

STABILITY ESTIMATES FOR THE INVERSE FRACTIONAL CONDUCTIVITY PROBLEM

GIOVANNI COVI, JESSE RAILO, TEEMU TYNI, AND PHILIPP ZIMMERMANN

ABSTRACT. We study the stability of an inverse problem for the fractional conductivity equation on bounded smooth domains. We obtain a logarithmic stability estimate for the inverse problem under suitable a priori bounds on the globally defined conductivities. The argument has three main ingredients: 1. the logarithmic stability of the related inverse problem for the fractional Schrödinger equation by Rüländ and Salo; 2. the Lipschitz stability of the exterior determination problem; 3. utilizing and identifying nonlocal analogies of Alessandrini's work on the stability of the classical Calderón problem. The main contribution of the article is the resolution of the technical difficulties related to the last mentioned step. Furthermore, we show the optimality of the logarithmic stability estimates, following the earlier works by Mandache on the instability of the inverse conductivity problem, and by Rüländ and Salo on the analogous problem for the fractional Schrödinger equation.

1. INTRODUCTION

Stability estimates for inverse problems give important information on theoretical limitations of different imaging techniques appearing in various medical, engineering, and scientific applications. They are also useful for development of numerical methods. A common feature of many inverse problems is that they are ill-posed, which means that small measurement errors may lead to large errors in the reconstructed images. One of the most popular model problems is the inverse conductivity problem, known as the Calderón problem [Cal80], where one aims to recover the (scalar) conductivity γ from the voltage/current measurements on the boundary $\partial\Omega$ of an object Ω . In mathematical terms, one defines the data as a Dirichlet-to-Neumann (DN) map $\Lambda_\gamma: f \mapsto \gamma\partial_\nu u_f|_{\partial\Omega}$, where ∂_ν is the outer boundary normal derivative, the electric potential u_f is the unique solution of the boundary value problem

$$\begin{aligned} \operatorname{div}(\gamma\nabla u) &= 0 && \text{in } \Omega, \\ u &= f && \text{on } \partial\Omega, \end{aligned}$$

and the voltage f is the given Dirichlet boundary condition. The Calderón problem asks to recover γ from the knowledge of Λ_γ , which corresponds

Date: March 19, 2024.

2020 Mathematics Subject Classification. Primary 35R30; secondary 26A33, 42B37, 46F12.

Key words and phrases. Fractional Laplacian, fractional gradient, Calderón problem, conductivity equation.

to knowing the outer normal fluxes (i.e. boundary currents) generated by imposing different boundary voltages f .

The Calderón problem serves both as a mathematical model for electrical impedance tomography [Uhl14], and more generally as a prototypical model for inverse problems. In fact, methods and techniques originally developed for the classical Calderón problem have applications in a wide range of other inverse problems, among which the anisotropic Calderón problem [APL05, DSFKSU09], hyperbolic problems [RS88, Sun90] and inverse problems related to the theory of elasticity [NU94]. The work of Sylvester and Uhlmann proved a fundamental uniqueness theorem for the classical Calderón problem in dimension $n \geq 3$, using a reduction to an analogous problem for the Schrödinger equation and constructing the so called complex geometrical optics (or CGO) solutions [SU87]. Nachman established a reconstruction method [Nac88], and Astala–Päivärinta showed a fundamental uniqueness result when $n = 2$ using methods from complex analysis and a reduction to the Beltrami equation [AP06]. We recall the stability theorem of Alessandrini [Ale88], which is an important motivation for our present work:

Theorem 1.1 (Alessandrini [Ale88, Theorem 1]). *Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 3$, with C^∞ boundary $\partial\Omega$. Given s and E , $s > n/2$, $E > 0$, let γ_1, γ_2 be any two functions in $H^{s+2}(\Omega)$ satisfying the following conditions*

$$\begin{aligned} E^{-1} &\leq \gamma_\ell(x), \quad \text{for every } x \text{ in } \Omega, \ell = 1, 2. \\ \|\gamma_\ell\|_{H^{s+2}(\Omega)} &\leq E, \quad \ell = 1, 2. \end{aligned}$$

The following estimate holds ¹

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C_E \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)}),$$

where the function ω is such that

$$\omega(t) \leq |\log t|^{-\delta}, \quad \text{for every } t, 0 < t < 1/e,$$

and δ , $0 < \delta < 1$, depends only on n and s .

Mandache showed that the logarithmic stability estimates are optimal up to the constants C, δ [Man01]. The works of Alessandrini and Mandache therefore show that the classical Calderón problem is ill-posed and furthermore accurately characterize this phenomenon. Mandache’s work was recently systematically studied and extended by Koch, Rüländ and Salo [KRS21] to many different settings. For the other recent works on the stability of the classical Calderón problem, we point to the following works [CDR16, CS14], where stability under partial data is obtained, and stability for recovery of anisotropic conductivities is considered. Under certain *a priori* assumptions, such as piecewise constant conductivities, the stronger result of Lipschitz stability holds [AV05]. Lipschitz stability is also possible with a finite number of measurements [AS22]. In a different direction, we mention [AN19] for an application of stability to the statistical Calderón problem.

In the present work, we study the stability properties of an inverse problem for a nonlocal analogue of the classical Calderón problem. There has

¹Given a bounded linear mapping $A: X \rightarrow Y$ between two Banach spaces, we denote its operator norm by $\|A\|_{X \rightarrow Y}$.

been growing interest towards establishing the theory of inverse problems for elliptic nonlocal variable coefficient operators. Other recent studies include inverse problems for the fractional powers of elliptic second order operators [GU21] and inverse problems for source-to-solutions maps related to fractional geometric operators on manifolds [FGKU21, QU22]. We note that the exterior value inverse problems considered in [GU21] for the operators $(\operatorname{div}(\gamma\nabla))^s$, $0 < s < 1$, generated by the heat semigroups, give another possibility to define a nonlocal conductivity equation which is presumably different from the equation we study here.

Let $s \in (0, 1)$ and consider the exterior value problem for the fractional conductivity equation

$$(1) \quad \begin{aligned} \operatorname{div}_s(\Theta_\gamma \nabla^s u) &= 0 && \text{in } \Omega, \\ u &= f && \text{in } \Omega_e. \end{aligned}$$

Here $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$ is the exterior of the domain Ω and the matrix $\Theta_\gamma: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n \times n}$ is defined as $\Theta_\gamma(x, y) := \gamma^{1/2}(x)\gamma^{1/2}(y)Id$. The operators $\operatorname{div}_s, \nabla^s$ appearing in (1) are the fractional divergence and the fractional gradient. These are nonlocal counterparts of the familiar differential operators $\operatorname{div}, \nabla$, and have the expected properties

$$(\operatorname{div}_s)^* = \nabla^s, \quad \operatorname{div}_s \nabla^s = (-\Delta)^s.$$

In Section 2.2 we will discuss these fractional operators in detail. Given the mapping properties of the fractional gradient and divergence (see Section 2.2), in the assumption that the conductivity γ is regular enough ([RZ22c]), we see that the fractional conductivity operator can be defined as a map

$$\operatorname{div}_s(\Theta_\gamma \nabla^s) : H^s(\mathbb{R}^n) \rightarrow H^{-s}(\mathbb{R}^n).$$

We say $u \in H^s(\mathbb{R}^n)$ is a (weak) solution of (1) if $u - f \in \tilde{H}^s(\Omega)$ and

$$B_\gamma(u, \phi) := \frac{C_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|x-y|^{n+2s}} (u(x) - u(y))(\phi(x) - \phi(y)) dx dy = 0$$

holds for all $\phi \in C_c^\infty(\Omega)$. We refer to Section 2.1 for the definition of the fractional Sobolev space $\tilde{H}^s(\Omega)$, as well as of all other function spaces. For all $f \in X := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ in the abstract trace space there is a unique weak solution $u_f \in H^s(\mathbb{R}^n)$ of the fractional conductivity equation (1). The fractional conductivity operator converges in the sense of distributions to the classical conductivity operator when applied to sufficiently regular functions when $s \uparrow 1$ [Cov20, Lemma 4.2]. Moreover, it is immediately seen that the fractional conductivity operator reduces to the fractional Laplacian in the case $\gamma \equiv 1$.

The exterior DN map $\Lambda_\gamma: X \rightarrow X^*$ is defined by

$$\langle \Lambda_\gamma f, g \rangle := B_\gamma(u_f, g).$$

The inverse problem for the fractional conductivity equation asks to recover the conductivity γ from Λ_γ , which maps as $\Lambda_\gamma: H^s(\Omega_e) \rightarrow H_{\Omega_e}^{-s}(\mathbb{R}^n)$ in the case of Lipschitz domains. We define $m_\gamma := \gamma^{1/2} - 1$ and call it the *background deviation* of γ . Let $\Omega \subset \mathbb{R}^n$ be bounded in one direction and $n \geq 1$. (We suppose additionally that $0 < s < 1/2$ when $n = 1$.) The

uniqueness properties of this inverse problem are studied extensively in the recent literature and the following list summarizes these advances:

- *Global uniqueness.* If $W \subset \Omega_e$ is an open nonempty set such that $\gamma_i|_W$ are continuous a.e., and $m_i \in H^{2s, \frac{n}{2s}}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, $j = 1, 2$, then $\gamma_1 = \gamma_2$ if and only if $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C_c^\infty(W)$ [CRZ22]. This result generalizes and expands the scope of the earlier works [Cov20, RZ22b], which solved the inverse problem in certain special cases by means of the fractional Liouville transformation. This is a technique used to reduce the fractional conductivity equation to the fractional Schrödinger equation introduced in [GSU20], which is in turn better understood.
- *Low regularity uniqueness.* If $W \subset \Omega_e$ is an open nonempty set such that $\gamma_i|_W$ are continuous a.e., and $m_i \in H^{s, n/s}(\mathbb{R}^n)$, then $\gamma_1 = \gamma_2$ if and only if $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C_c^\infty(W)$ [RZ22c]. This uses a general UCP result for the fractional Laplacians in [KRZ22].
- *Counterexamples for disjoint measurement sets.* For any nonempty open disjoint sets $W_1, W_2 \subset \Omega_e$ with $\text{dist}(W_1 \cup W_2, \Omega) > 0$ there exist two different conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$, $m_1, m_2 \in H^{s, n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, and $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$ [RZ22a, RZ22c].

In this article, we obtain a quantitative stability estimate for the global inverse fractional conductivity problem on *bounded smooth domains* with *full data*. This is based on one of the possible global uniqueness proofs presented in [CRZ22, RZ22c]. There remain some nontrivial challenges in order to obtain a quantitative version of the partial data uniqueness results in [CRZ22, RZ22c], as well as to remove the regularity/boundedness assumptions of the domain even for the full data case.

We will next recall two earlier stability results related to the fractional Calderón problems. The first one considers the stable recovery of γ in the exterior, based on [CRZ22, Proposition 1.4]. The second one considers the stability of the analogous inverse problem for the fractional Schrödinger equation $(-\Delta)^s + q$ due to Rüländ and Salo [RS20, Theorem 1.2]. The uniqueness properties of the Calderón problem for large classes of fractional Schrödinger type equations have been extensively studied starting from the seminal work of [GSU20]. These include perturbations to the fractional powers of elliptic operators [GLX17], first order perturbations [CLR20], nonlinear perturbations [LL22], higher order equations with local perturbations [CMR21, CMRU22], quasilocal perturbations [Cov21], and general theory for nonlocal elliptic equations [RS20, RZ22b].

In particular, the following results are needed in our proofs:

Theorem 1.2 ([RZ22c, Remark 3.3]). *Let $\Omega \subset \mathbb{R}^n$ be a domain bounded in one direction and $0 < s < 1$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ satisfy $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$, and are continuous a.e. in Ω_e . There exists a constant $C > 0$ depending only on s such that ²*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*.$$

²Here and in the rest of paper we use the notation $\|A\|_* := \|A\|_{H^s(\Omega_e) \rightarrow (H^s(\Omega_e))^*}$.

Given a Sobolev multiplier $q \in M(H^s \rightarrow H^{-s})$ (cf. [RS20, CMRU22]), we define the following bilinear form

$$B_q(u, v) := \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx + \langle qu, v \rangle, \quad u, v \in H^s(\mathbb{R}^n),$$

related to the fractional Schrödinger operator $(-\Delta)^s + q$.

Theorem 1.3 ([RS20, Theorem 1.2]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $0 < s < 1$, and $W_1, W_2 \subset \Omega$ be nonempty open sets. Assume that for some $\delta, M > 0$ the potentials $q_1, q_2 \in H^{\delta, \frac{n}{2s}}(\mathbb{R}^n)$ have the bounds*

$$\|q_j\|_{H^{\delta, \frac{n}{2s}}(\Omega)} \leq M, \quad j = 1, 2.$$

Suppose also that zero is not a Dirichlet eigenvalue for the exterior value problem

$$(2) \quad (-\Delta)^s u + q_j u = 0, \quad \text{in } \Omega$$

*with $u|_{\Omega_e} = 0$, for $j = 1, 2$. Then one has*³

$$\|q_1 - q_2\|_{L^{\frac{n}{2s}}(\Omega)} \leq \omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_{\tilde{H}^s(W_1) \rightarrow (\tilde{H}^s(W_2))^*}),$$

where $\Lambda_{q_j} : X \rightarrow X^$ with $\langle \Lambda_{q_j} f, g \rangle := B_{q_j}(u_f, g)$ is the DN map related to the exterior value problem for equation (2), and ω is a modulus of continuity satisfying*

$$\omega(x) \leq C |\log x|^{-\sigma}, \quad 0 < x \leq 1$$

for some C and σ depending only on $\Omega, n, s, W_1, W_2, \delta$ and M .

The following Lemma is fundamental for our arguments, as it relates the fractional conductivity operator with the fractional Schrödinger operator. We will discuss this result further in Section 2.3.

Lemma 1.4 (Liouville reduction, [RZ22c, Lemma 3.9]). *Let $0 < s < \min(1, n/2)$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with conductivity matrix Θ_γ and background deviation m_γ satisfies $\gamma(x) \geq \gamma_0 > 0$ and $m_\gamma \in H^{s, n/s}(\mathbb{R}^n)$. Let $q_\gamma := -\frac{(-\Delta)^s m_\gamma}{\gamma^{1/2}}$. Then there holds*

$$\begin{aligned} \langle \Theta_\gamma \nabla^s u, \nabla^s \phi \rangle_{L^2(\mathbb{R}^{2n})} &= \langle (-\Delta)^{s/2} (\gamma^{1/2} u), (-\Delta)^{s/2} (\gamma^{1/2} \phi) \rangle_{L^2(\mathbb{R}^n)} \\ &\quad + \langle q_\gamma (\gamma^{1/2} u), (\gamma^{1/2} \phi) \rangle \end{aligned}$$

for all $u, \phi \in H^s(\mathbb{R}^n)$.

In the sequel, we will call q_γ above a potential.

1.1. Main results. We next state our main result, whose proof is based on a reduction to Theorems 1.2 and 1.3.

Theorem 1.5. *Let $0 < s < \min(1, n/2)$, $\epsilon > 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Suppose that the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 fulfill the following conditions:*

$$(i) \quad \gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1} \text{ for some } 0 < \gamma_0 < 1,$$

³Note $\|A\|_{\tilde{H}^s(W_1) \rightarrow (\tilde{H}^s(W_2))^*} = \sup\{|\langle Au_1, u_2 \rangle| ; \|u_j\|_{H^s(\mathbb{R}^n)} = 1, u_j \in C_c^\infty(W_j)\}$.

(ii) $m_1, m_2 \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$ and there exists $C_1 > 0$ such that

$$(3) \quad \|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} \leq C_1$$

for $i = 1, 2$,

(iii) $m_1 - m_2 \in H^s(\mathbb{R}^n)$ and there exists $C_2 > 0$

$$(4) \quad \|(-\Delta)^s m_i\|_{L^1(\Omega_e)} \leq C_2$$

for $i = 1, 2$.

If $\theta_0 \in (\max(1/2, 2s/n), 1)$ and there holds $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* \leq 3^{-1/\delta}$ for some $0 < \delta < \frac{1-\theta_0}{2}$, then we have

$$\|\gamma_1 - \gamma_2\|_{L^q(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)$$

for all $1 \leq q \leq \frac{2n}{n-2s}$, where $\omega(x)$ is a logarithmic modulus of continuity satisfying

$$\omega(x) \leq C |\log x|^{-\sigma}, \quad \text{for } 0 < x \leq 1,$$

for some constants $\sigma, C > 0$ depending only on $s, \epsilon, n, \Omega, C_1, C_2, \theta_0$ and γ_0 .

Remark 1.6. We make several comments about Theorem 1.5 and its assumptions to clarify some interesting points:

(i) Theorems 1.2 and 1.5 together imply that for any compact set $K \subset \mathbb{R}^n$ there holds

$$\|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^q(K)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)$$

where ω is a logarithmic modulus of continuity with a constant C additionally depending on K . In general, one has L^∞ control in Ω_e and L^q control in Ω .

(ii) The L^1 assumption (4) is required due to the noncompact, global, setting of the problem, and L^∞ stability in the exterior. The stability estimate in the exterior forces us to impose the additional condition that the related potentials $q_i := -\frac{(-\Delta)^s m_i}{\gamma_i^{1/2}} \in (L^\infty(\Omega_e))^*$ with a priori bounds in their norms, and hence (4) is a natural assumption.

(iii) We assume that the domain has smooth boundary due to Theorem 1.3, as one has to impose the smoothness assumption in order to use the Vishik–Eskin estimates. In light of [RS20, Remark 7.1], the rest of their proof could be formulated under weaker regularity assumptions, as well as the one of Theorem 1.5 (see Section 5). This leaves the interesting open question of whether it is possible to obtain the stability results, i.e. Theorems 1.3 and 1.5, for less regular domains.

(iv) We impose the assumption (3) so that the related potentials satisfy $q_i \in H^{\delta, \frac{n}{2s}}(\Omega)$ for some $\delta > 0$, and Theorem 1.3 applies. The assumption (3) also implies that γ_1, γ_2 are continuous, so that Theorem 1.2 is known to apply. This assumption is much stronger than the ones required for the global uniqueness theorems [CRZ22, RZ22c].

(v) By formally taking $s = 1$ in Theorem 1.5 and comparing with Theorem 1.1, one sees that the latter has slightly sharper differentiability assumptions when $n = 3$, and the reverse is true for $n \geq 5$. In dimension $n = 4$, the assumptions of the two theorems are comparable.

In the sequel, $B_r \subset \mathbb{R}^n$ denotes the ball of radius $r > 0$ centered at the origin. We complement our stability estimate with the following statement on exponential instability under partial data.

Theorem 1.7. *For any $\ell \in \mathbb{R}_+ \setminus \mathbb{N}$ such that $\ell - 2s \in \mathbb{R}_+ \setminus \mathbb{N}$ as well there exists a constant $\beta > 0$ such that for all sufficiently small $\epsilon > 0$ there are conductivities $\gamma_1, \gamma_2 \in C^\ell(\mathbb{R}^n)$ such that*

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^s(B_3 \setminus \overline{B}_2) \rightarrow (H^s(B_3 \setminus \overline{B}_2))^*} \leq \exp\left(-\epsilon^{-\frac{n}{(2n+3)\ell}}\right),$$

$$\|\gamma_1 - \gamma_2\|_{L^\infty(B_1)} \geq \epsilon,$$

$$\|\gamma_i\|_{C^\ell(B_1)} \leq \beta, \quad 1 \leq \gamma_i \leq 2, \quad i = 1, 2.$$

Remark 1.8. *For the sake of simplicity, we restrict our analysis of instability to a very symmetric geometrical setting. This is convenient for the proof, as it is possible to explicitly construct a basis $\{f_{h,k,l}\}_{h,k \in \mathbb{N}, 0 \leq l \leq l_h}$ of $L^2(B_3 \setminus \overline{B}_2)$ with the special properties given in Lemma 2.1 of [RS18], where l_h is the number of spherical harmonics of order h on ∂B_1 . However, it would suffice to consider the exterior DN maps in any annulus $B_R \setminus \overline{B}_r$ with $1 < r < R$, as the rest of the construction can be easily adapted to this case. Whether instability holds in the case of full data remains to be proved.*

1.2. Organization of the article. The article is organized as follows. We begin Section 2 by defining the many needed function spaces and recalling the notation for the fractional conductivity equation. Section 3 concerns extension and multiplication lemmas for Sobolev functions. In Section 4, we prove our main stability estimate, Theorem 1.5. In Section 5, we discuss quantitative reduction to the Schrödinger problem with partial data. Finally, to complement the stability theorem, in Section 6 we prove the exponential instability result of Theorem 1.7. For clarity, some proofs of auxiliary results are postponed to Appendix A.

Acknowledgements. G.C. was supported by an Alexander-von-Humboldt postdoctoral fellowship. J.R. was supported by the Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters.

2. PRELIMINARIES

2.1. Function spaces. Throughout this article $\Omega \subset \mathbb{R}^n$ is always an open set. The classical Sobolev spaces of order $k \in \mathbb{N}$ and integrability exponent $p \in [1, \infty]$ are denoted by $W^{k,p}(\Omega)$ and for $k = 0$ we use the convention $W^{0,p}(\Omega) = L^p(\Omega)$. Moreover, we let $W^{s,p}(\Omega)$ stand for the fractional Sobolev spaces, when $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $1 \leq p < \infty$. These spaces are also called Slobodeckij spaces or Gagliardo spaces. If $1 \leq p < \infty$ and $s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$, then they are defined by

$$W^{s,p}(\Omega) := \{u \in W^{k,p}(\Omega); [\partial^\alpha u]_{W^{\sigma,p}(\Omega)} < \infty \quad \forall |\alpha| = k\},$$

where

$$[u]_{W^{\sigma,p}(\Omega)} := \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\sigma p}} dx dy \right)^{1/p}$$

is the so-called Gagliardo seminorm. The Slobodeckij spaces are naturally endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{W^{k,p}(\Omega)}^p + \sum_{|\alpha|=k} [\partial^\alpha u]_{W^{\sigma,p}(\Omega)}^p \right)^{1/p}.$$

Next we recall the definition of the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ and introduce several local variants of them. For the Fourier transform, we use the following convention

$$\mathcal{F}u(\xi) := \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x)e^{-ix \cdot \xi} dx,$$

whenever it is defined. Moreover, the Fourier transform acts as an isomorphism on the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ and by duality on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. The inverse of the Fourier transform is denoted in each case by \mathcal{F}^{-1} . The Bessel potential of order $s \in \mathbb{R}$ is the Fourier multiplier $\langle D \rangle^s : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$, that is

$$\langle D \rangle^s u := \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}),$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ is the Japanese-bracket. For any $s \in \mathbb{R}$ and $1 \leq p < \infty$, the Bessel potential space $H^{s,p}(\mathbb{R}^n)$ is defined by

$$H^{s,p}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n); \langle D \rangle^s u \in L^p(\mathbb{R}^n)\},$$

which we endow with the norm $\|u\|_{H^{s,p}(\mathbb{R}^n)} := \|\langle D \rangle^s u\|_{L^p(\mathbb{R}^n)}$. If $\Omega \subset \mathbb{R}^n$, $F \subset \mathbb{R}^n$ are given open and closed sets, then we define the following local Bessel potential spaces:

$$\begin{aligned} \tilde{H}^{s,p}(\Omega) &:= \text{closure of } C_c^\infty(\Omega) \text{ in } H^{s,p}(\mathbb{R}^n), \\ H_F^{s,p}(\mathbb{R}^n) &:= \{u \in H^{s,p}(\mathbb{R}^n); \text{supp}(u) \subset F\}, \\ H^{s,p}(\Omega) &:= \{u|_\Omega; u \in H^{s,p}(\mathbb{R}^n)\}. \end{aligned}$$

The space $H^{s,p}(\Omega)$ is equipped with the quotient norm

$$\|u\|_{H^{s,p}(\Omega)} := \inf\{\|w\|_{H^{s,p}(\mathbb{R}^n)}; w \in H^{s,p}(\mathbb{R}^n), w|_\Omega = u\}.$$

We see that $\tilde{H}^{s,p}(\Omega), H_F^{s,p}(\mathbb{R}^n)$ are closed subspaces of $H^{s,p}(\mathbb{R}^n)$. As customary, we set $H^s(\Omega) := H^{s,2}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$. If the boundary of the domain $\Omega \subset \mathbb{R}^n$ is regular enough then there is a close relation between the fractional Sobolev and Bessel potential spaces but also between two of the above introduced local Bessel potential spaces. For this purpose we next introduce the Hölder spaces and the notion of domains of class $C^{k,\alpha}$.

For all $k \in \mathbb{N}_0$ and $0 < \alpha \leq 1$, the space $C^{k,\alpha}(\Omega)$ consists of all functions $u \in C^k(\Omega)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\Omega)} := \|u\|_{C^k(\Omega)} + \sum_{|\beta|=k} [\partial^\beta u]_{C^{0,\alpha}(\Omega)}$$

is finite, where

$$\|u\|_{C^k(\Omega)} := \sum_{|\beta| \leq k} \|\partial^\beta u\|_{L^\infty(\Omega)} \quad \text{and} \quad [u]_{C^{0,\alpha}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We remark that the same notation will be used for \mathbb{R}^m -valued functions. We say that an open subset $\Omega \subset \mathbb{R}^n$ is of class $C^{k,\alpha}$ for $k \in \mathbb{N}_0$, $0 < \alpha \leq 1$ if there exists $C > 0$ such that for any $x \in \partial\Omega$ there exists a ball $B = B_r(x)$, $r > 0$, and a map $T: Q \rightarrow B$ satisfying

- (i) $T \in C^{k,\alpha}(Q)$, $T^{-1} \in C^{k,\alpha}(B)$,
- (ii) $\|T\|_{C^{k,\alpha}(Q)}, \|T^{-1}\|_{C^{k,\alpha}(B)} \leq C$,
- (iii) $T(Q_+) = \Omega \cap B$, $T(Q_0) = \partial\Omega \cap B$.

In the special case $k = 0, \alpha = 1$, we say that Ω is a Lipschitz domain. Moreover, we say that a domain is of class C^k if the above conditions hold for $\alpha = 0$ and it is smooth if it is of class C^k for any $k \in \mathbb{N}$. Above we used the following notation:

$$\begin{aligned} Q &:= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; |x'| < 1, |x_n| < 1\} \\ Q_+ &:= \{x = (x', x_n) \in Q; x_n > 0\} \\ Q_0 &:= \{x = (x', x_n) \in Q; x_n = 0\} \end{aligned}$$

One can prove the following equivalence of local Bessel potential spaces:

Lemma 2.1 ([McL00, Theorem 3.29]). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with bounded boundary and $s \in \mathbb{R}$ then $\tilde{H}^s(\Omega) = H_{\Omega}^s(\mathbb{R}^n)$.*

Next we note that the following embeddings hold between Bessel potential spaces $H^{s,p}$ and the fractional Sobolev spaces $W^{s,p}$:

Theorem 2.2. *Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $1 < p < \infty$ and assume $\Omega \subset \mathbb{R}^n$ is an open set.*

- (i) *If $1 < p \leq 2$, $s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$ and $\Omega = \mathbb{R}^n$ or Ω is of class $C^{k,1}$ with bounded boundary, then $W^{s,p}(\Omega) \hookrightarrow H^{s,p}(\Omega)$.*
- (ii) *If $2 \leq p < \infty$, then $H^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$.*

Remark 2.3. *A proof for the case $\Omega = \mathbb{R}^n$ can be found in [Ste70, Chapter V, Theorem 5], and by using extension theorems in Slobodeckij spaces the theorem follows for $C^{k,1}$ domains with bounded boundary.*

Remark 2.4. *In particular, the above theorem asserts that for all $s = k + \sigma$ with $k \in \mathbb{N}_0, 0 < \sigma < 1$ there holds $H^s(\Omega) = W^{s,2}(\Omega)$, when $\Omega \subset \mathbb{R}^n$ is an open set of class $C^{k,1}$ with bounded boundary.*

2.2. Fractional operators. For all $s \geq 0$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the fractional Laplacian of order s by

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s} \hat{u}),$$

whenever the right hand side is well-defined. One can easily show by using the Mihlin multiplier theorem that the fractional Laplacian is a bounded linear operator $(-\Delta)^s: H^{t,p}(\mathbb{R}^n) \rightarrow H^{t-2s,p}(\mathbb{R}^n)$ for all $t \in \mathbb{R}$ and $1 < p < \infty$. In the special case $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, the fractional Laplacian can be calculated as the following singular integrals (see e.g. [DNPV12, Section 3])

$$\begin{aligned} (-\Delta)^s u(x) &= C_{n,s} \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \\ &= -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \end{aligned}$$

where $C_{n,s} > 0$ is a normalization constant. One immediately sees that the above integral is in the range $s \in (0, 1/2)$ for (local) Lipschitz functions not really singular. The fractional Laplacian has a distinguished property which simplifies the analysis of the inverse fractional conductivity problem compared to the classical Calderón problem, namely the unique continuation property (UCP), which asserts that if $r \in \mathbb{R}$, $1 \leq p < \infty$, $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $u \in H^{r,p}(\mathbb{R}^n)$ satisfies $u|_V = (-\Delta)^s u|_V = 0$ in some nonempty open set $V \subset \mathbb{R}^n$, then there holds $u \equiv 0$ in \mathbb{R}^n (cf. [KRZ22, Theorem 2.2]).

Moreover, let us point out that a large part of the theory of the inverse fractional conductivity problem can be extended to a certain class of unbounded domains, which are called domains bounded in one direction (cf. [RZ22b, Definition 2.1]), since the fractional Laplacian satisfies on these domains a Poincaré inequality. However, in this work we restrict our attention to bounded domains, and the stability of the inverse fractional conductivity problem on unbounded domains remains open.

For the rest of this section, we fix $s \in (0, 1)$. The fractional gradient of order s is the bounded linear operator $\nabla^s: H^s(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{2n}; \mathbb{R}^n)$ given by (see [Cov20, DGLZ12, RZ22b])

$$\nabla^s u(x, y) := \sqrt{\frac{C_{n,s}}{2}} \frac{u(x) - u(y)}{|x - y|^{n/2+s+1}} (x - y),$$

and satisfies

$$\|\nabla^s u\|_{L^2(\mathbb{R}^{2n})} = \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)}$$

for all $u \in H^s(\mathbb{R}^n)$. The adjoint of ∇^s is called fractional divergence of order s and denoted by div_s . More concretely, the fractional divergence of order s is the bounded linear operator

$$\operatorname{div}_s: L^2(\mathbb{R}^{2n}; \mathbb{R}^n) \rightarrow H^{-s}(\mathbb{R}^n)$$

satisfying the identity

$$\langle \operatorname{div}_s u, v \rangle_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} = \langle u, \nabla^s v \rangle_{L^2(\mathbb{R}^{2n})}$$

for all $u \in L^2(\mathbb{R}^{2n}; \mathbb{R}^n)$, $v \in H^s(\mathbb{R}^n)$. A simple estimate shows that there holds (see [RZ22b, Section 8])

$$\|\operatorname{div}_s u\|_{H^{-s}(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^{2n})}$$

for all $u \in L^2(\mathbb{R}^{2n}; \mathbb{R}^n)$, and also a comparison with the quadratic form definition for the fractional Laplacian implies $(-\Delta)^s u = \operatorname{div}_s(\nabla^s u)$ for all $u \in H^s(\mathbb{R}^n)$ (see e.g. [Kwa17, Theorem 1.1] and [Cov20, Lemma 2.1]).

2.3. The fractional Liouville reduction. We conclude this section with a discussion of the fractional Liouville reduction from Lemma 1.4, which constitutes one of the main tools in our analysis of the fractional conductivity equation.

It is a well-known classical result that the conductivity equation

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 & \text{in } \Omega, \\ u &= f & \text{in } \partial\Omega \end{aligned}$$

can be restated as the Schrödinger equation

$$\begin{aligned} -\Delta w + qw &= 0 & \text{in } \Omega, \\ w &= g & \text{in } \partial\Omega, \end{aligned}$$

where $w := \gamma^{1/2}u$, $g := \gamma^{1/2}f$, and the new potential q is computed as $q := \frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}$. This transformation takes the name of *Liouville reduction*, and its significance resides in the fact that the principal part of the operator of the equivalent problem has constant coefficients, which makes it easier to study. The dependency of the transformed operator on the conductivity γ is only through the lower order perturbation q . In this respect, the local and nonlocal problems behave analogously: as stated in Lemma 1.4 and proved in [Cov20, RZ22c], the fractional conductivity equation

$$\begin{aligned} \operatorname{div}_s(\Theta_\gamma \nabla^s u) &= 0 & \text{in } \Omega, \\ u &= f & \text{in } \Omega_e \end{aligned}$$

can be restated as the fractional Schrödinger equation

$$\begin{aligned} (-\Delta)^s w + q_\gamma w &= 0 & \text{in } \Omega, \\ w &= g & \text{in } \Omega_e, \end{aligned}$$

where $w := \gamma^{1/2}u$, $g := \gamma^{1/2}f$ as in the classical case, while $q_\gamma := -\frac{(-\Delta)^s m_\gamma}{\gamma^{1/2}}$ with $m_\gamma = \gamma^{1/2} - 1$. This result is proved directly for a Schwartz function u , and it is then extended to $u \in H^s(\mathbb{R}^n)$ by density under mild regularity assumptions for the conductivity γ . With this tool at hand, the uniqueness results for the fractional Schrödinger equation are readily generalized to the case of the fractional conductivity equation. Observe that the technique from [GSU20] based on the use of the Runge approximation property and the Alessandrini identity can not be directly applied for the fractional conductivity equation, in light of the inherently nonlocal nature of the part of the operator with variable coefficients. This problem is however avoided by using the fractional Liouville reduction.

The fractional Liouville reduction has been used also in the study of the inverse problem for fractional isotropic elasticity [CdHS22] and of a fractional anisotropic Calderón problem [Cov22]. Furthermore, the fractional Liouville reduction has been applied in [Zim23] to study the simultaneous recovery of the conductivity and a lower order potential. These constitute an interesting novelty, given that the classical Liouville reduction is not usually applied for the local versions of these problems. Additionally, in the recent article [LZ23] it has been shown that for the uniqueness in the fractional conductivity problem only nonnegative test functions have to be used, which as a particular case lead to unique determination results in the inverse problem for nonlocal porous medium equations. Finally, in [LRZ22] the Liouville reduction has been extended to a space-time setting to establish uniqueness results in a parabolic version of the fractional conductivity problem.

3. EXTENSION AND MULTIPLICATION LEMMAS FOR FRACTIONAL SOBOLEV SPACES

In this section, we establish a higher extension theorem for the spaces $W^{s,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a sufficiently regular domain with bounded boundary. This is then used to extend a Gagliardo–Nirenberg inequality to these domains (cf. [BM01, Corollary 2, (iii)]). These results are needed to have access to suitable Hölder embeddings for the conductivities and have access to L^∞ estimates in Ω_e for the conductivities via concrete extension operators and the Gagliardo–Nirenberg inequality. This need is in turn related to having only L^∞ control of conductivities in the exterior via Theorem 1.2. The proofs of Lemmas 3.1 and 3.2 are found in Appendix A.

Lemma 3.1 (Multiplication by Hölder functions). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with bounded boundary, $1 \leq p < \infty$ and $s \in \mathbb{R}_+ \setminus \mathbb{N}$.*

(i) *Let $0 < s < 1$ and $s < \mu \leq 1$. If $u \in W^{s,p}(\Omega)$ and $\phi \in C^{0,\mu}(\Omega)$, then $\phi u \in W^{s,p}(\Omega)$ satisfies*

$$\|\phi u\|_{W^{s,p}(\Omega)} \leq C_1 \left(1 + \left(\frac{\mu}{s(\mu - s)} \right)^{1/p} \right) \|\phi\|_{C^{0,\mu}(\Omega)} \|u\|_{W^{s,p}(\Omega)}$$

for some $C_1 > 0$ only depending on n and p .

(ii) *Let $s = k + \sigma$ with $k \in \mathbb{N}$, $0 < \sigma < 1$ and $\sigma < \mu \leq 1$. If $u \in W^{s,p}(\Omega)$ and $\phi \in C^k(\Omega)$ with $\partial^\alpha \phi \in C^{0,\mu}(\Omega)$ for all $|\alpha| \leq k$, then $\phi u \in W^{s,p}(\Omega)$ satisfying*

$$\begin{aligned} \|\phi u\|_{W^{s,p}(\Omega)} &\leq C_2 \left(1 + \left(\frac{1}{\sigma(1 - \sigma)} \right)^{1/p} \right) \left(1 + \left(\frac{\mu}{\sigma(\mu - \sigma)} \right)^{1/p} \right) \\ &\quad \cdot \left(\sum_{\ell=0}^k \|\nabla^\ell \phi\|_{C^{0,\mu}(\Omega)} \right) \|u\|_{W^{s,p}(\Omega)} \end{aligned}$$

for some $C_2 > 0$ only depending on n, k, p, Ω .

The following statement is a generalization of [DNPV12, Lemma 5.1], where the support of u is not necessarily compact and s is allowed to be larger than one:

Lemma 3.2 (Zero extension). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $s > 0$ and $1 \leq p < \infty$. Assume that $u \in W^{s,p}(\Omega)$ satisfies $d := \text{dist}(\text{supp}(u), \partial\Omega) > 0$ and let $\bar{u}: \mathbb{R}^n \rightarrow \mathbb{R}$ be its zero extension, that is*

$$\bar{u}(x) := \begin{cases} u(x), & x \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

Then $\bar{u} \in W^{s,p}(\mathbb{R}^n)$ and there holds

$$\|\bar{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}.$$

Lemma 3.3 (Higher order extension theorem). *Let $1 \leq p < \infty$, $s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$ and assume that $\Omega \subset \mathbb{R}^n$ is a domain of class $C^{k,1}$ with bounded boundary. Then there exists an extension operator $E: W^{s,p}(\Omega) \rightarrow W^{s,p}(\mathbb{R}^n)$ such that $E u|_\Omega = u$ and $\|E u\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}$.*

Proof of Lemma 3.3. By assumption there is a finite collection of balls B_j , $j = 1, \dots, m$, and maps $T_j: Q \rightarrow B_j$ such that

- (i) $T_j \in C^{k,1}(Q)$, $T_j^{-1} \in C^{k,1}(B_j)$,
- (ii) $\|T_j\|_{C^{k,1}(Q)}, \|T_j^{-1}\|_{C^{k,1}(B_j)} \leq C$ for some $C > 0$,
- (iii) $T_j(Q_+) = \Omega \cap B_j$, $T_j(Q_0) = \partial\Omega \cap B_j$

for all $j = 1, \dots, m$. By [Bre11, Lemma 9.3] there exist $(\phi_j)_{j=0, \dots, m} \subset C^\infty(\mathbb{R}^n)$ such that

- (I) $0 \leq \phi_j \leq 1$ for all $j = 0, \dots, m$,
- (II) $\text{supp}(\phi_0) \subset \mathbb{R}^n \setminus \partial\Omega$,
- (III) $\phi_j \in C_c^\infty(B_j)$ for all $j = 1, \dots, m$
- (IV) and $\sum_{j=0}^m \phi_j = 1$ on \mathbb{R}^n .

Using the compactness of $\partial\Omega$ and the assertion (II) we see that

$$d := \text{dist}(\text{supp}(\phi_0|_\Omega), \partial\Omega) > 0.$$

On the other hand the properties (III), (IV) imply $\partial^\alpha \phi_0 \in C^{0,1}(\Omega)$ for all $\alpha \in \mathbb{N}_0^n$. Hence, by Lemma 3.1 we know $\phi_0 u \in W^{s,p}(\Omega)$ and therefore we deduce from Lemma 3.2 that $u_0 := \overline{\phi_0 u} \in W^{s,p}(\mathbb{R}^n)$. Next we want to extend the functions $\phi_j u$ to elements of $W^{s,p}(\mathbb{R}^n)$. In the proof of [Dob10, Satz 6.10, Satz 6.38], which establishes the result for bounded domains, it has been shown that there exists $u_j \in W^{s,p}(\mathbb{R}^n)$ such that $u_j|_\Omega = \phi_j u$ for all $j = 1, \dots, m$ and $\|u_j\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$. Therefore, the operator $E: W^{s,p}(\Omega) \rightarrow W^{s,p}(\mathbb{R}^n)$ given by $Eu := \sum_{j=0}^m u_j$ satisfies the asserted properties and we can conclude the proof. \square

Lemma 3.4 (Gagliardo–Nirenberg inequality). *Let $1 < p < \infty$, $s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$ and assume that $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ is a domain of class $C^{k,1}$ with bounded boundary. Then for any $0 < \theta < 1$ there holds*

$$\|u\|_{W^{\theta s, p/\theta}(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)}^\theta \|u\|_{L^\infty(\Omega)}^{1-\theta}$$

for all $u \in W^{s,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. In the case $\Omega = \mathbb{R}^n$ the result holds by [BM01, Corollary 2.c)]. If $\Omega \subset \mathbb{R}^n$ is a domain of class $C^{k,1}$ with bounded boundary then by Lemma 3.3 for all $u \in W^{s,p}(\Omega) \cap L^\infty(\Omega)$ there is an extension $Eu \in W^{s,p}(\mathbb{R}^n)$. Moreover, the proof in [Dob10, Satz 6.10, Satz 6.38] shows that one has $\|Eu\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{L^\infty(\Omega)}$ as the extensions u_j are obtained by a higher order reflection technique. Therefore, we deduce

$$\begin{aligned} \|u\|_{W^{\theta s, p/\theta}(\Omega)} &\leq \|Eu\|_{W^{\theta s, p/\theta}(\mathbb{R}^n)} \leq \|Eu\|_{W^{s,p}(\mathbb{R}^n)}^\theta \|Eu\|_{L^\infty(\mathbb{R}^n)}^{1-\theta} \\ &\leq C \|u\|_{W^{s,p}(\Omega)}^\theta \|u\|_{L^\infty(\Omega)}^{1-\theta}. \end{aligned}$$

Hence, we can conclude the proof. \square

4. STABILITY ESTIMATES

We prove Theorem 1.5 in this section. In the proof, we make use of the exterior determination result stated in Theorem 1.2. Then we establish Hölder estimates for the function $\gamma_1^{-1/2} - \gamma_2^{-1/2}$ in terms of $\gamma_1^{1/2} - \gamma_2^{1/2}$ and a quantitative version of [RZ22c, Corollary 3.6]. Afterwards, we prove a reduction theorem, which demonstrates that the difference of the DN maps

corresponding to two potentials q_1, q_2 can essentially be controlled by powers of the difference of the DN maps related to the conductivities γ_1, γ_2 . Finally, using the stability result stated in Theorem 1.3 for the fractional Schrödinger operators, we can prove Theorem 1.5.

4.1. Reduction Lemma.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $0 < \alpha \leq 1$. For all $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ satisfying $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$, we have*

$$(5) \quad \|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{L^\infty(\Omega)} \leq C \|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^\infty(\Omega)} \leq C \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}^{1/2}.$$

Moreover, under the additional assumption $\gamma_1^{1/2}, \gamma_2^{1/2} \in C^{0,\alpha}(\Omega)$, there holds $\gamma_i^{-1/2} \in C^{0,\alpha}(\Omega)$ with

$$(6) \quad \|\gamma_i^{-1/2}\|_{L^\infty(\Omega)} \leq 1/\gamma_0^{1/2}, \quad [\gamma_i^{-1/2}]_{C^{0,\alpha}(\Omega)} \leq \frac{[\gamma_i^{1/2}]_{C^{0,\alpha}(\Omega)}}{\gamma_0}$$

for $i = 1, 2$ and $\gamma_1^{-1/2} - \gamma_2^{-1/2} \in C^{0,\alpha}(\Omega)$ satisfying

$$(7) \quad \begin{aligned} & [\gamma_1^{-1/2} - \gamma_2^{-1/2}]_{C^{0,\alpha}(\Omega)} \leq \frac{[\gamma_1^{1/2} - \gamma_2^{1/2}]_{C^{0,\alpha}(\Omega)}}{\gamma_0} \\ & + \frac{\|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^\infty(\Omega)}}{\gamma_0^{3/2}} ([\gamma_2^{1/2}]_{C^{0,\alpha}(\Omega)} + [\gamma_1^{1/2}]_{C^{0,\alpha}(\Omega)}). \end{aligned}$$

Proof. We have

$$|\gamma_1^{-1/2}(x) - \gamma_2^{-1/2}(x)| = \left| \frac{\gamma_2^{1/2}(x) - \gamma_1^{1/2}(x)}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)} \right| \leq \gamma_0^{-1} \|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^\infty(\Omega)}$$

for all $x \in \Omega$ and therefore the first estimate in (5) follows. The second part in (5) follows from the estimate $|a^{1/2} - b^{1/2}| \leq |a - b|^{1/2}$ for all $a, b \in \mathbb{R}_+$.

From now on assume that the functions γ_1, γ_2 satisfy additionally $\gamma_1^{1/2}, \gamma_2^{1/2} \in C^{0,\alpha}(\Omega)$. Using the uniform ellipticity of γ_1, γ_2 , we have $\|\gamma_i^{-1/2}\|_{L^\infty(\Omega)} \leq \gamma_0^{-1/2}$ and

$$|\gamma_i^{-1/2}(x) - \gamma_i^{-1/2}(y)| = \frac{|\gamma_i^{1/2}(y) - \gamma_i^{1/2}(x)|}{|\gamma_i^{1/2}(x)\gamma_i^{1/2}(y)|} \leq \frac{[\gamma_i^{1/2}]_{C^{0,\alpha}(\Omega)}}{\gamma_0} |x - y|^\alpha.$$

This establishes the estimate (6) and hence $\gamma_i \in C^{0,\alpha}(\Omega)$ for $i = 1, 2$. Next we prove the bound (7). We have

$$[\gamma_1^{-1/2} - \gamma_2^{-1/2}]_{C^{0,\alpha}(\Omega)} = \left[\frac{\gamma_2^{1/2} - \gamma_1^{1/2}}{\gamma_1^{1/2} \gamma_2^{1/2}} \right]_{C^{0,\alpha}(\Omega)} = \left[\frac{\gamma_1^{1/2} - \gamma_2^{1/2}}{\gamma_1^{1/2} \gamma_2^{1/2}} \right]_{C^{0,\alpha}(\Omega)}.$$

We have

$$\begin{aligned}
 & \left| \frac{(\gamma_1^{1/2} - \gamma_2^{1/2})(x)}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)} - \frac{(\gamma_1^{1/2} - \gamma_2^{1/2})(y)}{\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)} \right| \\
 &= \frac{|(\gamma_1^{1/2} - \gamma_2^{1/2})(x)\gamma_1^{1/2}(y)\gamma_2^{1/2}(y) - (\gamma_1^{1/2} - \gamma_2^{1/2})(y)\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)|}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)} \\
 &= \left| \frac{((\gamma_1^{1/2} - \gamma_2^{1/2})(x) - (\gamma_1^{1/2} - \gamma_2^{1/2})(y))\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)} \right. \\
 & \quad \left. + \frac{(\gamma_1^{1/2} - \gamma_2^{1/2})(y)(\gamma_1^{1/2}(y)\gamma_2^{1/2}(y) - \gamma_1^{1/2}(x)\gamma_2^{1/2}(x))}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)} \right|
 \end{aligned}$$

for all $x, y \in \Omega$. Next observe that there holds

$$\begin{aligned}
 & \gamma_1^{1/2}(y)\gamma_2^{1/2}(y) - \gamma_1^{1/2}(x)\gamma_2^{1/2}(x) \\
 &= -(\gamma_1^{1/2}(y)(\gamma_2^{1/2}(x) - \gamma_2^{1/2}(y)) + \gamma_2^{1/2}(x)(\gamma_1^{1/2}(x) - \gamma_1^{1/2}(y))).
 \end{aligned}$$

By assumption we get

$$\begin{aligned}
 & \left| \frac{(\gamma_1^{1/2} - \gamma_2^{1/2})(x)}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)} - \frac{(\gamma_1^{1/2} - \gamma_2^{1/2})(y)}{\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)} \right| \\
 & \leq \frac{|(\gamma_1^{1/2} - \gamma_2^{1/2})(x) - (\gamma_1^{1/2} - \gamma_2^{1/2})(y)|}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)} \\
 & \quad + |(\gamma_1^{1/2} - \gamma_2^{1/2})(y)| \left(\frac{|\gamma_2^{1/2}(x) - \gamma_2^{1/2}(y)|}{\gamma_1^{1/2}(x)\gamma_2^{1/2}(x)\gamma_2^{1/2}(y)} + \frac{|\gamma_1^{1/2}(x) - \gamma_1^{1/2}(y)|}{\gamma_1^{1/2}(x)\gamma_1^{1/2}(y)\gamma_2^{1/2}(y)} \right) \\
 & \leq \left(\frac{[\gamma_1^{1/2} - \gamma_2^{1/2}]_{C^{0,\alpha}(\Omega)}}{\gamma_0} + \frac{\|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^\infty(\Omega)}}{\gamma_0^{3/2}} ([\gamma_2^{1/2}]_{C^{0,\alpha}(\Omega)} + [\gamma_1^{1/2}]_{C^{0,\alpha}(\Omega)}) \right) \\
 & \quad \cdot |x - y|^\alpha
 \end{aligned}$$

for all $x, y \in \Omega$ and hence there holds

$$\begin{aligned}
 & [\gamma_1^{-1/2} - \gamma_2^{-1/2}]_{C^{0,\alpha}(\Omega)} \\
 & \leq \frac{[\gamma_1^{1/2} - \gamma_2^{1/2}]_{C^{0,\alpha}(\Omega)}}{\gamma_0} + \frac{\|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^\infty(\Omega)}}{\gamma_0^{3/2}} ([\gamma_2^{1/2}]_{C^{0,\alpha}(\Omega)} + [\gamma_1^{1/2}]_{C^{0,\alpha}(\Omega)}).
 \end{aligned}$$

□

Lemma 4.2 (Multiplication by Sobolev functions). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $0 < s < 1$. If $u \in H^s(\Omega)$ and $\gamma \in L^\infty(\mathbb{R}^n)$ with background deviation $m \in H^{s,n/s}(\mathbb{R}^n)$ satisfies $\gamma(x) \geq \gamma_0 > 0$ then there holds*

$$(8) \quad \|\gamma^{1/2}u\|_{H^s(\Omega)} \leq C(1 + \|m\|_{L^\infty(\mathbb{R}^n)} + \|m\|_{H^{s,n/s}(\mathbb{R}^n)})\|u\|_{H^s(\Omega)}$$

and

$$(9) \quad \|\gamma^{-1/2}u\|_{H^s(\Omega)} \leq C(1 + \|m\|_{L^\infty(\mathbb{R}^n)} + \|m\|_{H^{s,n/s}(\mathbb{R}^n)})\|u\|_{H^s(\Omega)}.$$

Proof. Let $Eu \in H^s(\mathbb{R}^n)$ be an extension of u such that $\|Eu\|_{H^s(\mathbb{R}^n)} \leq 2\|u\|_{H^s(\Omega)}$. This extension exists by the quotient space definition of $H^s(\Omega)$. Thus, applying [RZ22c, Lemma 3.4] to $Eu \in H^s(\mathbb{R}^n)$, we deduce

$$\begin{aligned} \|\gamma^{1/2}u\|_{H^s(\Omega)} &\leq \|\gamma^{1/2}Eu\|_{H^s(\mathbb{R}^n)} \leq \|mEu\|_{H^s(\mathbb{R}^n)} + \|Eu\|_{H^s(\mathbb{R}^n)} \\ &\leq C(1 + \|m\|_{L^\infty(\mathbb{R}^n)} + \|m\|_{H^{s,n/s}(\mathbb{R}^n)})\|Eu\|_{H^s(\mathbb{R}^n)} \\ &\leq C(1 + \|m\|_{L^\infty(\mathbb{R}^n)} + \|m\|_{H^{s,n/s}(\mathbb{R}^n)})\|u\|_{H^s(\Omega)}. \end{aligned}$$

This establishes (8). Arguing as in the proof of [RZ22c, Lemma 3.7] we can write $\gamma^{-1/2} = 1 - \frac{m}{m+1}$ with $\frac{m}{m+1} \in H^{s,n/s}(\mathbb{R}^n)$ and $\|\frac{m}{m+1}\|_{H^{s,n/s}(\mathbb{R}^n)} \leq \|m\|_{H^{s,n/s}(\mathbb{R}^n)}$. Thus, we can repeat the above estimates to obtain (9). \square

Theorem 4.3. *Let $0 < s < \min(1, n/2)$, $\theta_0 \in (s/n, 1)$, $0 < \epsilon \ll 1$ and $k \in \mathbb{N}_0$ satisfy*

$$k < \frac{2s + \epsilon}{\theta_0} < k + 1 \quad \text{and} \quad \ell s + \epsilon \notin \mathbb{N} \quad \forall \ell = 1, 2.$$

Assume that $\Omega \subset \mathbb{R}^n$ is a domain of class $C^{k,1}$ with bounded boundary and the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 and potentials q_1, q_2 fulfill the following conditions:

- (i) $\gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1}$ for some $0 < \gamma_0 < 1$,
- (ii) $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n) \cap W^{2s+\epsilon, n/s}(\Omega_e)$ with $m_1 - m_2 \in W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\Omega_e)$,
- (iii) there exist $C_1, C_2, C_{\theta_0} > 0$ such that

$$(10) \quad \|m_i\|_{H^{s,n/s}(\mathbb{R}^n)} \leq C_1, \quad \|m_i\|_{W^{2s+\epsilon, n/s}(\Omega_e)} \leq C_2$$

for $i = 1, 2$ and

$$(11) \quad \|m_1 - m_2\|_{W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\Omega_e)}^{\theta_0} \leq C_{\theta_0}.$$

Then there holds

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \leq CC_{\theta_0} (\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}}).$$

Proof. Let $f, g \in H^s(\Omega_e)$ and for $i = 1, 2$ denote by $v_f^i \in H^s(\mathbb{R}^n)$ the unique solution to the fractional Schrödinger equation $(-\Delta)^s + q_i$ (see [RZ22c, Lemma 3.11]). Using Lemma 1.4, we deduce for $i = 1, 2$ and any extension $e_g \in H^s(\mathbb{R}^n)$ of $g \in H^s(\Omega_e)$ the identity

$$\langle \Lambda_{q_i} f, g \rangle = B_{q_i}(v_f^i, e_g) = B_{\gamma_i}(\gamma_i^{-1/2} v_f^i, \gamma_i^{-1/2} e_g) = \langle \Lambda_{\gamma_i}(\gamma_i^{-1/2} f), \gamma_i^{-1/2} g \rangle.$$

Therefore, we obtain

$$\begin{aligned} \langle (\Lambda_{q_1} - \Lambda_{q_2})f, g \rangle &= \langle \Lambda_{\gamma_1}(\gamma_1^{-1/2} f), \gamma_1^{-1/2} g \rangle - \langle \Lambda_{\gamma_2}(\gamma_2^{-1/2} f), \gamma_2^{-1/2} g \rangle \\ &= \langle \Lambda_{\gamma_1}(\gamma_1^{-1/2} f), (\gamma_1^{-1/2} - \gamma_2^{-1/2})g \rangle + \langle \Lambda_{\gamma_1}(\gamma_1^{-1/2} f), \gamma_2^{-1/2} g \rangle \\ &\quad - \langle \Lambda_{\gamma_2}(\gamma_2^{-1/2} - \gamma_1^{-1/2})f, \gamma_2^{-1/2} g \rangle - \langle \Lambda_{\gamma_2}(\gamma_1^{-1/2} f), \gamma_2^{-1/2} g \rangle \\ &= \langle \Lambda_{\gamma_1}(\gamma_1^{-1/2} f), (\gamma_1^{-1/2} - \gamma_2^{-1/2})g \rangle + \langle \Lambda_{\gamma_2}(\gamma_1^{-1/2} - \gamma_2^{-1/2})f, \gamma_2^{-1/2} g \rangle \\ &\quad + \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\gamma_1^{-1/2} f), \gamma_2^{-1/2} g \rangle \\ &=: I_1 + I_2 + I_3 \end{aligned}$$

for all $f, g \in H^s(\Omega_e)$. Next note that the assumption (i) and the fact that solutions to the homogeneous fractional conductivity equation depend continuously on the data imply

$$(12) \quad \|\Lambda_{\gamma_i}\|_* \leq C$$

for $i = 1, 2$ and some $C > 0$. On the other hand, using Lemma 4.2, the uniform ellipticity (i) and the uniform bound (10), we deduce

$$(13) \quad \|\gamma^{1/2}f\|_{H^s(\Omega_e)} \leq C\|f\|_{H^s(\Omega_e)} \quad \text{and} \quad \|\gamma^{-1/2}f\|_{H^s(\Omega_e)} \leq C\|f\|_{H^s(\Omega_e)}$$

for all $f \in H^s(\Omega_e)$ and some $C > 0$. Using (12), (13) and Lemma 3.1, we can estimate I_1 as follows:

$$\begin{aligned} |I_1| &\leq \|\Lambda_{\gamma_1}\|_* \|\gamma_1^{-1/2}f\|_{H^s(\Omega_e)} \|(\gamma_1^{-1/2} - \gamma_2^{-1/2})g\|_{H^s(\Omega_e)} \\ &\leq \|\Lambda_{\gamma_1}\|_* \|\gamma_1^{-1/2}f\|_{H^s(\Omega_e)} \|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{C^{0,s+\epsilon}(\Omega_e)} \|g\|_{H^s(\Omega_e)} \\ &\leq C \|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{C^{0,s+\epsilon}(\Omega_e)} \|f\|_{H^s(\Omega_e)} \|g\|_{H^s(\Omega_e)}. \end{aligned}$$

By Lemma 4.1 we can upper bound the Hölder norm by

$$\begin{aligned} (14) \quad &\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{C^{0,s+\epsilon}(\Omega_e)} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{1/2} \\ &+ \frac{[\gamma_1^{1/2} - \gamma_2^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)}}{\gamma_0} + \frac{\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{1/2}}{\gamma_0^{3/2}} ([\gamma_2^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)} + [\gamma_1^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)}) \\ &\leq C(1 + [\gamma_2^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)} + [\gamma_1^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)}) \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{1/2} \\ &+ C[\gamma_1^{1/2} - \gamma_2^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)}. \end{aligned}$$

By the (supercritical) Sobolev embedding in Slobodeckij spaces (cf. [DDE12, Theorem 4.57]) and the second estimate in (iii) we have

$$(15) \quad [\gamma_i^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)} = [m_i]_{C^{0,s+\epsilon}(\Omega_e)} \leq C\|m_i\|_{W^{2s+\epsilon, n/s}(\Omega_e)} \leq C$$

for $i = 1, 2$. On the other hand [DDE12, Theorem 4.57], Lemma 3.4 and Lemma 4.1 imply

$$\begin{aligned} [\gamma_1^{1/2} - \gamma_2^{1/2}]_{C^{0,s+\epsilon}(\Omega_e)} &\leq C\|m_1 - m_2\|_{W^{2s+\epsilon, n/s}(\Omega_e)} \\ &\leq C\|m_1 - m_2\|_{W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\Omega_e)}^{\theta_0} \|m_1 - m_2\|_{L^\infty(\Omega_e)}^{1-\theta_0} \\ &\leq C\|m_1 - m_2\|_{W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\Omega_e)}^{\theta_0} \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{\frac{1-\theta_0}{2}} \end{aligned}$$

for all $s/n < \theta_0 < 1$. Note that by assumption we have $s/n < 1/2$. Therefore, using the assertion (11) and (15) we deduce from the estimate (14) the following bound:

$$\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{C^{0,s+\epsilon}(\Omega_e)} \leq CC_{\theta_0} (\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{\frac{1}{2}} + \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{\frac{1-\theta_0}{2}}).$$

Hence, we have shown

$$|I_1| \leq CC_{\theta_0} (\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{\frac{1}{2}} + \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{\frac{1-\theta_0}{2}}) \|f\|_{H^s(\Omega_e)} \|g\|_{H^s(\Omega_e)}.$$

Clearly the same estimate holds for I_2 . Finally, for the expression I_3 we use (13) to obtain

$$|I_3| \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* \|f\|_{H^s(\Omega_e)} \|g\|_{H^s(\Omega_e)}.$$

Therefore, using exterior stability (cf. Theorem 1.2) we have

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \leq CC_{\theta_0} (\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}}).$$

□

4.2. Proof of Theorem 1.5. Using the reduction lemma from Section 4.1, we give here a proof of Theorem 1.5. Throughout this section, we will assume without loss of generality that $\epsilon > 0$ is such that $0 < \epsilon \ll 1$ and $\ell s + \epsilon \notin \mathbb{N}$ for $\ell = 1, 2$. We split the proof into three smaller technical lemmas. The first lemma states that under assumptions of Theorem 1.5 the function $\tilde{m} := m/\gamma_1^{1/2}$, where $m := m_1 - m_2$, satisfies a fractional conductivity equation connected to the conductivities γ_i and the difference of the potentials q_i . This lemma is our main tool for connecting the fractional conductivity equation to the fractional Schrödinger equation, which will allow us to use Theorem 1.3, once the potentials q_i are shown to be regular enough.

Lemma 4.4. *Let $0 < s < \min(1, n/2)$, $\epsilon > 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Suppose that the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 fulfill the following conditions:*

- (i) $\gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1}$ for some $0 < \gamma_0 < 1$,
- (ii) $m_1, m_2 \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$ and there exists $C_1 > 0$ such that

$$\|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} \leq C_1$$

for $i = 1, 2$,

- (iii) $m := m_1 - m_2 \in H^s(\mathbb{R}^n)$ and there exists $C_2 > 0$

$$(16) \quad \|(-\Delta)^s m_i\|_{L^1(\Omega_e)} \leq C_2$$

for $i = 1, 2$.

Then there holds

$$(17) \quad \operatorname{div}_s(\Theta_{\gamma_1} \nabla^s \tilde{m}) = \gamma_1^{1/2} \gamma_2^{1/2} (q_2 - q_1) \quad \text{in } \mathbb{R}^n,$$

where $\tilde{m} := m/\gamma_1^{1/2}$.

Proof. First note that by assumption we have $m_i \in H^{2s, \frac{n}{2s}}(\mathbb{R}^n)$ for $i = 1, 2$ and thus we can calculate as in [RZ22b, Proof of Lemma 8.13]:

$$\begin{aligned} \gamma_1^{1/2} \gamma_2^{1/2} (q_2 - q_1) &= -\gamma_1^{1/2} \gamma_2^{1/2} \left(\frac{(-\Delta)^s m_2}{\gamma_2^{1/2}} - \frac{(-\Delta)^s m_1}{\gamma_1^{1/2}} \right) \\ &= \gamma_2^{1/2} (-\Delta)^s m_1 - \gamma_1^{1/2} (-\Delta)^s m_2 \\ &= (1 + m_2) (-\Delta)^s m_1 - (1 + m_1) (-\Delta)^s m_2 \\ &= (1 + m_2) (-\Delta)^s m_1 + (1 + m_1) (-\Delta)^s m \\ &\quad - (1 + m_1) (-\Delta)^s m_1 \\ &= \gamma_1^{1/2} (-\Delta)^s m - m (-\Delta)^s m_1. \end{aligned}$$

Setting $\tilde{m} := m/\gamma_1^{1/2}$, we obtain

$$(18) \quad \gamma_1^{1/2}(-\Delta)^s(\gamma_1^{1/2}\tilde{m}) + \gamma_1^{1/2}(\gamma_1^{1/2}\tilde{m})q_1 = \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1).$$

Using $m_1 \in H^{2s, \frac{n}{2s}}(\mathbb{R}^n)$ then we deduce from [RZ22b, Corollary A.8] that there holds $\gamma_1^{1/2}\psi, \gamma_1^{-1/2}\psi \in H^s(\mathbb{R}^n)$ for all $\psi \in H^s(\mathbb{R}^n)$ and in particular $\tilde{m} \in H^s(\mathbb{R}^n)$. We next observe that by the Gagliardo–Nirenberg inequality in Bessel potential spaces (cf. [RZ22b, Corollary A.3,(iii)]) and the monotonicity of Bessel potential spaces, we have

$$(19) \quad \|m_i\|_{H^{2s+\epsilon, n/s}(\mathbb{R}^n)} \leq \|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)}^{1/2} \|m_i\|_{L^\infty(\mathbb{R}^n)}^{1/2}$$

for $i = 1, 2$. By the uniform ellipticity of γ_i , $i = 1, 2$, this immediately implies $\gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \in L^{n/s}(\mathbb{R}^n)$. By the assumptions $n/s > 2$, (16) and the uniform ellipticity as well as interpolation in L^p spaces, we see that $\gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \in L^2(\Omega_e)$. On the other hand, the boundedness of Ω and $n/s > 2$ gives $\gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \in L^2(\Omega)$. Therefore, we have $\gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \in L^2(\mathbb{R}^n)$. Hence, multiplying (18) by $\phi \in \mathcal{S}(\mathbb{R}^n)$ and integrating over \mathbb{R}^n shows

$$\begin{aligned} & \int_{\mathbb{R}^n} (-\Delta)^s(\gamma_1^{1/2}\tilde{m})(\gamma_1^{1/2}\phi) dx + \int_{\mathbb{R}^n} (\gamma_1^{1/2}\tilde{m})q_1(\gamma_1^{1/2}\phi) dx \\ &= \int_{\mathbb{R}^n} \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1)\phi dx. \end{aligned}$$

Now the first integral is finite since $m \in H^{2s, \frac{n}{2s}}(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\gamma_i \in L^\infty(\mathbb{R}^n)$ for $i = 1, 2$, the second integral by [RZ22b, Lemma A.10] and Hölder's inequality and the integral on the right hand side by the fact that $\gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \in L^2(\mathbb{R}^n)$. Next let $(\rho_\epsilon)_{\epsilon>0}$ be the standard mollifiers and let $m_\epsilon := \rho_\epsilon * m$. It is well-known that $m_\epsilon \rightarrow m$ in $H^{2s, \frac{n}{2s}}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ as m satisfies $m \in H^{2s, \frac{n}{2s}}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. On the other hand since the Bessel potential commutes with mollification, we deduce $m_\epsilon \in H^t(\mathbb{R}^n)$ for all $t \in \mathbb{R}$ as $m \in L^2(\mathbb{R}^n)$. Therefore, we can calculate

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^s(\gamma_1^{1/2}\tilde{m})(\gamma_1^{1/2}\phi) dx &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} (-\Delta)^s m_\epsilon (\gamma_1^{1/2}\phi) dx \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} (-\Delta)^{s/2} m_\epsilon (-\Delta)^{s/2} (\gamma_1^{1/2}\phi) dx \\ &= \int_{\mathbb{R}^n} (-\Delta)^{s/2} m (-\Delta)^{s/2} (\gamma_1^{1/2}\phi) dx \\ &= \int_{\mathbb{R}^n} (-\Delta)^{s/2} (\gamma_1^{1/2}\tilde{m}) (-\Delta)^{s/2} (\gamma_1^{1/2}\phi) dx. \end{aligned}$$

In the first equality we used the convergence $m_\epsilon \rightarrow m$ in $H^{2s, \frac{n}{2s}}(\mathbb{R}^n)$ as $\epsilon \rightarrow 0$, the continuity of the fractional Laplacian and that $\gamma_1^{1/2}\phi \in L^{\frac{n}{n-2s}}(\mathbb{R}^n)$, in the second equality that $m_\epsilon \in H^{2s}(\mathbb{R}^n)$, $\gamma_1^{1/2}\phi \in H^s(\mathbb{R}^n)$ and Plancherel's theorem, in the third equality that $m_\epsilon \rightarrow m$ in $H^s(\mathbb{R}^n)$ as $\epsilon \rightarrow 0$ and finally

the definition of \tilde{m} . Therefore, we obtain

$$(20) \quad \begin{aligned} & \langle (-\Delta)^{s/2}(\gamma_1^{1/2}\tilde{m}), (-\Delta)^{s/2}(\gamma_1^{1/2}\phi) \rangle_{L^2(\mathbb{R}^n)} + \langle q_1(\gamma_1^{1/2}\tilde{m}), (\gamma_1^{1/2}\phi) \rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1), \phi \rangle_{L^2(\mathbb{R}^n)} \end{aligned}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Now, if $\phi \in H^s(\mathbb{R}^n)$ then we can choose a sequence $(\phi_k)_{k \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ such that $\phi_k \rightarrow \phi$ in $H^s(\mathbb{R}^n)$. By (20) we have

$$\begin{aligned} & \langle (-\Delta)^{s/2}(\gamma_1^{1/2}\tilde{m}), (-\Delta)^{s/2}(\gamma_1^{1/2}\phi_k) \rangle_{L^2(\mathbb{R}^n)} + \langle q_1(\gamma_1^{1/2}\tilde{m}), (\gamma_1^{1/2}\phi_k) \rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1), \phi_k \rangle_{L^2(\mathbb{R}^n)} \end{aligned}$$

for all $k \in \mathbb{N}$. Since $\gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \in L^2(\mathbb{R}^n)$, there holds

$$\langle \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1), \phi_k \rangle_{L^2(\mathbb{R}^n)} \rightarrow \langle \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1), \phi \rangle_{L^2(\mathbb{R}^n)}$$

as $k \rightarrow \infty$. Again by [RZ22b, Lemma A.10], Hölder's inequality and the Sobolev embedding we see that

$$\langle q_1(\gamma_1^{1/2}\tilde{m}), (\gamma_1^{1/2}\phi_k) \rangle_{L^2(\mathbb{R}^n)} \rightarrow \langle q_1(\gamma_1^{1/2}\tilde{m}), (\gamma_1^{1/2}\phi) \rangle_{L^2(\mathbb{R}^n)}$$

as $k \rightarrow \infty$. Finally, by [RZ22b, Corollary A.7] it follows that $\gamma_1^{1/2}\phi_k \rightarrow \gamma_1^{1/2}\phi$ in $H^s(\mathbb{R}^n)$ and hence $(-\Delta)^{s/2}(\gamma_1^{1/2}\phi_k) \rightarrow (-\Delta)^{s/2}(\gamma_1^{1/2}\phi)$ in $L^2(\mathbb{R}^n)$, but then by the Cauchy–Schwartz inequality it follows that

$$\begin{aligned} & \langle (-\Delta)^{s/2}(\gamma_1^{1/2}\tilde{m}), (-\Delta)^{s/2}(\gamma_1^{1/2}\phi_k) \rangle_{L^2(\mathbb{R}^n)} \\ & \rightarrow \langle (-\Delta)^{s/2}(\gamma_1^{1/2}\tilde{m}), (-\Delta)^{s/2}(\gamma_1^{1/2}\phi) \rangle_{L^2(\mathbb{R}^n)} \end{aligned}$$

as $k \rightarrow \infty$. Hence, (20) holds for all $\phi \in H^s(\mathbb{R}^n)$. Therefore, by the fractional Liouville reduction (Lemma 1.4), we see that $\tilde{m} \in H^s(\mathbb{R}^n)$ satisfies

$$\operatorname{div}_s(\Theta_{\gamma_1}\nabla^s\tilde{m}) = \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \quad \text{in } \mathbb{R}^n$$

as claimed. \square

Next we show that the assumptions of Theorem 1.5 imply the required regularity and a priori bounds for the potentials q_i , allowing us to apply Theorem 1.3.

Lemma 4.5. *Let $0 < s < \min(1, n/2)$, $\epsilon > 0$. Suppose that the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 fulfill the following conditions:*

- (i) $\gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1}$ for some $0 < \gamma_0 < 1$,
- (ii) $m_1, m_2 \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$ and there exists $C_1 > 0$ such that

$$\|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} \leq C_1$$

for $i = 1, 2$,

Then $q_i \in H^{\delta, \frac{n}{2s}}(\mathbb{R}^n)$ for $\delta = 2\epsilon/3$ with

$$\|q_i\|_{H^{\delta, \frac{n}{2s}}(\mathbb{R}^n)} \leq M,$$

where $M > 0$ depends only on γ_0, C_1, n, s and ϵ .

Proof. First observe that we can write

$$\begin{aligned} q_i &= -\frac{(-\Delta)^s m_i}{\gamma_i^{1/2}} = -(-\Delta)^s m_i \left(1 - \frac{m_i}{m_i + 1}\right) \\ &= -(-\Delta)^s m_i + (-\Delta)^s m_i \frac{m_i}{m_i + 1} \end{aligned}$$

for $i = 1, 2$. By the assumption (ii) the first term belongs to $H^{2s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$ and hence it is sufficient to show that the second term is in $H^{\delta, \frac{n}{2s}}(\mathbb{R}^n)$ for some $\delta > 0$. Now using $m_i \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset H^{2s+\epsilon, \frac{n}{s}}(\mathbb{R}^n)$ (see (19)), the mapping properties of the fractional Laplacian and the Sobolev embedding $H^{2s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ we see that $(-\Delta)^s m_i \in H^{\epsilon, \frac{n}{s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Now we claim that $\frac{m_i}{m_i+1} \in H^{\frac{2\epsilon}{3}, \frac{n}{2s}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for all $\frac{n}{2s} \leq q \leq \infty$. That $\frac{m_i}{m_i+1} \in L^q(\mathbb{R}^n)$ follows from the uniform ellipticity of γ_i , $m_i \in L^{\frac{n}{2s}}(\mathbb{R}^n)$ and interpolation in L^p spaces. Next define $\Gamma_0 := \min(0, \gamma_0^{1/2} - 1)$ and choose $\Gamma \in C_b^\infty(\mathbb{R})$ such that $\Gamma(t) = \frac{t}{t+1}$ for $t \geq \Gamma_0$. By [AF92, p. 156] and $m_i \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we deduce for $i = 1, 2$ that $\Gamma(m_i) \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$, but since $m \geq \gamma_0^{1/2} - 1 > -1$ it follows that $\frac{m_i}{m_i+1} \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Moreover, [AF92, p. 156] gives the estimate

$$(21) \quad \left\| \frac{m_i}{m_i + 1} \right\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} \leq C(\|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} + \alpha \|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)}^{4s+2\epsilon}) \leq C,$$

where $\alpha = 0$ when $4s + 2\epsilon \leq 1$ and otherwise $\alpha = 1$. Hence, the claim is proved. Next define

$$p_1 := \frac{n}{s}, \quad p_2 := \frac{n}{2s}, \quad r_2 := \frac{3n}{4s}, \quad s_1 := \epsilon \quad \text{and} \quad \theta := \frac{2}{3}.$$

Then there holds $1 < p_1, p_2, r_2 < \infty$ and

$$\frac{1}{p_2} = \frac{\theta}{p_1} + \frac{1}{r_2}.$$

Moreover, since $(-\Delta)^s m_i \in H^{s_1, p_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\frac{m_i}{m_i+1} \in H^{\theta s_1, p_2}(\mathbb{R}^n) \cap L^{r_2}(\mathbb{R}^n)$ we deduce from [RZ22b, Lemma A.6] that there holds

$$(-\Delta)^s m_i \frac{m_i}{m_i + 1} \in H^{\theta s_1, p_2}(\mathbb{R}^n) = H^{\frac{2\epsilon}{3}, \frac{n}{2s}}(\mathbb{R}^n).$$

Hence, we have shown $q_i \in H^{\frac{2\epsilon}{3}, \frac{n}{2s}}(\mathbb{R}^n)$ for $i = 1, 2$ as previously asserted. Moreover, [RZ22b, Lemma A.6 (i)] yields the estimate

$$\begin{aligned} & \left\| (-\Delta)^s m_i \frac{m_i}{m_i + 1} \right\|_{H^{\frac{2\epsilon}{3}, \frac{n}{2s}}(\mathbb{R}^n)} \\ & \leq C \left(\|(-\Delta)^s m_i\|_{L^\infty(\mathbb{R}^n)} \left\| \frac{m_i}{m_i + 1} \right\|_{H^{\frac{2\epsilon}{3}, \frac{n}{2s}}(\mathbb{R}^n)} \right. \\ & \quad \left. + \left\| \frac{m_i}{m_i + 1} \right\|_{L^{\frac{3n}{4s}}(\mathbb{R}^n)} \|(-\Delta)^s m_i\|_{H^{\epsilon, \frac{n}{s}}(\mathbb{R}^n)}^\theta \|(-\Delta)^s m_i\|_{L^\infty(\mathbb{R}^n)}^{1-\theta} \right) \leq C \end{aligned}$$

where in the second inequality we used (21) and the assumption $m_i \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$. \square

Our final technical lemma is an interpolation statement, required for the use of Theorem 4.3.

Lemma 4.6. *Let $0 < s < \min(1, n/2)$, $\epsilon > 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Assume that $\theta_0 \in (\max(1/2, 2s/n), 1)$. Suppose that the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 fulfill the following conditions:*

- (i) $\gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1}$ for some $0 < \gamma_0 < 1$,
- (ii) $m_1, m_2 \in H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$ and there exists $C_1 > 0$ such that

$$\|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} \leq C_1$$

for $i = 1, 2$.

Then there holds

$$\|m_i\|_{W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\Omega_e)} \leq C$$

for $i = 1, 2$ and some constant $C > 0$ depending only on n, s, ϵ, θ_0 and C_1 .

Proof. Define

$$s_1 := 2s + \epsilon, \quad s_2 := 4s + 2\epsilon, \quad p_1 := \frac{n}{s}, \quad p_2 := \frac{n}{2s} \quad \text{and} \quad \theta := 2 - 1/\theta_0.$$

Since, $1/2 < \theta_0 < 1$ we have $0 < \theta < 1$. Moreover, there holds $0 < s_1 < s_2 < \infty$, $1 < p_1, p_2 < \infty$ and

$$\frac{2s + \epsilon}{\theta_0} = \theta s_1 + (1 - \theta) s_2 \quad \text{and} \quad \frac{1}{\theta_0 \frac{n}{s}} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}.$$

Therefore, [RZ22b, Corollary A.3] implies

$$\|u\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 \frac{n}{s}}(\mathbb{R}^n)} \leq C \|u\|_{H^{2s+\epsilon, n/s}(\mathbb{R}^n)}^\theta \|u\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)}^{1-\theta}$$

for all $u \in H^{2s+\epsilon, n/s}(\mathbb{R}^n) \cap H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)$. By the assumptions (i)-(ii) and the uniform estimate (19) this ensures

$$\|m_i\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 \frac{n}{s}}(\mathbb{R}^n)} \leq C$$

for $i = 1, 2$. On the other hand the condition $\theta_0 > \frac{2s}{n}$ ensures $\theta_0 \frac{n}{s} > 2$ and therefore Theorem 2.2 shows

$$\|m_i\|_{W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\Omega_e)} \leq \|m_i\|_{W^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\mathbb{R}^n)} \leq \|m_i\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\mathbb{R}^n)} \leq C$$

for $i = 1, 2$. \square

We are finally ready to complete the proof of Theorem 1.5:

Proof of Theorem 1.5. We start by recalling that the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is defined as the space of tempered distributions whose Fourier transform belongs to $L^1_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty,$$

see [BCD11, Section 1.3.1]. Since $\dot{H}^s(\mathbb{R}^n)$ continuously embeds into $L^{2n/(n-2s)}(\mathbb{R}^n)$ for all $0 \leq s < n/2$ (see [BCD11, Theorem 1.38]), we find for any $1 \leq q \leq \frac{2n}{n-2s}$

$$\begin{aligned} \|m\|_{L^q(\Omega)} &\leq C \|m\|_{L^{\frac{2n}{n-2s}}(\Omega)} \leq C \|\tilde{m}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\ &\leq C \|\tilde{m}\|_{\dot{H}^s(\mathbb{R}^n)} \leq C \langle \Theta_{\gamma_1} \nabla^s \tilde{m}, \nabla^s \tilde{m} \rangle. \end{aligned}$$

Testing (17) with $\tilde{m} \in H^s(\mathbb{R}^n)$ and applying Lemma 4.4 we can estimate

$$\begin{aligned} \|m\|_{L^q(\Omega)} &\leq C \langle \Theta_{\gamma_1} \nabla^s \tilde{m}, \nabla^s \tilde{m} \rangle \leq C \left| \int_{\mathbb{R}^n} \gamma_1^{1/2} \gamma_2^{1/2} (q_2 - q_1) \tilde{m} \, dx \right| \\ &\leq C \left(\left| \int_{\Omega} \gamma_1^{1/2} \gamma_2^{1/2} (q_1 - q_2) \tilde{m} \, dx \right| \right. \\ &\quad \left. + \left| \int_{\Omega_e} \gamma_1^{1/2} \gamma_2^{1/2} (q_1 - q_2) \tilde{m} \, dx \right| \right) \\ &=: I_1 + I_2. \end{aligned}$$

Observe that we have $q_i \in L^{\frac{n}{2s}}(\mathbb{R}^n)$ for $i = 1, 2$, since $m_i \in H^{2s, \frac{n}{2s}}(\mathbb{R}^n)$ and the conductivities are uniformly elliptic. Thus, using Hölder's inequality and then the Cauchy–Schwarz inequality, we get

$$\begin{aligned} I_1 &\leq C \|q_1 - q_2\|_{L^{\frac{n}{2s}}(\Omega)} \|\gamma_2^{1/2} m\|_{L^{\frac{n}{n-2s}}(\Omega)} \\ &\leq C \|q_1 - q_2\|_{L^{\frac{n}{2s}}(\Omega)} \|\gamma_2^{1/2}\|_{L^{\frac{2n}{n-2s}}(\Omega)} \|m\|_{L^{\frac{2n}{n-2s}}(\Omega)} \\ &\leq C \|q_1 - q_2\|_{L^{\frac{n}{2s}}(\Omega)}, \end{aligned}$$

where in the last estimate we have used that Ω is bounded and the assumption (i). On the other hand the second integral can be estimated by

$$\begin{aligned} I_2 &\leq C \|m \gamma_2^{1/2}\|_{L^\infty(\Omega_e)} \|q_1 - q_2\|_{L^1(\Omega_e)} \\ &\leq C \frac{\|\gamma_2\|_{L^\infty(\Omega_e)}^{1/2}}{\gamma_0^{1/2}} (\|(-\Delta)^s m_1\|_{L^1(\Omega_e)} + \|(-\Delta)^s m_2\|_{L^1(\Omega_e)}) \|m\|_{L^\infty(\Omega_e)} \\ &\leq C \|m\|_{L^\infty(\Omega_e)}, \end{aligned}$$

where we have used the assumptions (i) and (iii). Thus, for all $1 \leq q \leq 2n/(n-2s)$ there holds

$$(22) \quad \|m\|_{L^q(\Omega)} \leq C \|m\|_{L^{\frac{2n}{n-2s}}(\Omega)} \leq C (\|q_1 - q_2\|_{L^{\frac{n}{2s}}(\Omega)} + \|m\|_{L^\infty(\Omega_e)}).$$

By Lemma 4.5 the assumptions in Theorem 1.3 on the potentials q_i , $i = 1, 2$, namely that $q_i \in H^{\delta, \frac{n}{2s}}(\mathbb{R}^n)$ with an a priori bound $\|q_i\|_{H^{\delta, \frac{n}{2s}}(\mathbb{R}^n)} \leq M$, are satisfied and we can estimate the first term in the right-hand side of (22) as

$$\|q_1 - q_2\|_{L^{\frac{n}{2s}}(\Omega)} \leq C \omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*),$$

for some logarithmic modulus of continuity satisfying $\omega(x) \leq C |\log(x)|^{-\sigma}$ for all $0 < x \leq 1$, where $C, \sigma > 0$. The second term of (22) can be estimated

by first using Lemma 4.1 as

$$\|m\|_{L^\infty(\Omega_e)} = \|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^\infty(\Omega_e)} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{1/2}$$

and then applying the exterior stability result of Theorem 1.2 to obtain

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)}^{1/2} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{1/2}.$$

In summary, we have shown that there holds

$$\|m_1 - m_2\|_{L^q(\Omega)} \leq C\left(\omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*) + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{1/2}\right).$$

Next, we show that m_1, m_2 satisfy the conditions in Theorem 4.3. Similarly, as for the estimate (19), by the Gagliardo–Nirenberg inequality in Bessel potential spaces (cf. [RZ22b, Corollary A.3,(iii)]) and the monotonicity of Bessel potential spaces, we have

$$\begin{aligned} \|m_i\|_{H^{s,n/s}(\mathbb{R}^n)} &\leq \|m_i\|_{H^{s+\epsilon,n/s}(\mathbb{R}^n)} \leq \|m_i\|_{H^{2s+2\epsilon,\frac{n}{2s}}(\mathbb{R}^n)}^{1/2} \|m_i\|_{L^\infty(\mathbb{R}^n)}^{1/2} \\ &\leq \|m_i\|_{H^{4s+2\epsilon,\frac{n}{2s}}(\mathbb{R}^n)}^{1/2} \|m_i\|_{L^\infty(\mathbb{R}^n)}^{1/2} \leq C\|m_i\|_{L^\infty(\mathbb{R}^n)}^{1/2} \end{aligned}$$

for $i = 1, 2$, where in the last step we used the assumption (ii). Moreover, by the fact that $n/s > 2$, Theorem 2.2, (ii) and Lemma 3.4, we have

$$\begin{aligned} \|m_i\|_{W^{2s+\epsilon,\frac{n}{s}}(\Omega_e)} &\leq \|m_i\|_{W^{2s+\epsilon,\frac{n}{s}}(\mathbb{R}^n)} \leq \|m_i\|_{H^{2s+\epsilon,\frac{n}{s}}(\mathbb{R}^n)} \\ &\leq C\|m_i\|_{H^{4s+2\epsilon,\frac{n}{2s}}(\mathbb{R}^n)}^{1/2} \|m_i\|_{L^\infty(\mathbb{R}^n)}^{1/2} \end{aligned}$$

for $i = 1, 2$. Now, using the uniform bound (3) and the uniform ellipticity of γ_i we deduce

$$\|m\|_{W^{2s+\epsilon,n/s}(\Omega_e)} \leq C.$$

Applying Lemma 4.6 we also see that

$$\|m_i\|_{W^{\frac{2s+\epsilon}{\theta_0},\theta_0 n/s}(\Omega_e)} \leq C.$$

This now demonstrates that $m_1 - m_2$ satisfies the condition (11) in Theorem 4.3. Therefore, we can apply Theorem 4.3 to deduce the estimate

$$\begin{aligned} \|m_1 - m_2\|_{L^q(\Omega)} &\leq C\left(\omega\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1}{2}}\right.\right. \\ &\quad \left.\left.+ \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}}\right) + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1}{2}}\right) \end{aligned}$$

for all $1 \leq q \leq \frac{2n}{n-2s}$. Since $\theta_0 \in (1/2, 1)$ we have $\frac{1-\theta_0}{2} \in (0, 1/4)$. Hence, there holds $x \leq x^{1/2} \leq x^{\frac{1-\theta_0}{2}}$ for all $0 < x \leq 1$. Therefore, we obtain

$$\begin{aligned} &\omega\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}}\right) \\ &\leq \omega\left(3\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}}\right). \end{aligned}$$

By the assumptions $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* \leq 3^{-1/\delta}$, $0 < \delta < \frac{1-\theta_0}{2}$, we have

$$3\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}} \leq \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2} - \delta}.$$

Using the fact that $\omega(x) \leq C|\log x|^{-\sigma}$ for $0 < x \leq 1$, we deduce

$$\begin{aligned} & \omega \left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{1-\theta_0}{2}} \right) \\ & \leq C\omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*). \end{aligned}$$

Next we observe that there holds $x^{1/2} \leq C|\log x|^{-\sigma}$ for all $0 < x \leq 1$ and some $C > 0$. To see this, observe that this estimate is equivalent to $Cx^{-1/2} \geq 2^{-\sigma}|\log x^{-1/2}|^\sigma$. Since $0 < x \leq 1$, this is the same as $\log x^{-1/2} \leq \frac{C^{1/\sigma}}{2}(x^{-1/2})^{1/\sigma}$, but it is well-known that there holds $\log(y) \leq y^r/r$ for all $y > 0$ and $r > 0$. In fact, the last assertion is a straightforward consequence of the inequality $\log(z) \leq z - 1$ for all $z > 0$ by applying it to $z = y^r$ with $y > 0$, $r > 0$. Hence, the above estimate holds with $C = (2\sigma)^\sigma$. This finally shows

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* \leq C|\log(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)|^{-\sigma} = C\omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*),$$

and we can conclude the proof. \square

5. PARTIAL DATA REDUCTION WITH MINIMAL REGULARITY ASSUMPTIONS ON THE DOMAIN

For the sake of completeness and possible future work on low regularity settings, we record a proposition considering a quantitative partial data reduction to the Schrödinger case with minimal assumptions on the domain Ω . These improvements come with the cost of having certain restrictions on the considered sets of measurements but on the other hand also allow a simpler proof. Furthermore, Proposition 5.1 is strong enough to conclude Theorem 1.5 after minor changes to the proof and using the partial data stability result for the Schrödinger case in [RS20]. This argument however does not establish the quantitative reduction "up to the boundary" like the proof of Theorem 4.3. In particular, this approach avoids using $W^{s,p}$ spaces and the explicit extension operators, which in part explains why the boundary regularity questions are not encountered and the proof is considerably simpler.

Finally, we emphasize that the partial data reduction to the Schrödinger case does not directly imply partial data stability for the conductivity case as the partial data uniqueness result is based on an additional unique continuation argument for the conductivities (see [CRZ22, RZ22c]) and the authors are not aware of quantitative unique continuation results of the following type: Let $1 \leq p, q, r \leq \infty$, $s, t \geq 0$ and $W \subset \mathbb{R}^n$ be a nonempty open set. For all $u \in X \subset H^{t,q}(\mathbb{R}^n)$ there holds

$$(23) \quad \|u\|_{L^p(\mathbb{R}^n)} \leq F(\|(-\Delta)^s u\|_{L^r(W)}, \|u\|_{L^\infty(W)})$$

for some continuous function $F : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ with $F(0, 0) = 0$ and independent of $u \in X$ where the set X encodes possible a priori assumptions. For the relevant regularity assumptions and choices of p, q, r, s, t , see [CRZ22, RZ22c] where $u = m_1 - m_2$ is the choice of u in our possible application. Such estimates for the special case $u|_W = 0$, i.e. $\|u\|_{L^\infty(W)} = 0$, could already give new results towards the stability of the partial data problems

for the fractional conductivity equation. In general, estimates of the type (23) would be interesting also in other function spaces and norms.

Proposition 5.1. *Let $0 < s < \min(1, n/2)$, $s/n < \theta_0 < 1$ and $\epsilon > 0$. Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set bounded in one direction. Suppose that the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 and potentials q_1, q_2 fulfill the following conditions:*

- (i) $\gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1}$ for some $0 < \gamma_0 < 1$,
- (ii) There exists $C_0 > 0$ such that

$$\|m_i\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\mathbb{R}^n)} \leq C_0.$$

for $i = 1, 2$.

Let $W_1, W_2, W \Subset \Omega_\epsilon$ be nonempty open sets such that $W_1 \cup W_2 \Subset W$. Then there holds

$$\begin{aligned} & \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\tilde{H}^s(W_1) \rightarrow (\tilde{H}^s(W_2))^*} \\ & \leq C(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*}^{\frac{1-\theta_0}{2}}) \end{aligned}$$

where $C > 0$ depending only on $s, \epsilon, n, \Omega, C_0, \theta_0, W_1, W_2, W$ and γ_0 .

Proof. Suppose that $f \in C_c^\infty(W_1)$ and $g \in C_c^\infty(W_2)$. Choose a smooth cutoff function $\eta|_{W_1 \cup W_2} = 1$, $0 \leq \eta \leq 1$ and $\text{supp}(\eta) \subset W$. We explain next how to modify the proof of Theorem 4.3, in order to obtain the quantitative partial data reduction result. To do so, we will establish sufficient estimates next. We first note that $m_1, m_2 \in H^{2s+\epsilon, n/s}(\mathbb{R}^n) \supset H^{s, n/s}(\mathbb{R}^n)$ with explicit bounds for the norms by (i), (ii), the Gagliardo–Nirenberg inequality in Bessel potential spaces and the monotonicity of Bessel potential spaces. Therefore we may continue the proof of Theorem 4.3 up to the point where we have the terms I_1, I_2, I_3 as in the proof of Theorem 4.3.

Since $W_1, W_2 \subset W$, we have

$$|I_3| \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*} \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)}.$$

as in the earlier proof.

We may suppose, by taking ϵ smaller if necessary, that $0 < \epsilon < 1 - s$ and $2s + \epsilon$ is not an integer by the monotonicity of Bessel potential spaces. For the term I_1 , we may calculate that

$$\begin{aligned} |I_1| & \leq \|\Lambda_{\gamma_1}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*} \|\gamma_1^{-1/2} f\|_{H^s(\mathbb{R}^n)} \|(\gamma_1^{-1/2} - \gamma_2^{-1/2})g\|_{H^s(\mathbb{R}^n)} \\ & \leq C\|\Lambda_{\gamma_1}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*} \|f\|_{H^s(\mathbb{R}^n)} \|\eta(\gamma_1^{-1/2} - \gamma_2^{-1/2})\|_{C^{0, s+\epsilon}(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

since $g = \eta g$. We may then estimate using the embeddings to Hölder spaces, Gagliardo–Nirenberg inequality in Bessel potential spaces, the formula (5),

boundedness of the multiplication with η and support conditions, that

$$\begin{aligned}
 & \|\eta(\gamma_1^{-1/2} - \gamma_2^{-1/2})\|_{C^{0,s+\epsilon}(\mathbb{R}^n)} \\
 & \leq C \|\eta(\gamma_1^{-1/2} - \gamma_2^{-1/2})\|_{H^{2s+\epsilon, n/s}(\mathbb{R}^n)} \\
 & \leq C \|\eta(\gamma_1^{-1/2} - \gamma_2^{-1/2})\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\mathbb{R}^n)}^{\theta_0} \|\eta(\gamma_1^{-1/2} - \gamma_2^{-1/2})\|_{L^\infty(\mathbb{R}^n)}^{1-\theta_0} \\
 & \leq C \|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\mathbb{R}^n)}^{\theta_0} \|\gamma_1^{-1/2} - \gamma_2^{-1/2}\|_{L^\infty(W)}^{1-\theta_0} \\
 & \leq C \left\| \frac{m_1}{m_1+1} - \frac{m_2}{m_2+1} \right\|_{H^{\frac{2s+\epsilon}{\theta_0}, \theta_0 n/s}(\mathbb{R}^n)}^{\theta_0} \|\gamma_1 - \gamma_2\|_{L^\infty(W)}^{\frac{1-\theta_0}{2}} \\
 & \leq C \|\gamma_1 - \gamma_2\|_{L^\infty(W)}^{\frac{1-\theta_0}{2}}
 \end{aligned}$$

where in the last step we used the triangle inequality and a composition estimate for functions in Bessel potential spaces (see e.g. [AF92] and references therein). In fact, for any $t > 0$, $1 < p < \infty$, $\delta > 0$, there exists a polynomial function $P: \mathbb{R}^2 \rightarrow \mathbb{R}$ (of degree at most $t+1$ and with nonnegative coefficients) such that

$$\left\| \frac{m}{m+1} \right\|_{H^{t,p}(\mathbb{R}^n)} \leq P(\|m\|_{H^{t,p}(\mathbb{R}^n)}, \|m\|_{L^\infty(\mathbb{R}^n)})$$

for all $m \in H^{t,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $m+1 \geq \delta$. This can be argued similarly as (21) in the proof of Lemma 4.5 but we decided to recall this alternative estimate here. This let us conclude that

$$|I_1| \leq C \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)} \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*}^{\frac{1-\theta_0}{2}}$$

by the exterior stability estimate (Theorem 1.2) since γ_1, γ_2 are continuous by (ii). This is possible since the exterior stability estimate also holds for the considered partial data, i.e. $\|\gamma_1 - \gamma_2\|_{L^\infty(W)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\tilde{H}^s(W) \rightarrow (\tilde{H}^s(W))^*}$. We may argue similarly with I_2 , which completes the proof. \square

Remark 5.2. *Theorem 4.3 could be also adapted into the partial data setting as in Proposition 5.1. We omit presenting these details.*

6. EXPONENTIAL INSTABILITY

In this Section we complement the above considerations about stability with an instability result in the flavour of [KRS21]. We start by recalling the definitions of ϵ -discrete sets and δ -nets. These can be given in the setting of a generic metric space (X, d) .

Definition 6.1. *Let (X, d) be a metric space, and assume $\epsilon, \delta > 0$. A set $Y \subset X$ is said to be ϵ -discrete if for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$ it holds $d(y_1, y_2) \geq \epsilon$. A set $Z \subset X$ is said to be a δ -net for a set $X_1 \subset X$ if for all $x \in X_1$ there exists $z \in Z$ such that $d(x, z) \leq \delta$.*

We can now prove Theorem 1.7, which shows that the exponential stability obtained in Section 4 can not be improved. For this result it will suffice to consider conductivities whose exterior value is the constant 1.

Proof of Theorem 1.7. Define the set

$$X_{\ell\epsilon\beta} := \{f \in C_c^\ell(B_1); \|f\|_{L^\infty} \leq \epsilon, \|f\|_{C^\ell} \leq \beta\},$$

and let $\tilde{X}_{\ell\epsilon\beta} := 1 + X_{\ell\epsilon\beta}$. By [Man01, Lemma 2] (see also [RS18, Lemma 3.4]) we deduce the existence of an ϵ -discrete set $\tilde{Z} \subset \tilde{X}_{\ell\epsilon\beta}$ of cardinality

$$|\tilde{Z}| \geq \exp\left(C(\beta/\epsilon)^{n/\ell}\right),$$

with $C = C(\ell, n) > 0$, where $\tilde{X}_{\ell\epsilon\beta}$ is seen as a metric space with respect to the L^∞ norm. By careful construction, it is also possible to ensure that $1 \leq \gamma \leq 2$ for all $\gamma \in \tilde{Z}$ (see [Man01, Proof of Corollary 1]). Let now $\gamma \in \tilde{Z}$, and let q be the corresponding transformed potential. Since $\gamma \geq 1$, for all $v \in \tilde{H}^s(B_1)$ we get

$$\begin{aligned} \langle \operatorname{div}_s \Theta_\gamma \nabla^s v, v \rangle_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} &= B_\gamma(v, v) \\ &\geq \|(-\Delta)^{s/2} v\|_{L^2(\mathbb{R}^n)}^2 \\ &\geq \lambda_{1,s} \|v\|_{L^2(B_1)}^2, \end{aligned}$$

where $\lambda_{1,s}$ is the first Dirichlet eigenvalue of $(-\Delta)^s$ in B_1 (see e.g. [RS18, Proof of Lemma 3.2]). Therefore, by the fractional Liouville reduction

$$\begin{aligned} \langle ((-\Delta)^s + q)v, v \rangle_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} &= \langle \operatorname{div}_s \Theta_\gamma \nabla^s (\gamma^{-1/2} v), \gamma^{-1/2} v \rangle_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} \\ &\geq \lambda_{1,s} \|\gamma^{-1/2} v\|_{L^2(B_1)}^2 \\ &\geq \frac{\lambda_{1,s}}{2} \|v\|_{L^2(B_1)}^2, \end{aligned}$$

and eventually $\|((-\Delta)^s + q)^{-1}\|_{L^2(B_1) \rightarrow L^2(B_1)} \leq \frac{2}{\lambda_{1,s}}$.

Since $m := \gamma^{1/2} - 1 \geq 0$, for the potential we compute

$$\|q\|_{L^\infty(B_1)} = \left\| \frac{(-\Delta)^s m}{1+m} \right\|_{L^\infty(B_1)} \lesssim \|(-\Delta)^s m\|_{L^\infty(B_1)} \leq \|(-\Delta)^s m\|_{C^{\ell-2s}(\mathbb{R}^n)}.$$

Observe now that the fractional Laplacian $(-\Delta)^s$ acts as a bounded operator between C^r and C^{r-2s} for any $r, r-2s \in \mathbb{R}^+ \setminus \mathbb{N}$. In order to see this, write the symbol as

$$|\xi|^{2s} = \psi(\xi)|\xi|^{2s} + (1 - \psi(\xi))|\xi|^{2s},$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ is 1 near the origin. The second term on the right hand side belongs to Hörmander's class $S_{1,0}^{2s}$, and thus has the correct mapping properties (see e.g. [Tay11]). The first term on the right hand side corresponds to a convolution operator with kernel $k = \mathcal{F}^{-1}(\psi) * \mathcal{F}^{-1}(|\xi|^{2s})$, which is L^1 as a convolution of a Schwartz function with a homogeneous function of order $-n - 2s$. Therefore $u \mapsto k * u$ is bounded between any two Hölder spaces by the Fourier characterization of Hölder spaces ([Tri83, Section 2.3.7]). Thus

$$\|q\|_{L^\infty(B_1)} \lesssim \|(-\Delta)^s m\|_{C^{\ell-2s}(\mathbb{R}^n)} \lesssim \|m\|_{C^\ell(\mathbb{R}^n)}.$$

Moreover, $\gamma \in \tilde{Z} \subset \tilde{X}_{\ell\epsilon\beta}$ implies

$$m^2 + 2m = (m+1)^2 - 1 = \gamma - 1 \in X_{\ell\epsilon\beta},$$

and thus in particular $m^2 + 2m \in C_c^\ell(B_1)$. Define the function $F: x \mapsto \sqrt{1+x} - 1$, which is smooth for non-negative x and has the property that $F(m^2 + 2m) = m$. Since C_c^ℓ is closed under composition with smooth functions, we have $m \in C_c^\ell(B_1)$ as well, with $\|m\|_{C^\ell(B_1)} \lesssim \|m^2 + 2m\|_{C^\ell(B_1)}$.

Eventually

$$\|q\|_{L^\infty(B_1)} \lesssim \|m\|_{C^\ell(\mathbb{R}^n)} \lesssim \|\gamma - 1\|_{C^\ell(B_1)} \leq \beta.$$

We shall now apply [RS18, Proposition 2.4]. Observe that the eigenvalue condition of the said proposition is not needed here, because in this case the potential q comes from a fractional Liouville reduction, and so the Dirichlet problem for the transformed operator is already known to be well-posed.

Recall that $\{f_{h,k,l}\}$ is the basis of $L^2(B_3 \setminus \overline{B_2})$ constructed in [RS18, Lemma 2.1]. The operator $\Gamma(q) := \Lambda_q - \Lambda_0$ mapping $L^2(B_3 \setminus \overline{B_2})$ to itself is completely characterized by the quantities

$$a_{h_1, k_1, l_1}^{h_2, k_2, l_2}(q) := \langle \Gamma(q) f_{h_1, k_1, l_1}, f_{h_2, k_2, l_2} \rangle_{L^2(B_3 \setminus \overline{B_2})}.$$

Let

$$X := \{ \Gamma(q) ; q \in \mathcal{Q}, \|\Gamma(q)\|_X < \infty \},$$

where \mathcal{Q} is the class of all transformed potentials, and

$$\|\Gamma(q)\|_X := \sup_{h_i, k_i, l_i} (1 + \max\{h_1 + k_1, h_2 + k_2\})^{n+2} |a_{h_1, k_1, l_1}^{h_2, k_2, l_2}(q)|.$$

By [RS18, Proposition 2.4] we obtain

$$\begin{aligned} |a_{h_1, k_1, l_1}^{h_2, k_2, l_2}(q)| &\leq C_{n,s} e^{-c \max\{h_1 + k_1, h_2 + k_2\}} \|q\|_{L^\infty(B_1)} \|((-\Delta)^s + q)^{-1}\|_{L^2(B_1) \rightarrow L^2(B_1)} \\ &\leq C'_{n,s} \beta e^{-c \max\{h_1 + k_1, h_2 + k_2\}}, \end{aligned}$$

which means that if $\gamma \in \tilde{X}_{\ell\epsilon\beta}$ then $\Gamma(q) \in X$.

With this in mind, we can follow the proof of [Man01, Lemma 3] (see also [RS18, Lemma 3.2]) to construct a δ -net \overline{Y} for the image under $\Gamma: q \mapsto \Lambda_q - \Lambda_0$ of the set of potentials corresponding to conductivities in $\tilde{X}_{\ell\epsilon\beta}$. Here $\delta := \exp\left(-\epsilon^{-\frac{n}{(2n+3)\ell}}\right)$, and the cardinality of \overline{Y} is

$$|\overline{Y}| \leq \beta \exp\left(C'' \epsilon^{-n/\ell}\right).$$

It is clear that for β large enough it must hold that $|\tilde{Z}| > |\overline{Y}|$, which means that there exists two conductivities $\gamma_1, \gamma_2 \in \tilde{X}_{\ell\epsilon\beta}$ with $\|\gamma_1 - \gamma_2\|_{L^\infty(B_1)} \geq \epsilon$ and

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_{L^2(B_3 \setminus \overline{B_2}) \rightarrow L^2(B_3 \setminus \overline{B_2})} \lesssim \|\Gamma(q_1) - \Gamma(q_2)\|_X \lesssim \delta,$$

in light of the fact that Λ_q is a bounded operator $L^2(B_3 \setminus \overline{B_2}) \rightarrow L^2(B_3 \setminus \overline{B_2})$ (see [RS18, Remarks 2.2, 2.5]) and the related estimate [RS18, eq. (21)].

Since $\gamma_1 = \gamma_2 = 1$ in $\mathbb{R}^n \setminus \overline{B_1}$, by [CRZ22, Lemma 4.1] we deduce $\Lambda_q = \Lambda_\gamma$ as operators on $H^s(\mathbb{R}^n \setminus \overline{B_1}) \rightarrow (H^s(\mathbb{R}^n \setminus \overline{B_1}))^*$. This lets us conclude that

$$\begin{aligned} \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^s(B_3 \setminus \overline{B_2}) \rightarrow (H^s(B_3 \setminus \overline{B_2}))^*} &\leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^s(B_3 \setminus \overline{B_2}) \rightarrow (H^s(B_3 \setminus \overline{B_2}))^*} \\ &\leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{L^2(B_3 \setminus \overline{B_2}) \rightarrow L^2(B_3 \setminus \overline{B_2})} \\ &\lesssim \delta. \end{aligned} \quad \square$$

APPENDIX A. PROOFS OF AUXILIARY RESULTS

Proof of Lemma 3.1. Statement (i) has been essentially proved in [DNPV12, Lemma 5.3] but under the assumptions $0 \leq \phi \leq 1$ and $\mu = 1$. We give a proof of this slightly more general result here for the sake of completeness. Since $C^{0,\mu}(\Omega) \subset L^\infty(\Omega)$, we have $\|\phi u\|_{L^p(\Omega)} \leq \|\phi\|_{L^\infty(\Omega)} \|u\|_{L^p(\Omega)}$ and hence it remains to control the Gagliardo seminorm. We have

$$\begin{aligned} [\phi u]_{W^{s,p}(\Omega)}^p &= \int_{\Omega} \int_{\Omega} \frac{|(\phi u)(x) - (\phi u)(y)|^p}{|x - y|^{n+sp}} dx dy \\ &= \int_{\Omega} \int_{\Omega} \frac{|\phi(x)(u(x) - u(y)) + (\phi(x) - \phi(y))u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\leq 2^{p-1} \left(\int_{\Omega} \int_{\Omega} \frac{|\phi(x)|^p |u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy + \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p |u(y)|^p}{|x - y|^{n+sp}} dx dy \right) \\ &\leq 2^{p-1} \left(\|\phi\|_{L^\infty(\Omega)}^p [u]_{W^{s,p}(\Omega)}^p + \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p |u(y)|^p}{|x - y|^{n+sp}} dx dy \right). \end{aligned}$$

Now we write

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p |u(y)|^p}{|x - y|^{n+sp}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p |u(y)|^p}{|x - y|^{n+sp}} \chi_{B_1(y)}(x) dx dy \\ &+ \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p |u(y)|^p}{|x - y|^{n+sp}} \chi_{B_1(y)^c}(x) dx dy =: I_1 + I_2, \end{aligned}$$

where χ_A denotes the characteristic function of the set $A \subset \mathbb{R}^n$. Next we estimate I_1, I_2 . We have

$$\begin{aligned} I_1 &\leq [\phi]_{C^{0,\mu}(\Omega)}^p \int_{\Omega} |u(y)|^p \left(\int_{B_1(y)} \frac{dx}{|x - y|^{n+(s-\mu)p}} \right) dy \\ &= [\phi]_{C^{0,\mu}(\Omega)}^p \|u\|_{L^p(\Omega)}^p \int_{B_1(0)} \frac{dz}{|z|^{n+(s-\mu)p}} = \frac{\omega_n}{(\mu - s)p} [\phi]_{C^{0,\mu}(\Omega)}^p \|u\|_{L^p(\Omega)}^p \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq 2^p \|\phi\|_{L^\infty(\Omega)}^p \int_{\Omega} |u(y)|^p \left(\int_{B_1(y)^c} \frac{dx}{|x - y|^{n+sp}} \right) dy \\ &= 2^p \|\phi\|_{L^\infty(\Omega)}^p \|u\|_{L^p(\Omega)}^p \int_{B_1(0)^c} \frac{dz}{|z|^{n+sp}} = \frac{2^p \omega_n}{sp} \|\phi\|_{L^\infty(\Omega)}^p \|u\|_{L^p(\Omega)}^p, \end{aligned}$$

where ω_n is the area of the unit sphere. Therefore, we get

$$\begin{aligned} [\phi u]_{W^{s,p}(\Omega)}^p &\leq 2^{p-1} \left(1 + \frac{2^p \omega_n}{sp} \|\phi\|_{L^\infty(\Omega)}^p + \frac{\omega_n}{(\mu - s)p} [\phi]_{C^{0,\mu}(\Omega)}^p \right) \|u\|_{W^{s,p}(\Omega)}^p \\ &\leq C \left(1 + \frac{\mu}{s(\mu - s)} \right) \|\phi\|_{C^{0,\mu}(\Omega)}^p \|u\|_{W^{s,p}(\Omega)}^p, \end{aligned}$$

where C only depends on n and p . This establishes the assertion (i).

Now let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in (0, 1)$. By classical results we have $\phi u \in W^{k,p}(\Omega)$ with $\|\phi u\|_{W^{k,p}(\Omega)} \leq C \|\phi\|_{C^k(\Omega)} \|u\|_{W^{k,p}(\Omega)}$ for some $C > 0$ only depending on n, k and p . Thus it remains to estimate the Gagliardo

seminorm of $\partial^\alpha(\phi u)$ for all multi-indices α of order k . By the Leibniz rule we have

$$\begin{aligned}
 [\partial^\alpha(\phi u)]_{W^{\sigma,p}(\Omega)} &\leq C \sum_{\beta \leq \alpha} [\partial^{\alpha-\beta} \phi \partial^\beta u]_{W^{\sigma,p}(\Omega)} \\
 &\leq C \sum_{\beta \leq \alpha: \alpha \neq \beta} [\partial^{\alpha-\beta} \phi \partial^\beta u]_{W^{\sigma,p}(\Omega)} + C [\phi \partial^\alpha u]_{W^{\sigma,p}(\Omega)} \\
 &\leq C \sum_{\beta \leq \alpha: \alpha \neq \beta} \left(1 + \frac{\mu}{\sigma(\mu - \sigma)}\right)^{1/p} \|\partial^{\beta-\alpha} \phi\|_{C^{0,\mu}(\Omega)} \|\partial^\beta u\|_{W^{\sigma,p}(\Omega)} \\
 &\quad + C \left(1 + \frac{\mu}{\sigma(\mu - \sigma)}\right)^{1/p} \|\phi\|_{C^{0,\mu}(\Omega)} \|\partial^\alpha u\|_{W^{\sigma,p}(\Omega)} \\
 &\leq C \left(1 + \frac{1}{\sigma(1 - \sigma)}\right)^{1/p} \left(1 + \frac{\mu}{\sigma(\mu - \sigma)}\right)^{1/p} \sum_{\beta \leq \alpha: \alpha \neq \beta} \|\partial^{\beta-\alpha} \phi\|_{C^{0,\mu}(\Omega)} \|\partial^\beta u\|_{W^{1,p}(\Omega)} \\
 &\quad + C \left(1 + \frac{\mu}{\sigma(\mu - \sigma)}\right)^{1/p} \|\phi\|_{C^{0,\mu}(\Omega)} \|\partial^\alpha u\|_{W^{\sigma,p}(\Omega)} \\
 &\leq C \left(1 + \frac{1}{\sigma(1 - \sigma)}\right)^{1/p} \left(1 + \frac{\mu}{\sigma(\mu - \sigma)}\right)^{1/p} \left(\sum_{\ell=0}^k \|\nabla^\ell \phi\|_{C^{0,\mu}(\Omega)}\right) \|u\|_{W^{s,p}(\Omega)}
 \end{aligned}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ and some constant $C > 0$ only depending on n, k and p . Here, we used the bound from the case $0 < s < 1$ and the coefficient in the forth inequality comes from the embedding $W^{1,p}(\Omega) \hookrightarrow W^{\sigma,p}(\Omega)$ (see [DNPV12, Proposition 2.2]). \square

Proof of Lemma 3.2. First note that $\text{dist}(\text{supp}(u), \Omega^c) \geq d > 0$. In fact, for $x \in \text{supp}(u) \subset \Omega$, $y \in \Omega_e$ consider the curve $\gamma: [0, 1] \rightarrow \mathbb{R}^n$ with $\gamma(t) := x + t(y - x)$. Then there exists $t_0 \in (0, 1)$ such that $\gamma(t_0) \in \partial\Omega$ because otherwise one would have $[0, 1] = \gamma^{-1}(\Omega) \cup \gamma^{-1}(\Omega_e)$ which is not possible. But then $|x - y| \geq |x - \gamma(t_0)| \geq d > 0$. Next we distinguish the cases $0 < s < 1$, $s = k \in \mathbb{N}$ and $s = k + \sigma$ with $k \in \mathbb{N}$, $0 < \sigma < 1$.

Case $0 < s < 1$: Clearly, we have $\|\bar{u}\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\Omega)}$ and thus it remains to show $[\bar{u}]_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}$. By symmetry we can split $[\bar{u}]_{W^{s,p}(\mathbb{R}^n)}$ as

$$\begin{aligned}
 &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\bar{u}(x) - \bar{u}(y)|^p}{|x - y|^{n+sp}} dx dy = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \\
 &\quad + \int_{\Omega^c} \int_{\Omega^c} \frac{|\bar{u}(x) - \bar{u}(y)|^p}{|x - y|^{n+sp}} dx dy + 2 \int_{\Omega} \int_{\Omega^c} \frac{|u(x)|^p}{|x - y|^{n+sp}} dx dy \\
 &= [u]_{W^{s,p}(\Omega)}^p + 2 \int_{\text{supp}(u)} |u(x)|^p \left(\int_{\Omega^c} \frac{dy}{|x - y|^{n+sp}} dy \right) dx.
 \end{aligned}$$

For any $x \in \text{supp}(u)$ there holds $B_{d/2}(x) \subset \Omega$ and hence we have

$$\int_{\Omega^c} \frac{dy}{|x - y|^{n+sp}} dy \leq \int_{B_{d/2}(x)^c} \frac{dy}{|x - y|^{n+sp}} = \omega_n \int_{d/2}^{\infty} \frac{r^{n-1}}{r^{n+sp}} dr = \frac{\omega_n}{sp} \left(\frac{d}{2}\right)^{-sp}.$$

Therefore we get

$$[\bar{u}]_{W^{s,p}(\mathbb{R}^n)}^p \leq [u]_{W^{s,p}(\Omega)}^p + 2^{1+sp} \frac{\omega_n}{sp} d^{-sp} \|u\|_{L^p(\Omega)}^p.$$

This shows

$$\|\bar{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \left(1 + \frac{2^s}{(sp)^{1/p}} d^{-s} \right) \|u\|_{W^{s,p}(\Omega)}.$$

Case $s = k \in \mathbb{N}$: Let $\Omega_d \subset \Omega$ be the $d/2$ -neighborhood of $\text{supp}(u)$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then $\text{supp}(\phi) \cap \overline{\Omega_d} = \emptyset$ or $\text{supp}(\phi) \cap \overline{\Omega_d} \subset \Omega$ is a nonempty compact set. In the latter case choose a cutoff function $\eta_d \in C_c^\infty(\Omega)$, $0 \leq \eta_d \leq 1$ with $\eta|_V = 1$, where $V \Subset \Omega$ such that $\text{supp}(\phi) \cap \overline{\Omega_d} \subset V$. Then for any $u \in W^{1,p}(\Omega)$ there holds

$$\begin{aligned} \int_{\mathbb{R}^n} \bar{u} \partial_i \phi \, dx &= \int_{\Omega} u \eta_d \partial_i \phi \, dx = \int_{\Omega} u \partial_i (\eta_d \phi) \, dx - \int_{\Omega} u \phi \partial_i \eta_d \, dx \\ &= - \int_{\Omega} \partial_i u (\eta_d \phi) \, dx = - \int_{\Omega} \partial_i u (\eta_d - 1) \phi \, dx - \int_{\Omega} (\partial_i u) \phi \, dx \\ &= - \int_{\Omega} (\partial_i u) \phi \, dx = - \int_{\mathbb{R}^n} \overline{\partial_i u} \phi \, dx \end{aligned}$$

for all $1 \leq i \leq n$. This identity clearly also holds if the intersection is empty. Thus $\partial_i \bar{u} = \overline{\partial_i u} \in L^p(\mathbb{R}^n)$ for all $1 \leq i \leq n$. Hence, if $u \in W^{1,p}(\Omega)$ then $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ and by the previous case there holds $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = \|u\|_{W^{1,p}(\Omega)}$. By induction we see that for any $k \in \mathbb{N}$ we have $\bar{u} \in W^{k,p}(\mathbb{R}^n)$ whenever $u \in W^{k,p}(\Omega)$. Moreover, there holds $\|\bar{u}\|_{W^{k,p}(\mathbb{R}^n)} = \|u\|_{W^{k,p}(\Omega)}$.

Case $s = k + \sigma$ with $k \in \mathbb{N}$, $0 < \sigma < 1$: This follows immediately from the previous two cases. \square

REFERENCES

- [AF92] David R. Adams and Michael Frazier. Composition operators on potential spaces. *Proc. Amer. Math. Soc.*, 114(1):155–165, 1992.
- [Ale88] Giovanni Alessandrini. Stable determination of conductivity by boundary measurements. *Appl. Anal.*, 27(1-3):153–172, 1988.
- [AN19] Kweku Abraham and Richard Nickl. On statistical Calderón problems. *Math. Stat. Learn.*, 2(2):165–216, 2019.
- [AP06] Kari Astala and Lassi Päiväranta. Calderón’s inverse conductivity problem in the plane. *Ann. of Math. (2)*, 163(1):265–299, 2006.
- [APL05] Kari Astala, Lassi Päiväranta, and Matti Lassas. Calderón’s inverse problem for anisotropic conductivity in the plane. *Comm. Partial Differential Equations*, 30(1-3):207–224, 2005.
- [AS22] Giovanni S. Alberti and Matteo Santacesaria. Infinite-dimensional inverse problems with finite measurements. *Arch. Ration. Mech. Anal.*, 243(1):1–31, 2022.
- [AV05] Giovanni Alessandrini and Sergio Vessella. Lipschitz stability for the inverse conductivity problem. *Adv. in Appl. Math.*, 35(2):207–241, 2005.
- [BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.
- [BM01] Haïm Brezis and Petru Mironescu. Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces. *J. Evol. Equ.*, 1(4):387–404, 2001. Dedicated to the memory of Tosio Kato.

- [Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [Cal80] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.
- [CdHS22] Giovanni Covi, Maarten de Hoop, and Mikko Salo. Uniqueness in an inverse problem of fractional elasticity, 2022.
- [CDR16] Pedro Caro, David Dos Santos Ferreira, and Alberto Ruiz. Stability estimates for the calderón problem with partial data. *Journal of Differential Equations*, 260(3):2457–2489, 2016.
- [CLR20] Mihajlo Cekić, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for the fractional Schrödinger equation with drift. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 91, 46, 2020.
- [CMR21] Giovanni Covi, Keijo Mönkkönen, and Jesse Railo. Unique continuation property and Poincaré inequality for higher order fractional Laplacians with applications in inverse problems. *Inverse Probl. Imaging*, 15(4):641–681, 2021.
- [CMRU22] Giovanni Covi, Keijo Mönkkönen, Jesse Railo, and Gunther Uhlmann. The higher order fractional Calderón problem for linear local operators: Uniqueness. *Adv. Math.*, 399:Paper No. 108246, 2022.
- [Cov20] Giovanni Covi. Inverse problems for a fractional conductivity equation. *Nonlinear Anal.*, 193:111418, 18, 2020.
- [Cov21] Giovanni Covi. Uniqueness for the fractional Calderón problem with quasilo-cal perturbations. *SIAM J. Math. Anal. (to appear)*, 2021. arXiv:2110.11063.
- [Cov22] Giovanni Covi. Uniqueness for the anisotropic fractional conductivity equation, 2022.
- [CRZ22] Giovanni Covi, Jesse Railo, and Philipp Zimmermann. The global inverse fractional conductivity problem, 2022. arXiv:2204.04325.
- [CS14] Pedro Caro and Mikko Salo. Stability of the calderón problem in admissible geometries. *Inverse Problems and Imaging*, 8(4):939–957, 2014.
- [DDE12] Françoise Demengel, Gilbert Demengel, and Reinie Ern . *Functional spaces for the theory of elliptic partial differential equations*. Springer, 2012.
- [DGLZ12] Qiang Du, Max Gunzburger, R.B. Lehoucq, and Kun Zhou. Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM rev 54, No 4:667-696*, 2012.
- [DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
- [Dob10] Manfred Dobrowolski. *Angewandte Funktionalanalysis: Funktionalanalysis, Sobolev-R ume und Elliptische Differentialgleichungen*. Springer-Verlag, 2010.
- [DSFKSU09] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann. Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.*, 178(1):119–171, 2009.
- [FGKU21] Ali Feizmohammadi, Tuhin Ghosh, Katya Krupchyk, and Gunther Uhlmann. Fractional anisotropic Calderón problem on closed Riemannian manifolds, 2021. arxiv:2112.03480.
- [GLX17] Tuhin Ghosh, Yi-Hsuan Lin, and Jingni Xiao. The Calderón problem for variable coefficients nonlocal elliptic operators. *Comm. Partial Differential Equations*, 42(12):1923–1961, 2017.
- [GSU20] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann. The Calderón problem for the fractional Schrödinger equation. *Anal. PDE*, 13(2):455–475, 2020.
- [GU21] Tuhin Ghosh and Gunther Uhlmann. The Calderón problem for nonlocal operators, 2021. arxiv:2110.09265.
- [KRS21] Herbert Koch, Angkana Rüland, and Mikko Salo. On instability mechanisms for inverse problems. *Ars Inven. Anal.*, pages Paper No. 7, 93, 2021.

- [KRZ22] Manas Kar, Jesse Railo, and Philipp Zimmermann. The fractional p -biharmonic systems: optimal poincaré constants, unique continuation and inverse problems, 2022. arXiv:2208.09528.
- [Kwa17] Mateusz Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017.
- [LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Anal.*, 216:Paper No. 112699, 2022.
- [LRZ22] Yi-Hsuan Lin, Jesse Railo, and Philipp Zimmermann. The calderón problem for a nonlocal diffusion equation with time-dependent coefficients, 2022. arxiv:2211.07781.
- [LZ23] Yi-Hsuan Lin and Philipp Zimmermann. Unique determination of coefficients and kernel in nonlocal porous medium equations with absorption term, 2023. arxiv:2305.16282.
- [Man01] Nicolae Mandache. Exponential instability in an inverse problem for the Schrödinger equation. *Inverse Problems*, 17(5):1435–1444, 2001.
- [McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
- [Nac88] Adrian I. Nachman. Reconstructions from boundary measurements. *Ann. of Math. (2)*, 128(3):531–576, 1988.
- [NU94] Gen Nakamura and Gunther Uhlmann. Global uniqueness for an inverse boundary problem arising in elasticity. *Invent. Math.*, 118(3):457–474, 1994.
- [QU22] Hadrian Quan and Gunther Uhlmann. The Calderón problem for the fractional Dirac operator, 2022. arXiv:2204.00965.
- [RS88] Rakesh and William W. Symes. Uniqueness for an inverse problem for the wave equation. *Comm. Partial Differential Equations*, 13(1):87–96, 1988.
- [RS18] Angkana Rüland and Mikko Salo. Exponential instability in the fractional Calderón problem. *Inverse Problems*, 34(4):045003, 21, 2018.
- [RS20] Angkana Rüland and Mikko Salo. The fractional Calderón problem: low regularity and stability. *Nonlinear Anal.*, 193:111529, 56, 2020.
- [RZ22a] Jesse Railo and Philipp Zimmermann. Counterexamples to uniqueness in the inverse fractional conductivity problem with partial data. *Inverse Probl. Imaging (to appear)*, 2022. arXiv:2203.02442.
- [RZ22b] Jesse Railo and Philipp Zimmermann. Fractional Calderón problems and Poincaré inequalities on unbounded domains. *J. Spectr. Theory (to appear)*, 2022. arXiv:2203.02425.
- [RZ22c] Jesse Railo and Philipp Zimmermann. Low regularity theory for the inverse fractional conductivity problem, 2022. arXiv:2208.11465.
- [Ste70] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
- [SU87] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math.*, 125(1):153–169, 1987.
- [Sun90] Zi Qi Sun. On continuous dependence for an inverse initial-boundary value problem for the wave equation. *J. Math. Anal. Appl.*, 150(1):188–204, 1990.
- [Tay11] Michael E. Taylor. *Partial differential equations III. Nonlinear equations*, volume 117 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.
- [Tri83] H. Triebel. *Theory of function spaces*, volume 38 of *Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology]*. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.
- [Uhl14] Gunther Uhlmann. 30 years of Calderón’s problem. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2012–2013*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIII, 25. École Polytech., Palaiseau, 2014.
- [Zim23] Philipp Zimmermann. Inverse problem for a nonlocal diffuse optical tomography equation, 2023. arxiv:2302.08610.

INSTITUT FÜR ANGEWANDTE MATHEMATIK, RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 205, 69120 HEIDELBERG, GERMANY
Email address: `giovanni.covi@uni-heidelberg.de`

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, CAMBRIDGE CB3 0WB, UK
Email address: `jr891@cam.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, CANADA
Email address: `teemu.tyni@oulu.fi`

DEPARTMENT OF MATHEMATICS, ETH ZÜRICH, ZÜRICH, SWITZERLAND
Email address: `philipp.zimmermann@math.ethz.ch`