q(x)|\tau|^r\tau$ for $r\geq 0$, which satisfies (3.10). The regularity conditions are clearly fulfilled. Moreover, one easily checks that there holds

$$\partial_\tau f(x,\tau)=q(x)(r+1)|\tau|^r \tag{3.11}$$

and

$$\frac{d}{d\tau}|\tau|^{r+2}=(r+2)|\tau|^r\tau \tag{3.12}$$

for $\tau\neq 0$. It is not hard to see that the first identity (3.11) implies (3.8). On the other hand, from the second identity (3.12), we can get

$$F(x,\tau)=q(x)\int_0^\tau|\rho|^r\rho\,d\rho=\frac{q(x)}{r+2}\int_0^\tau\frac{d}{d\rho}|\rho|^{r+2}\,d\rho=q(x)\frac{|\tau|^{r+2}}{r+2}\geq 0$$

and hence the condition (ii) is satisfied as well. We also point out the condition (3.9) of the exponent p satisfies the assumption in Theorem 3.1.

[2] The symbols $\lesssim$ denotes the inequality holds up to a positive constant whose value is irrelevant for our arguments.

Theorem 3.6 (*Well-Posedness of Nonlinear Nonlocal Wave Equation*). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$ and $s > 0$ a non-integer. Suppose that f, F and g satisfy Assumption 3.4. Then for any pair $(u_0, u_1) \in \widetilde{H}^s(\Omega) \times L^2(\Omega)$ and $h \in L^2(\Omega_T)$ there exists a unique solution*

$$u \in C([0,T]; \widetilde{H}^s(\Omega)) \quad \textit{and} \quad \partial_t u \in C([0,T]; L^2(\Omega))$$

of

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) + g(x, \partial_t u) = h & \textit{in } \Omega_T, \\ u = 0 & \textit{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \textit{in } \Omega. \end{cases} \tag{3.13}$$

More concretely, this means that there holds

$$\begin{aligned} &\frac{d}{dt} \langle \partial_t u, v \rangle_{L^2(\Omega)} + \left\langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v \right\rangle_{L^2(\mathbb{R}^n)} \\ &\quad + \langle f(\cdot, u), v \rangle_{L^2(\Omega)} + \langle g(\cdot, \partial_t u), v \rangle_{L^2(\Omega)} = \langle h, v \rangle_{L^2(\Omega)} \end{aligned} \tag{3.14}$$

for all $v \in \widetilde{H}^s(\Omega)$ in the sense of $\mathscr{D}'((0,T))$. Furthermore, u satisfies the following energy identity

$$\begin{aligned} &\|\partial_t u(t)\|_{L^2(\Omega)}^2 + \left\|(-\Delta)^{s/2} u(t)\right\|_{L^2(\mathbb{R}^n)}^2 \\ &+ 2 \int_\Omega F(x, u(t))dx + 2 \langle g(\cdot, \partial_t u), \partial_t u \rangle_{L^2(\Omega_t)} \\ &= \|u_1\|_{L^2(\Omega)}^2 + \left\|(-\Delta)^{s/2} u_0\right\|_{L^2(\mathbb{R}^n)}^2 + 2 \langle h, \partial_t u \rangle_{L^2(\Omega_t)} + 2 \int_\Omega F(x, u_0)\, dx \end{aligned} \tag{3.15}$$

for all $t \in [0,T]$.

Proof. Let $\psi \in C([0,T]; L^2(\Omega))$. First we show that $g(\cdot, \psi)$, $f(\cdot, \psi)$ are in $L^2(\Omega_T)$. As g is a Carathéodory function the composition $g(x, \psi(x,t))$ is measurable for every $t \in [0,T]$. Additionally, the Lipschitz continuity of $g(x, \cdot)$ directly implies that

$$|g(x, \psi(x,t))| \leq C(|g(x,0)| + |\psi(x,t)|)$$

and we have

$$\|g(\cdot, \psi)\|_{L^2(\Omega_T)} \leq C T^{1/2} \left(\|g(\cdot, 0)\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(0,T;L^2(\Omega))} \right), \tag{3.16}$$

for some constant $C > 0$. By the same argument as for g the composition $f(x, \psi(x,t))$ is measurable for every $t \in [0,T]$. Moreover, by the fundamental theorem of calculus and Assumption 3.4 we have

$$|f(x,s) - f(x,0)| \leq \left| \int_0^s \partial_\tau f(x,\tau)\, d\tau \right| \leq C \left(|a(x)||s| + |s|^{r+1} \right) \tag{3.17}$$

for a.e. $x \in \Omega$, $s \in \mathbb{R}$. This implies

$$|f(x, \psi(x,t))| \leq C \left(|f(x,0)| + |a(x)||\psi(x,t)| + |\psi(x,t)|^{r+1} \right).$$

This guarantees that

$$\|f(\cdot, \psi(t))\|_{L^2(\Omega)} \leq C \left(\|f(\cdot, 0)\|_{L^2(\Omega)} + \|a\psi(t)\|_{L^2(\Omega)} + \|\psi(t)\|_{L^{2(r+1)}(\Omega)}^{r+1} \right),$$

for $0 \leq t \leq T$. From now on let us additionally assume that $\psi \in C([0,T]; H^s(\mathbb{R}^n))$ and without loss of generality we can assume $r > 0$. Then the computation in (3.7) allows to estimate

$$\|a\psi(t)\|_{L^2(\Omega)} \leq C \|a\|_{L^p(\Omega)} \left\|(-\Delta)^{s/2} \psi(t)\right\|_{L^2(\mathbb{R}^n)}.$$

Next note that

$$\begin{cases} 1 \leq 1 + r < \infty, & \text{if } 2s \geq n, \\ 1 \leq 1 + r \leq \frac{n}{n-2s} & \text{if } 2s < n. \end{cases}$$

If $2s > n$, then the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ implies

$$\|\psi(t)\|_{L^{2(r+1)}(\Omega)}^{r+1} \leq C \|\psi(t)\|_{H^s(\mathbb{R}^n)}^{r+1}.$$

In the critical case $2s = n$, we can apply [38] to obtain

$$\|\psi(t)\|_{L^{2(r+1)}(\Omega)}^{r+1} \leq C \|(-\Delta)^{s/2} \psi(t)\|_{L^2(\mathbb{R}^n)}^r \|\psi(t)\|_{L^2(\mathbb{R}^n)}.$$

In the subcritical case $2s < n$, we apply the Hardy–Littlewood–Sobolev lemma to deduce

$$\|\psi(t)\|_{L^{2(r+1)}(\Omega)}^{r+1} \leq C \|\psi(t)\|_{L^{\frac{2n}{n-2s}}(\Omega)}^{r+1} \leq C \left\|(-\Delta)^{s/2} \psi(t)\right\|_{L^2(\mathbb{R}^n)}^{r+1}.$$

As $\psi \in C([0,T];H^s(\mathbb{R}^n))$, we get by the continuity of the fractional Laplacian the estimate

$$\|f(\cdot,\psi)\|_{L^2(\Omega_T)} \lesssim T^{1/2}\left(\|f(\cdot,0)\|_{L^2(\Omega)} + \|a\|_{L^p(\Omega)}\|\psi\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\psi\|^{r+1}_{L^\infty(0,T;H^s(\mathbb{R}^n))}\right). \tag{3.18}$$

Next, let us set

$$A := \max\left(\|u_0\|_{\widetilde{H}^s(\Omega)}, \|u_1\|_{L^2(\Omega)}\right), \tag{3.19}$$

and for constants $0 < T_0 \le T$ and $R \ge A$, which will be fixed later. Consider the space

$$X_{T_0,R} := \left\{ u \in C([0,T_0];\widetilde{H}^s(\Omega)) \cap C^1([0,T_0];L^2(\Omega)) : \|u\|_{T_0} \le R \right\},$$

where $\|\cdot\|_{T_0}$ is given by

$$\|u\|_{T_0} := \max\left(\|u\|_{L^\infty(0,T_0;\widetilde{H}^s(\Omega))}, \|\partial_t u\|_{L^\infty(0,T_0;L^2(\Omega))}\right).$$

It is a known fact that under the given norm $C([0,T_0];\widetilde{H}^s(\Omega)) \cap C^1([0,T_0];L^2(\Omega))$ becomes a Banach space and thus $X_{T_0,R}$ is a complete metric space. Now, for given $v \in X_{T_0,R}$, let us consider the linear problem

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s\right) u = h - f(x,v) - g(x,\partial_t v) & \text{in } \Omega_{T_0}, \\ u = 0 & \text{in } (\Omega_e)_{T_0}, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases} \tag{3.20}$$

Theorem 3.1 yields the well-posedness of (3.20), that is, there exists a unique solution $u \in C([0,T_0];\widetilde{H}^s(\Omega)) \cap C^1([0,T_0];L^2(\Omega))$ to (3.20), and we can define a map

$$S : X_{T_0,R} \to C([0,T_0];\widetilde{H}^s(\Omega)) \cap C^1([0,T_0];L^2(\Omega)),$$

which maps any $v \in X_{T_0,R}$ to its unique solution $S(v)$ of (3.20). The rest of the proof is divided into four steps.

Step 1. First we show that there exists $R_0 \ge A$ and for all $R \ge R_0$ there exists $T_0 = T_0(R_0) > 0$ such that $S(X_{T_0,R}) \subset X_{T_0,R}$.

By the energy identity of Theorem 3.1, Hölder's inequality and Young's inequality, there holds

$$\begin{aligned} &\|\partial_t u(t)\|^2_{L^2(\Omega)} + \left\|(-\Delta)^{s/2}u(t)\right\|^2_{L^2(\mathbb{R}^n)} \\ &= \|u_1\|^2_{L^2(\Omega)} + \left\|(-\Delta)^{s/2}u_0\right\|^2_{L^2(\mathbb{R}^n)} + 2\left\langle h - f(\cdot,v) - g(\cdot,\partial_t v), \partial_t u\right\rangle_{L^2(\Omega_t)} \\ &\le \|u_1\|^2_{L^2(\Omega)} + \left\|(-\Delta)^{s/2}u_0\right\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))} \\ &\quad + 2T_0\|h - f(\cdot,v) - g(\cdot,\partial_t v)\|^2_{L^2(\Omega_{T_0})} \\ &\le \|u_1\|^2_{L^2(\Omega)} + \left\|(-\Delta)^{s/2}u_0\right\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))} \\ &\quad + CT_0\left(\|h\|^2_{L^2(\Omega_{T_0})} + \|f(\cdot,v)\|^2_{L^2(\Omega_{T_0})} + \|g(\cdot,\partial_t v)\|^2_{L^2(\Omega_{T_0})}\right) \end{aligned}$$

for all $0 \le t \le T_0$ and $\varepsilon > 0$. Taking the supremum over $[0,T_0]$, absorbing the third term on the left-hand side and using (3.19) we get

$$\begin{aligned} &\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))} + \left\|(-\Delta)^{s/2}u\right\|^2_{L^\infty(0,T_0;L^2(\mathbb{R}^n))} \\ &\le 4A^2 + CT_0\left(\|h\|^2_{L^2(\Omega_{T_0})} + \|f(\cdot,v)\|^2_{L^2(\Omega_{T_0})} + \|g(\cdot,\partial_t v)\|^2_{L^2(\Omega_{T_0})}\right). \end{aligned}$$

By (3.16), (3.18), Lemma 2.3 and the definition of the space $X_{T_0,R}$, we get

$$\begin{aligned} &\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))} + \left\|(-\Delta)^{s/2}u\right\|^2_{L^\infty(0,T_0;L^2(\mathbb{R}^n))} \\ &\le 4A^2 + CT_0\|h\|^2_{L^2(\Omega_T)} \\ &\quad + CT_0^2\left(\|f(\cdot,0)\|^2_{L^2(\Omega)} + \|a\|^2_{L^p(\Omega)}R^2 + R^{2(r+1)} + \|g(\cdot,0)\|^2_{L^2(\Omega)} + R^2\right). \end{aligned}$$

This implies

$$\|u\|_{T_0} \le 2A + C_0 T_0^{1/2} + C_1 T_0(1 + R + R^{r+1}),$$

where $C_0, C_1 > 0$ are constants only depending on n, s, Ω and the norms $\|f(\cdot,0)\|_{L^2(\Omega)}$, $\|g(\cdot,0)\|_{L^2(\Omega)}$, $\|h\|_{L^2(\Omega_T)}$ as well as the constants appearing in Assumption 3.4. By choosing

$$R_0 \ge 4A \quad \text{and} \quad T_0 \le \min\left(\frac{R^2}{4C_0}, \frac{R}{4C_1(1 + R + R^{r+1})}\right), \tag{3.21}$$

we get $\|u\| \leq R$, whenever $R \geq R_0$. Note that T_0 only depends on n, s, Ω, the functions f, g, h and A measuring the size of the initial conditions.

Step 2. Next we prove that by making $T_0(R) > 0$ possibly smaller, the map $S : X_{T_0,R} \to X_{T_0,R}$ is a (strict) contraction.

Assume that R_0 and T_0 are chosen as in (3.21). Let us start by observing the estimate

$$|f(x,t) - f(x,s)| \leq C\left(|a(x)| + |t|^r + |s|^r\right)|t-s| \tag{3.22}$$

for all $s,t \in \mathbb{R}$ and a.e. $x \in \Omega$. Let $u^j = S(v^j) \in C([0,T_0]; \widetilde{H}^s(\Omega)) \cap C^1([0,T_0]; L^2(\Omega))$ for $j = 1,2$. Then we may estimate

$$\begin{aligned} &\left\|f(\cdot, v^1(t)) - f(\cdot, v^2(t))\right\|_{L^2(\Omega)} \\ &\leq C\left(\left\|a(v^1(t) - v^2(t))\right\|_{L^2(\Omega)} + \left\|(|v^1(t)|^r + |v^2(t)|^r)|v^1(t) - v^2(t)|\right\|_{L^2(\Omega)}\right) \\ &\leq C\left(\|a\|_{L^p(\Omega)}\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} + \left\|(|v^1(t)|^r + |v^2(t)|^r)|v^1(t) - v^2(t)|\right\|_{L^2(\Omega)}\right), \end{aligned} \tag{3.23}$$

for all $0 \leq t \leq T_0$, where we used in the second inequality (3.7) and (3.22). Next suppose that $2s < n$, then Hölder's inequality with

$$\frac{1}{2} = \frac{n-2s}{2n} + \frac{s}{n}$$

and the Sobolev embedding imply

$$\begin{aligned} &\left\|(|v^1(t)|^r + |v^2(t)|^r)|v^1(t) - v^2(t)|\right\|_{L^2(\Omega)} \\ &\leq \left\||v^1(t)|^r + |v^2(t)|^r\right\|_{L^{n/s}(\Omega)} \|v^1(t) - v^2(t)\|_{L^{\frac{2n}{n-2s}}(\Omega)} \\ &\leq C\left(\left\||v^1(t)|^r\right\|_{L^{n/s}(\Omega)} + \left\||v^2(t)|^r\right\|_{L^{n/s}(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)}. \end{aligned} \tag{3.24}$$

If $r \geq s/n$, then $1 \leq rn/s \leq \frac{2n}{n-2s}$ and thus the boundedness of Ω as well as the Sobolev embedding ensure that

$$\begin{aligned} &\left\|(|v^1(t)|^r + |v^2(t)|^r)|v^1(t) - v^2(t)|\right\|_{L^2(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{L^{rn/s}(\Omega)} + \left\|v^2(t)\right\|^r_{L^{rn/s}(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{L^{\frac{2n}{n-2s}}(\Omega)} + \left\|v^2(t)\right\|^r_{L^{\frac{2n}{n-2s}}(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{\widetilde{H}^s(\Omega)} + \left\|v^2(t)\right\|^r_{\widetilde{H}^s(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)}, \end{aligned} \tag{3.25}$$

for some constant $C > 0$ independent of v^1 and v^2.

Next assume that $0 \leq r < s/n$. For $r = 0$ the previous estimate still holds and so we can assume $0 < r < s/n$. In this situation, we choose $z \geq 1$ such that $1 \leq rz \leq 2$. This implies $n/s < 1/r \leq z$ and thus (3.24), Hölder's inequality and Poincaré's inequality ensure

$$\begin{aligned} &\left\|(|v^1(t)|^r + |v^2(t)|^r)|v^1(t) - v^2(t)|\right\|_{L^2(\Omega)} \\ &\leq C\left(\left\||v^1(t)|^r\right\|_{L^{n/s}(\Omega)} + \left\||v^2(t)|^r\right\|_{L^{n/s}(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \\ &\leq C\left(\left\||v^1(t)|^r\right\|_{L^z(\Omega)} + \left\||v^2(t)|^r\right\|_{L^z(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{L^{rz}(\Omega)} + \left\|v^2(t)\right\|^r_{L^{rz}(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{L^2(\Omega)} + \left\|v^2(t)\right\|^r_{L^2(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{\widetilde{H}^s(\Omega)} + \left\|v^2(t)\right\|^r_{\widetilde{H}^s(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)}. \end{aligned}$$

Hence, for $2s < n$ we have for all $0 \leq r \leq \frac{2}{n-2s}$ the estimate (3.25). Next, suppose $2s > n$, then the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and the Poincaré inequality directly give

$$\begin{aligned} &\left\|(|v^1(t)|^r + |v^2(t)|^r)|v^1(t) - v^2(t)|\right\|_{L^2(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{L^\infty(\Omega)} + \left\|v^2(t)\right\|^r_{L^\infty(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{L^2(\Omega)} \\ &\leq C\left(\left\|v^1(t)\right\|^r_{\widetilde{H}^s(\Omega)} + \left\|v^2(t)\right\|^r_{\widetilde{H}^s(\Omega)}\right)\left\|v^1(t) - v^2(t)\right\|_{\widetilde{H}^s(\Omega)} \end{aligned}$$

and so again obtain (3.25). Finally, assume that the critical case $2s = n$ holds. If $r > 0$, then we choose $z > 2$ such that $rz > 2$, use the Hölder inequality with $2 < p_0 < \infty$ satisfying

$$\frac{1}{2} = \frac{1}{z} + \frac{1}{p_0}$$

and [38] together with Poincaré's inequality to get

$$\begin{aligned}
&\left\|\left(|v^1(t)|^r+|v^2(t)|^r\right)\left|v^1(t)-v^2(t)\right|\right\|_{L^2(\Omega)}\\
&\leq C\left(\left\|v^1(t)\right\|^r_{L^{rz}(\Omega)}+\left\|v^2(t)\right\|^r_{L^{rz}(\Omega)}\right)\left\|v^1(t)-v^2(t)\right\|_{L^{p_0}(\Omega)}\\
&\leq C\left(\left\|v^1(t)\right\|^r_{\widetilde{H}^s(\Omega)}+\left\|v^2(t)\right\|^r_{\widetilde{H}^s(\Omega)}\right)\left\|v^1(t)-v^2(t)\right\|_{\widetilde{H}^s(\Omega)}.
\end{aligned}$$

Therefore, also in this case we have the final estimate in (3.25). The remaining case $r=0$ is immediate. Hence, we have in all cases (3.25) and therefore inserting this into (3.23), we deduce the bound

$$\begin{aligned}
&\left\|f(\cdot,v^1(t))-f(\cdot,v^2(t))\right\|_{L^2(\Omega)}\\
&\leq C\left(\|a\|_{L^p(\Omega)}+\left\|v^1(t)\right\|^r_{\widetilde{H}^s(\Omega)}+\left\|v^2(t)\right\|^r_{\widetilde{H}^s(\Omega)}\right)\left\|v^1(t)-v^2(t)\right\|_{\widetilde{H}^s(\Omega)}.
\end{aligned} \tag{3.26}$$

Next, note that by Theorem 3.1 the function $u=u^1-u^2$ is the unique solution to

$$\begin{cases}
\left(\partial_t^2+(-\Delta)^s\right)u=-(f(x,v^1)-f(x,v^2))-(g(x,\partial_t v^1)-g(x,\partial_t v^2)) & \text{in } \Omega_T,\\
u=0 & \text{in } (\Omega_e)_T,\\
u(0)=0,\quad \partial_t u(0)=0 & \text{in } \Omega
\end{cases}$$

and satisfies the energy estimate

$$\begin{aligned}
&\|\partial_t u(t)\|^2_{L^2(\Omega)}+\left\|(-\Delta)^{s/2}u(t)\right\|^2_{L^2(\mathbb{R}^n)}\\
&\leq C\left(\left\|f(\cdot,v^1)-f(\cdot,v^2)\right\|^2_{L^1(0,T_0;L^2(\Omega))}+\left\|g(\cdot,\partial_t v^1)-g(\cdot,\partial_t v^2)\right\|^2_{L^1(0,T_0;L^2(\Omega))}\right)\\
&\quad+\frac{1}{2}\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))}\\
&\leq CT_0^2\left(\left\|f(\cdot,v^1)-f(\cdot,v^2)\right\|^2_{L^\infty(0,T_0;L^2(\Omega))}+\left\|g(\cdot,\partial_t v^1)-g(\cdot,\partial_t v^2)\right\|^2_{L^\infty(0,T_0;L^2(\Omega))}\right)\\
&\quad+\frac{1}{2}\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))}
\end{aligned}$$

for all $0\leq t\leq T_0$. For the energy estimate we used additionally the Hölder and Young inequalities. Taking the supremum over $0\leq t\leq T_0$, absorbing the third term on the left-hand side, using the estimate (3.26) and the Lipschitz continuity of g, we deduce

$$\begin{aligned}
&\|\partial_t u\|^2_{L^\infty(0,T_0;L^2(\Omega))}+\|u\|^2_{L^\infty(0,T_0;\widetilde{H}^s(\Omega))}\\
&\leq CT_0^2\left(\|a\|^2_{L^p(\Omega)}+\left\|v^1\right\|^{2r}_{L^\infty(0,T_0;\widetilde{H}^s(\Omega))}+\left\|v^2\right\|^{2r}_{L^\infty(0,T_0;\widetilde{H}^s(\Omega))}\right)\\
&\qquad\cdot\left\|v^1-v^2\right\|^2_{L^\infty(0,T_0;\widetilde{H}^s(\Omega))}\\
&\quad+CT_0^2\left\|\partial_t v^1-\partial_t v^2\right\|^2_{L^\infty(0,T_0;L^2(\Omega))}.
\end{aligned}$$

This implies

$$\left\|S(v^1)-S(v^2)\right\|_{T_0}\leq C_2T_0\left(1+R^r\right)\left\|v^1-v^2\right\|_{T_0}$$

as $u^1,u^2\in X_{T_0,R}$. Note that $C_2>0$ only depends on the constants in Assumption 3.4, the parameters n,s,r,p, Ω and a. Thus, choosing

$$T_0\leq\min\left(\frac{R^2}{4C_0},\frac{R}{4C_1(1+R+R^{r+1})},\frac{1}{2C_2(1+R^r)}\right) \tag{3.27}$$

we see that

$$\|S(v^1)-S(v^2)\|_{T_0}\leq\frac{1}{2}\left\|v^1-v^2\right\|_{T_0},$$

making $S:X_{T_0,R}\to X_{T_0,R}$ a (strict) contraction.

Step 3. Next we show that there exists a unique local solution to problem (3.13).

Now, if we fix R_0 as in (3.21), take $R\geq R_0$ and let T_0 satisfy (3.27), then we know that S is a strict contraction from the complete metric space $X_{T_0,R}$ to itself. Thus, by the Banach contraction principle there is a unique fixed point $u\in X_{T_0,R}$, that is a solution to (3.13) with $T=T_0$. Next, we assert that if $\widetilde{u}\in C([0,T_0];\widetilde{H}^s(\Omega))\cap C^1([0,T_0];L^2(\Omega))$ is any other solution to (3.13) on $[0,T_0]$, then we have $\widetilde{u}=u$ on $[0,T_0]$. First note that since $\widetilde{u}(0)=u_0,\partial_t\widetilde{u}(0)=u_1$, we can pick $0<T_1\leq T_0$ such that $\|\widetilde{u}\|_{T_1}\leq R_0$. Next fix any $R\geq R_0$. Then by Steps 1 and 2, we can find a $0<T_2\leq T_1$ such that $S:X_{T_2,R}\to X_{T_2,R}$ is a contraction. But then the contraction principle implies that $\widetilde{u}=u$ on $[0,T_2]$. We next claim that we have $T_2=T_0$. For this purpose, let us define

$$I=\left\{t\in[0,T_0]\,:\,\widetilde{u}(t)=u(t)\right\}.$$

We already know that I is non-empty and closed, which follows from the continuity of the functions $\widetilde{u}$ and u. Next, we show that it is also open. Assume that $t_0 \in I$ and set

$$A' = \max\left(\|u(\cdot,t_0)\|_{\widetilde{H}^s(\Omega)}, \|\partial_t u(\cdot,t_0)\|_{L^2(\Omega)}\right).$$

Let $R' > 0$ satisfy $R' \geq R'_0 = 4A'$ and $R' > R$. Then Step 1, 2 imply that there exists $t_0 < T'_0 \leq T_0$ such that there is a unique fixed point of $S_{t_0} : X^{t_0}_{T'_0,R'} \to X^{t_0}_{T'_0,R'}$. Here the sub/subscripts indicate that we are considering everything over the time interval $[t_0, t_0 + T'_0]$. Making T'_0 possibly smaller, we have $\widetilde{u}, u \in X^{t_0}_{T'_0,R'}$ and hence it follows that $u = \widetilde{u}$ on $\left[t_0, t_0 + T'_0\right]$. Using the time-reversal symmetry of the wave operator, the same holds on some interval of the form $\left[t_0 - T''_0, t_0\right]$. This shows that there is an open neighborhood of t_0 contained I. Therefore, I is also open and by connectedness we have $I = [0, T_0]$. Hence, the constructed local solution is unique.

Step 4. Finally, we show that the constructed local solution can be extended to the unique global solution and the energy identity (3.15) holds.

First note that by Step 3, we can extend the local solution to a map $u : \Omega \times [0, T_*) \to \mathbb{R}$, where $T_* \leq T$ denotes the maximal time of existence having the property that for any $T' < T_*$ the function $u : \Omega \times [0, T'] \to \mathbb{R}$ solves (3.13).

Now, let us fix any $0 < T' < T_*$. Then we have $u \in H^1(0, T'; L^2(\Omega))$ and Fubini's theorem ensures $u(x,\cdot) \in H^1([0,T'])$ for a.e. $x \in \Omega$. Next, let us fix $x \in \Omega$ such that $u(x,\cdot) \in H^1([0,T'])$. By the embedding $H^1([0,T']) \hookrightarrow L^\infty([0,T'])$ and $F(x,\cdot) \in C^{0,1}_{loc}(\mathbb{R})$ (see (3.17)), we can modify $\tau \mapsto F(x,\tau)$ outside a neighborhood of the compact set $\left[-\|u(x,\cdot)\|_{L^\infty([0,T'])}, \|u(x,\cdot)\|_{L^\infty([0,T'])}\right]$ such that $F(x,\cdot) \in C^{0,1}(\mathbb{R})$. Then [40, Theorem 2.1.11] guarantees $F(x, u(x,\cdot)) \in H^1((0,T'))$ and there holds

$$\partial_t F(x, u(x,t)) = f(x, u(x,t))\partial_t u(x,t).$$

Thus, the fundamental theorem of calculus shows

$$F(x, u(x,t)) = F(x, u_0(x)) + \int_0^t f(x, u(x,s))\partial_t u(x,s)\,ds$$

for all $0 \leq t \leq T'$. Integrating this identity over Ω and using Fubini's theorem, we get

$$\int_\Omega F(x, u(x,t))\,dx = \int_\Omega F(x, u_0(x))\,dx + \int_0^t \int_\Omega f(x, u(x,s))\partial_t u(x,s)\,dxds \tag{3.28}$$

for all $0 \leq t \leq T'$. As $u : \Omega \times [0,T'] \to \mathbb{R}$ solves the linear problem

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s\right) u = h - f(\cdot,u) - g(\cdot,\partial_t u) & \text{in } \Omega_{T'}, \\ u = 0 & \text{in } (\Omega_e)_{T'}, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases}$$

By assumption and Step 1, we know that the right-hand side is in $L^2(\Omega_{T'})$ and we deduce from Theorem 3.1 that u satisfies the energy identity

$$\begin{aligned} &\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{\widetilde{H}^s(\Omega)} \\ &= \|u_1\|^2_{L^2(\Omega)} + \|u_0\|^2_{\widetilde{H}^s(\Omega)} + 2\langle h - f(\cdot,u) - g(\cdot,\partial_t u), \partial_t u\rangle_{L^2(\Omega_t)} \end{aligned}$$

for all $0 \leq t \leq T'$. By (3.28) this can be rewritten as

$$\begin{aligned} &\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{\widetilde{H}^s(\Omega)} + 2\int_\Omega F(x,u(x,t))\,dx + 2\langle g(x,\partial_t u), \partial_t u\rangle_{L^2(\Omega_t)} \\ &= \|u_1\|^2_{L^2(\Omega)} + \|u_0\|^2_{\widetilde{H}^s(\Omega)} + 2\langle h, \partial_t u\rangle_{L^2(\Omega_t)} + 2\int_\Omega F(x, u_0(x))\,dx \end{aligned} \tag{3.29}$$

for all $0 \leq t \leq T'$. Next, recall that $\inf_{\tau\in\mathbb{R}} F(x,\tau) \geq -C$ and therefore we obtain by Hölder's inequality the estimate

$$\begin{aligned} &\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{\widetilde{H}^s(\Omega)} \\ &\leq 2\langle h, \partial_t u\rangle_{L^2(\Omega_t)} - 2\langle g(x,\partial_t u), \partial_t u\rangle_{L^2(\Omega_t)} + C \\ &\leq C' \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\,ds + C \end{aligned}$$

for all $0 \leq t \leq T'$, where $C, C' > 0$ are independent of T'. Now, Grönwall's inequality implies

$$\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{\widetilde{H}^s(\Omega)} \leq C(1 + C' t e^{C't})$$

for all $0 \leq t \leq T'$. This shows that there exists $C > 0$ such that

$$\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{\widetilde{H}^s(\Omega)} \leq C$$

for all $0 \leq t < T_*$. Next recall that the extension time T_0 only depends on the size of the initial condition and therefore, choosing T' sufficiently close to T_* we can extend the solution beyond T_*. For this recall the local uniqueness of solutions. This contradicts the claim that $T_* < T$ is maximal. Therefore we can conclude that $T_* = T$. Now, since $u \in C([0,T]; \widetilde{H}^s(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ we can pass to the limit $T' \to T$ in (3.29) and obtain (3.15). □

Proposition 3.7 (*Well-Posedness of Nonlinear Nonlocal Wave Equation with Nonzero Exterior Condition*). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$ and $s > 0$ a non-integer. Suppose that f and g satisfy Assumption 3.4. Then for any pair $(u_0, u_1) \in \widetilde{H}^s(\Omega) \times L^2(\Omega)$, $\varphi \in C^2([0,T]; \widetilde{H}^{2s}(\Omega_e))$ and $h \in L^2(\Omega_T)$, there exists a unique solution u of*

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) + g(x,\partial_t u) = h & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases} \tag{3.30}$$

This means

(i) $u \in C([0,T]; H^s(\mathbb{R}^n))$ *with* $\partial_t u \in C([0,T]; L^2(\mathbb{R}^n))$,
(ii) $u = \varphi$ *a.e. in* $(\Omega_e)_T$,
(iii) (3.14) *holds for all* $v \in \widetilde{H}^s(\Omega)$ *in the sense of* $\mathscr{D}'((0,T))$
(iv) *and* $u(0) = u_0$, $\partial_t u(0) = u_1$.

Proof. If $u \in C([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n))$ and $u = \varphi$ a.e. in $(\Omega_e)_T$, then $v = u - \varphi \in C([0,T]; \widetilde{H}^s(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. Hence, u solves (3.30) if and only if v solves (3.13) with right-hand side given by $h - (-\Delta)^s\varphi$. The latter problem is according to Theorem 3.6 well-posed and hence we obtain well-posedness for the problem (3.30). □

Remark 3.8. Let us adopt the assumptions of Proposition 3.7. Assume additionally $g = 0$ and $\inf_{\tau\in\mathbb{R}} F(x,\tau) \geq 0$. Then the solution u of (3.30) satisfies the energy estimate

$$\begin{aligned} &\|u-\varphi\|^2_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t(u-\varphi)\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C\left(\|u_0\|^2_{\widetilde{H}^s(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|h - (-\Delta)^s\varphi\|^2_{L^2(\Omega_T)} + \int_\Omega F(x,u_0(x))\,dx \right). \end{aligned}$$

Particularly, when the initial data vanish the energy of the solution is bounded by the source and exterior data.

Proof of Remark 3.8. Let u satisfy the equation

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) = h & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases}$$

Then $v := u - \varphi$ satisfies

$$\begin{cases} \partial_t^2 v + (-\Delta)^s v + f(x,v) = h - (-\Delta)^s\varphi & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases}$$

The energy equality of Theorem 3.6 yields

$$\begin{aligned} &\|\partial_t v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{\widetilde{H}^s(\Omega)} + 2\int_\Omega F(x,v(x,t))\,dx \\ &= \|u_1\|^2_{L^2(\Omega)} + \|u_0\|^2_{\widetilde{H}^s(\Omega)} + 2\langle h - (-\Delta)^s\varphi, \partial_t v\rangle_{L^2(\Omega_t)} + 2\int_\Omega F(x,u_0(x))\,dx. \end{aligned}$$

Assuming that $\inf_{\tau\in\mathbb{R}} F(x,\tau) \geq 0$ we can estimate by Hölder's inequality

$$\begin{aligned} &\|\partial_t v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{\widetilde{H}^s(\Omega)} \\ &\leq \|u_1\|^2_{L^2(\Omega)} + \|u_0\|^2_{\widetilde{H}^s(\Omega)} \\ &\quad + 2\langle h - (-\Delta)^s\varphi, \partial_t v\rangle_{L^2(\Omega_t)} + 2\int_\Omega F(x,u_0(x))\,dx \\ &\leq \|u_1\|^2_{L^2(\Omega)} + \|u_0\|^2_{\widetilde{H}^s(\Omega)} \\ &\quad + 2\|h - (-\Delta)^s\varphi\|_{L^2(\Omega_T)} \|\partial_t v(t)\|_{L^2(\Omega_T)} + 2\int_\Omega F(x,u_0(x))\,dx. \end{aligned}$$

Note that since $\varphi \in C^2([0,T]; \widetilde{H}^{2s}(\Omega_e))$ we have $v|_{\Omega_T} = (u-\varphi)|_{\Omega_T} = u|_{\Omega_T}$. As the right-hand side is independent of t we may take supremum over $t \in [0,T]$. Using the elementary inequality $2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ and $\|\partial_t v(t)\|^2_{L^2(\Omega_T)} \leq C\|\partial_t v(t)\|^2_{L^\infty(0,T;L^2(\Omega))}$ we may absorb the norm $\|\partial_t v(t)\|^2_{L^\infty(0,T;L^2(\Omega))}$ on the left-hand side at the cost of a larger constant and obtain the claim. □

3.2. The DN map

With the well-posedness of the nonlinear nonlocal wave equation at hand, we can rigorously define the DN map.

Definition 3.9 (*The DN Map*)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and $s > 0$ a non-integer. Suppose the nonlinearities f, g satisfy the conditions in Assumption 3.4 and $(u_0, u_1) \in \widetilde{H}^s(\Omega) \times L^2(\Omega)$. Then we define the DN map $\Lambda^{f,g}_{u_0,u_1}$ related to (1.1) by

$$\left\langle \Lambda^{f,g}_{u_0,u_1} \varphi, \psi \right\rangle := \int_{\mathbb{R}^n_T} (-\Delta)^{s/2} u \, (-\Delta)^{s/2} \psi \, dxdt,$$

for all $\varphi, \psi \in C_c^\infty((\Omega_e)_T)$, where $u \in C([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\Omega))$ is the unique solution of

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) + g(x, \partial_t u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) & \text{in } \Omega \end{cases}$$

(see Proposition 3.7).

4. Inverse problems for the nonlinear nonlocal wave equation

In this section we move on the inverse problem for the (nonlinear) nonlocal wave equation. First, in Section 4.1 we establish the Runge approximation property for the linear nonlocal wave equation with nonnegative potentials, and arbitrary initial data. Then, in Lemma 4.3 we prove the main results of this work (Theorems 1.1 and 1.2).

4.1. Runge approximation

As usual we will deduce the Runge approximation property from the Hahn–Banach theorem and the UCP of the fractional Laplacian.

Proposition 4.1 (*Runge Approximation*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $W \subset \Omega_e$ an arbitrary open set, $s > 0$ a non-integer and $T > 0$. Suppose that $q \in L^p(\Omega)$ is nonnegative,*[3] *where p is given by (3.9), and $(u_0, u_1) \in \widetilde{H}^s(\Omega) \times L^2(\Omega)$ are fixed initial conditions. Consider the Runge set*

$$\mathcal{R}_W^{u_0,u_1} := \left\{ u_\varphi \big|_{\Omega_T} \; : \; \varphi \in C_c^\infty(W_T) \right\},$$

where $u_\varphi \in C([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n))$ is the unique solution to

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Then $\mathcal{R}_W^{u_0,u_1}$ is dense in $L^2(\Omega_T)$.

Proof. Let us first observe that without loss of generality it suffices to show that $\mathcal{R}_W := \mathcal{R}_W^{0,0}$ is dense in $L^2(\Omega_T)$. Indeed, if $g \in L^2(\Omega_T)$ and $u_0^{u_0,u_1}$ denotes the unique solution of (4.1) with $\varphi = 0$, then the function $\widetilde{g} := g - u_0^{u_0,u_1}$ belongs to $L^2(\Omega_T)$. Now, assuming $\mathcal{R}_W$ is dense in $L^2(\Omega_T)$, we can find a sequence $(v_k|_{\Omega_T})_{k=1}^\infty \subset \mathcal{R}_W$ such that $v_k|_{\Omega_T} \to \widetilde{g}|_{\Omega_T}$ in $L^2(\Omega_T)$ as $k \to \infty$. Defining $w_k := v_k + u_0^{u_0,u_1}$, we immediately see that $w_k|_{\Omega_T} \in \mathcal{R}_W^{u_0,u_1}$ and

$$w_k|_{\Omega_T} = \left(v_k + u_0^{u_0,u_1} \right)\Big|_{\Omega_T} \to \left(\widetilde{g} + u_0^{u_0,u_1} \right)\Big|_{\Omega_T} = g \quad \text{as} \quad k \to \infty$$

in $L^2(\Omega_T)$.

Thus, we may now suppose that $u_0 = u_1 = 0$. We combine the strategy of [34, Theorem 3.1] and [37, Theorem 4.3]. Since $\mathcal{R}_W \subset L^2(\Omega_T)$ is a subspace it is enough by the Hahn–Banach theorem to show that if $F \in L^2(\Omega_T)$ vanishes on $\mathcal{R}_W$, then $F = 0$. Hence, choose any $F \in L^2(\Omega_T)$ and assume that

$$\left\langle F, u_\varphi - \varphi \right\rangle_{L^2(\Omega_T)} = 0 \quad \text{for all} \quad \varphi \in C_c^\infty(W_T).$$

Next, let

$$w_F \in C([0,T]; \widetilde{H}^s(\Omega)) \text{ with } \partial_t w_F \in C([0,T]; L^2(\Omega)) \text{ and } \partial_t^2 w_F \in L^2(0,T; H^{-s}(\Omega))$$

be the unique solution to the adjoint equation

$$\begin{cases} \partial_t^2 w + (-\Delta)^s w + qw = F & \text{in } \Omega_T, \\ w = 0 & \text{in } (\Omega_e)_T, \\ w(T) = \partial_t w(T) = 0 & \text{in } \Omega, \end{cases}$$

[3] This assumption is included for simplicity and the result remains true, for example, if one assumes instead $q \in L^\infty(\Omega)$.

which can be obtained by [36, Chapter 3,Theorem 8.1–8.2] and a subsequent time reversal (i.e., $t \mapsto T-t$). Next, we show the following assertion.

Claim 4.2. *There holds*

$$\int_0^T \left\langle \partial_t^2(u_\varphi-\varphi), w_F \right\rangle dt = \int_0^T \langle \partial_t^2 w_F, (u_\varphi-\varphi)\rangle\, dt.$$

Proof (*Proof of Claim 4.2*)**.** Let us set $v_\varphi = u_\varphi - \varphi$ and consider for each $\varepsilon > 0$ the following regularized problem

$$\begin{cases} \partial_t^2 v + \varepsilon((-\Delta)^s + q)\partial_t v + (-\Delta)^s v + qv = -(-\Delta)^s\varphi & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(0) = \partial_t v(0) = 0 & \text{in } \Omega \end{cases} \tag{4.2}$$

and

$$\begin{cases} \partial_t^2 w - \varepsilon((-\Delta)^s + q)\partial_t w + (-\Delta)^s w + qw = F & \text{in } \Omega_T, \\ w = 0 & \text{in } (\Omega_e)_T, \\ w(T) = \partial_t w(T) = 0 & \text{in } \Omega. \end{cases} \tag{4.3}$$

It is well-known that by [36, Chapter 3,Theorem 8.3], the above problems have unique solutions

$$v_\varphi^\varepsilon \in C([0,T];\widetilde{H}^s(\Omega)) \text{ with } \begin{cases} \partial_t v_\varphi^\varepsilon \in L^2(0,T;\widetilde{H}^s(\Omega)) \cap C([0,T];L^2(\Omega)) \\ \partial_t^2 v_\varphi^\varepsilon \in L^2(0,T;H^{-s}(\Omega)) \end{cases}$$

$$w_F^\varepsilon \in C([0,T];\widetilde{H}^s(\Omega)) \text{ with } \begin{cases} \partial_t w_F^\varepsilon \in L^2(0,T;\widetilde{H}^s(\Omega)) \cap C([0,T];L^2(\Omega)) \\ \partial_t^2 w_F^\varepsilon \in L^2(0,T;H^{-s}(\Omega)) \end{cases}$$

(see also [35]) and as $\varepsilon \to 0$, one has

$$\begin{aligned} v_\varphi^\varepsilon &\to v_\varphi \text{ in } C([0,T];\widetilde{H}^s(\Omega)), \\ \partial_t v_\varphi^\varepsilon &\to \partial_t v_\varphi \text{ in } C([0,T];L^2(\Omega)), \\ \partial_t^2 v_\varphi^\varepsilon &\rightharpoonup \partial_t^2 v_\varphi \text{ in } L^2(0,T;H^{-s}(\Omega)). \end{aligned} \tag{4.4}$$

Note that here we use the assumption $q \geq 0$, and otherwise we would have to include a term $\varepsilon\lambda$ for some $\lambda \in \mathbb{R}$ in the regularized problems (4.2) and (4.3) such that $(-\Delta)^s + q + \lambda$ is coercive. The last convergence in (4.4) follows by [36, Chapter 3, eq. (8.74)] and in particular it implies that

$$\partial_t^2 v_\varphi^\varepsilon \overset{*}{\rightharpoonup} \partial_t^2 v_\varphi \text{ in } L^2(0,T;H^{-s}(\Omega)) \text{ as } \varepsilon \to 0. \tag{4.5}$$

The convergence results in (4.4) clearly hold for the functions w_F^ε and w_F, too. Now, using an integration by parts, we can easily get

$$\int_0^T \left\langle \partial_t^2 v_\varphi^\varepsilon, w_F^\varepsilon \right\rangle dt = \int_0^T \left\langle \partial_t^2 w_F^\varepsilon, v_\varphi^\varepsilon \right\rangle dt$$

for any $\varepsilon > 0$ (see [35, eq.(4.1)]). The convergence in (4.4) and (4.5) allow us to pass to the limit $\varepsilon \to 0$, which yields

$$\int_0^T \left\langle \partial_t^2 v_\varphi, w_F \right\rangle dt = \int_0^T \left\langle \partial_t^2 w_F, v_\varphi \right\rangle dt.$$

Recalling that $v_\varphi = u_\varphi - \varphi$, we can conclude the proof. □

Then we can test the equation for $u_\varphi - \varphi$ against w_F and correspondingly test the equation for w_F by $u_\varphi - \varphi$. By Claim 4.2 we may compute

$$\begin{aligned} 0 &= \left\langle F, u_\varphi - \varphi \right\rangle_{L^2(\Omega_T)} \\ &= \left\langle (\partial_t^2 + (-\Delta)^s + q) w_F, u_\varphi - \varphi \right\rangle \\ &= \left\langle (\partial_t^2 + (-\Delta)^s + q)(u_\varphi - \varphi), w_F \right\rangle \\ &= \langle (-\Delta)^s \varphi, w_F \rangle, \end{aligned}$$

for all $\varphi \in C_c^\infty(W_T)$. This implies that w_F satisfies

$$(-\Delta)^s w_F(x,t) = w_F(x,t) = 0 \quad \text{for} \quad x \in W,$$

for a.e. $0 < t < T$. By the UCP of the fractional Laplacian [3, Theorem 1.2], this gives $w_F = 0$ in $\mathbb{R}^n_T$. This in turn implies $F = 0$ as we wish. □

4.2. Proofs related to the inverse problems

To study the inverse problem of recovering the nonlinear terms f_j we need certain mapping properties of the Nemytskii operators.

Lemma 4.3 (*Continuity of Nemytskii Operators, [35, Lemma 3.6]*). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $T > 0$, and $1 \leq q, p < \infty$. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying*

$$|f(x,\tau)| \leq a + b|\tau|^\alpha$$

for some constants $a, b \geq 0$ and $0 < \alpha \leq \min(p,q)$. Then the Nemytskii operator f, defined by

$$f(u)(x,t) := f(x,u(x,t))$$

for all measurable functions $u : \Omega_T \to \mathbb{R}$, maps continuously $L^q(0,T;L^p(\Omega))$ into $L^{q/\alpha}(0,T;L^{p/\alpha}(\Omega))$.

Proof of Theorem 1.2.

Unique determination of initial data.

First note that the condition (1.3) is equivalent to

$$(-\Delta)^s u_0^{(1)} = (-\Delta)^s u_0^{(2)} \text{ in } (W_2)_T,$$

where $u_0^{(j)}$ is the unique solution to (1.2) for $j = 1,2$ with the same exterior condition $\varphi = 0$. Then the UCP for the fractional Laplacian (Proposition 2.1) yields that $u_0^{(1)} = u_0^{(2)}$ in $\mathbb{R}^n_T$. This in turn implies

$$u_0 := u_{0,1} = u_{0,2} \quad \text{and} \quad u_1 := u_{1,1} = u_{1,2} \quad \text{in} \quad \Omega.$$

Here we do not need to use any information of the nonlinearity f_j that satisfies Assumption 3.4, for $j = 1,2$. This concludes the proof. □

Remark 4.4. Even if the exterior data $\varphi = 0$, the corresponding solution of (1.4) does not need to vanish, since the initial data may be nonzero.

Proof of Corollary 1.4. Similarly as above, by the UCP it follows that the potential terms are equal:

$$a_1 u^{(1)} = a_2 u^{(1)}. \tag{4.6}$$

This implies that there holds

$$\int_{\Omega_T} a_1 u \, dx dt = \int_{\Omega_T} a_2 u \, dx dt$$

for all solutions u of a linear wave equation with potential a_j. Since we assume $0 \leq a_j \in L^p(\Omega)$ with (3.9) (implying $p \geq 2$) or $a_j \in L^\infty(\Omega)$, it follows from Proposition 4.1 that the solutions u are dense in $L^2(\Omega_T)$. Finally, using Hölder's inequality we get

$$\int_{\Omega_T} (a_1 - a_2)\psi \, dx dt = 0$$

for all $\psi \in C_c^\infty(\Omega_T)$ and hence $a_1 = a_2$. □

Last but not least, let us prove the recovery of the nonlinear term.

Proof of Theorem 1.1. *Unique determination of the nonlinear term $f(x,u)$.* Let $\varepsilon > 0$. We start by observing that the solution u_ε of

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) = 0 & \text{in } \Omega_T, \\ u = \varepsilon\varphi & \text{in } (\Omega_e)_T, \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega, \end{cases}$$

can be expanded as $u_\varepsilon = \varepsilon v + R_\varepsilon$, where

$$\begin{cases} \partial_t^2 v + (-\Delta)^s v = 0 & \text{in } \Omega_T, \\ v = \varphi & \text{in } (\Omega_e)_T, \\ v(0) = \partial_t v(0) = 0 & \text{in } \Omega \end{cases} \tag{4.7}$$

and

$$\begin{cases} \partial_t^2 R_\varepsilon + (-\Delta)^s R_\varepsilon = -f(x,u_\varepsilon) & \text{in } \Omega_T, \\ R_\varepsilon = 0 & \text{in } (\Omega_e)_T, \\ R_\varepsilon(0) = \partial_t R_\varepsilon(0) = 0 & \text{in } \Omega. \end{cases} \tag{4.8}$$

Indeed, the function $R_\varepsilon = u_\varepsilon - \varepsilon v$ satisfies (4.8). Note that by (3.18) for $u \in C([0,T]; H^s(\mathbb{R}^n))$ and $f(x,0) = 0$ we have the estimate

$$\left\| f_j(\cdot, u) \right\|_{L^2(\Omega_T)} \lesssim \|u\|^{r+1}_{L^\infty(0,T;H^s(\mathbb{R}^n))}.$$

Since $u_\varepsilon \in C([0,T]; H^s(\mathbb{R}^n))$ and $f(u_\varepsilon) \in L^2(\Omega_T)$, then $R_\varepsilon \in C([0,T]; \widetilde{H}^s(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and the energy identity (3.3) implies:

$$\begin{aligned} \|R_\varepsilon\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t R_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} &\lesssim \|f(u_\varepsilon)\|_{L^2(\Omega_T)} \\ &\lesssim \|u_\varepsilon\|^{r+1}_{L^\infty(0,T;H^s(\mathbb{R}^n))}. \end{aligned} \tag{4.9}$$

By Remark 3.8, we get

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\partial_t u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} \\ \lesssim\ &\|u_\varepsilon - \varepsilon\varphi\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t(u_\varepsilon - \varepsilon\varphi)\|_{L^\infty(0,T;L^2(\Omega))} \\ &+ \|\varepsilon\varphi\|_{L^\infty(0,T;\widetilde{H}^s(\Omega_e))} + \|\varepsilon\partial_t\varphi\|_{L^\infty(0,T;L^2(\Omega_e))} \\ \lesssim\ &\varepsilon\left(\|(-\Delta)^s\varphi\|_{L^2(\Omega_T)} + \|\varphi\|_{L^\infty(0,T;\widetilde{H}^s(\Omega_e))} + \|\partial_t\varphi\|_{L^\infty(0,T;L^2(\Omega_e))} \right). \end{aligned} \tag{4.10}$$

Therefore, by combining (4.9) with (4.10) we obtain

$$\begin{aligned} &\|R_\varepsilon\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t R_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \\ \lesssim\ &\varepsilon^{r+1}\left(\|(-\Delta)^s\varphi\|_{L^2(\mathbb{R}^n_T)} + \|\varphi\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t\varphi\|_{L^\infty(0,T;L^2(\Omega_e))} \right)^{r+1}. \end{aligned} \tag{4.11}$$

From (4.6) we know

$$f_1(x, u_\varepsilon) = f_2(x, u_\varepsilon)$$

which is equivalent to

$$f_1(x, \varepsilon v + R_\varepsilon) = f_2(x, \varepsilon v + R_\varepsilon).$$

By the $(r+1)$-homogeneity of f_j we then find that

$$f_1(x, v + \varepsilon^{-1}R_\varepsilon) = f_2(x, v + \varepsilon^{-1}R_\varepsilon).$$

Next, we may use the estimate (4.11) for $R_\varepsilon \in C([0,T]; \widetilde{H}^s(\Omega)) \subset L^2(\Omega_T)$ with the bound

$$\|R_\varepsilon\|_{L^2(\Omega_T)} \lesssim \varepsilon^{r+1}$$

to deduce that

$$\varepsilon^{-1}R_\varepsilon \to 0 \text{ in } L^2(\Omega_T) \text{ as } \varepsilon \to 0.$$

Therefore by continuity of the Nemytskii operator f_j from $L^2(\Omega_T)$ to $L^{\frac{2}{r+1}}(\Omega_T)$ (see Lemma 4.3) we get

$$f_j(v + \varepsilon^{-1}R_\varepsilon) \to f_j(v) \text{ in } L^{\frac{2}{r+1}}(\Omega_T), \quad \text{as} \quad \varepsilon \to 0,$$

for all $0 < r \le 1$.

Thus we have obtained that $f_1(x,v) = f_2(x,v)$ in $L^{\frac{2}{r+1}}(\Omega_T)$ for all solutions v of the linear wave Eq. (4.7). Finally, we can apply the Runge approximation property of solutions v to linear wave equations with $q = 0$. Let $(v_k)_{k=1}^\infty \subset \mathcal{R}_{W_1}$ be a sequence of solutions to the linear nonlocal wave equations with $v_k\big|_{\Omega_T} \to 1$ in $L^2(\Omega_T)$ given by Proposition 4.1. Then continuity of $f_j : L^2(\Omega_T) \to L^{\frac{2}{r+1}}(\Omega_T)$, when $0 < r \le 1$, yields

$$\left\| f_j(x,1) - f_j(x, v_k) \right\|_{L^{\frac{2}{r+1}}(\Omega_T)} \to 0 \quad \text{as } k \to \infty.$$

Thus we obtain

$$f_1(x,1) = f_2(x,1)$$

for almost all $x \in \Omega$. Therefore, by homogeneity we recover $f_1(x,\tau) = f_2(x,\tau)$ for a.e. $x \in \Omega$ and $\tau \in \mathbb{R}$. □

Remark 4.5. It is noteworthy that we do not need to use the (in local case) commonly used *higher order linearization* (popularized for inverse problems in [25,41] for both hyperbolic and elliptic equations). In the local case, the higher order linearization is used to builds products $v_1 \cdots v_k$ of solutions to linear equations. Then one can show that these products become dense in suitable function spaces. In the nonlocal case the UCP of the fractional Laplacian directly establishes that $f_1(u) = f_2(u)$ for any solution u of the nonlinear nonlocal wave equation. Then it suffices to linearize once to recover $f_1(v) = f_2(v)$ for any solution v of a linear wave equation. Finally, the Runge approximation property yields the equality $f_1 = f_2$. In a sense, this is consequence of the strength of nonlocality and its UCP. This technique applies to many equations of similar type, including as a typical example the fractional Schrödinger equation $(-\Delta)^s u + q(u) = 0$, since all one needs is the UCP and the Runge approximation.

Data availability

No data was used for the research described in the article.

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