The Ramanujan Journal (2024) 63:1–12
https://doi.org/10.1007/s11139-023-00776-4

On restricted approximation measures of Jacobi's triple product

Leena Leinonen[1] · Marko Leinonen[1]

Received: 18 October 2017 / Accepted: 17 July 2023 / Published online: 11 September 2023

Abstract
We obtain rational approximations for Jacobi's triple product

$$\Pi_q(t) := \prod_{m=1}^{\infty}(1 - q^{2m})(1 + q^{2m-1}t)(1 + q^{2m-1}t^{-1}),$$

when $t = a/b \in \mathbb{Q}$ is non-zero and $q = 1/d$ with $d \in \mathbb{Z}\setminus\{0, \pm 1\}$. Especially we give effective and restricted approximation for the values of Jacobi's triple product and for the values of Euler's infinite product.

Keywords Diophantine approximation · Restricted approximation exponent · Irrationality exponent · q-Exponential series

Mathematics Subject Classification 11J82

1 Introduction and results

In the following $\|x\|$ denotes the distance of a real number x to the nearest integer. Let ξ be an irrational real number. Then the irrationality exponent $\mu(\xi)$ of ξ is defined by setting $\mu(\xi) = v(\xi) + 1$, where $v(\xi)$ is the infimum of the real numbers u for which the inequality

$$\|N\xi\| > N^{-u}$$

✉ Marko Leinonen
Marko.Leinonen@oulu.fi

Leena Leinonen
lkoivula@student.oulu.fi

[1] Research Unit of Mathematical Sciences, University of Oulu, PL 8000, 90014 Oulu, Finland

holds for every sufficiently large positive integer N. By restricting the set of positive integers N in the above definition to a certain infinite subset of positive integers, we get the definition of so-called restricted irrationality exponents. Let now d be an integer, $|d| \geq 2$. We will follow Bennett and Bugeaud [3] by defining $v_d(\xi)$ to be the infimum of the real numbers u for which the inequality

$$\left\| d^s \xi \right\| > |d|^{-su}$$

holds for every sufficiently large positive integer s. Likewise, $v_d^{\text{eff}}(\xi)$ denotes the infimum of the real numbers u for which there exists a computable constant $c(\xi, d)$ such that the condition

$$\left\| d^s \xi \right\| > c(\xi, d)|d|^{-su}$$

holds for every sufficiently large positive integer s. Further, we call $v_d(\xi) + 1$ and $v_d^{\text{eff}}(\xi) + 1$ restricted irrationality exponents of ξ.

Amou and Bugeaud [1] noted that $v_d(\xi) \geq 0$ for all irrational real numbers ξ, and furthermore $v_d(\xi) = 0$ for almost all irrational real numbers ξ, provided that $d \geq 2$. However, if ξ is a classical mathematical constant like $\sqrt{2}$, e or π, we do not even know whether $v_d(\xi) = 0$ for any d. On the other hand, for certain explicit numbers there are already results which give upper bounds for restricted irrationality exponents. Namely, Rivoal [10] proved that $v_d(\log r)$ is arbitrarily close to 0 for certain integers d, when $r \in \mathbb{Q}$ is sufficiently close to 1. See also Dubickas [6]. Recently, Bennett and Bugeaud [3] proved that there exists an effectively computable positive constant $\tau_1 = \tau_1(p)$ such that

$$v_p^{\text{eff}}\left(\sqrt{p^2 + 1}\right) \leq 1 - \tau_1$$

for every prime number p. They also noted that one can deduce the existence of an effectively computable positive constant $\tau_2 = \tau_2(d, k)$ such that

$$v_d^{\text{eff}}\left(\sqrt{d^{2k} + 1}\right) \leq 1 - \tau_2$$

for every positive integer k and $d \geq 2$.

In the present work, we investigate restricted rational approximations for the values of Jacobi's triple product

$$\Pi_q(t) := \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1}t)(1 + q^{2m-1}t^{-1})$$

at $t = a/b \in \mathbb{Q} \setminus \{0\}$ and $q = 1/d$, where $d \in \mathbb{Z}\setminus\{0, \pm 1\}$. Particularly, we consider determining effective exponents $v_d^{\text{eff}}(\Pi_{\frac{1}{d}}(a/b))$. Furthermore, we obtain that $v_d(\Pi_{\frac{1}{d}}(a/b)) = 0$.

Theorem 1 *Let* $t = a/b \in \mathbb{Q} \setminus \{0\}$, $\gcd(a, b) = 1$, $d \in \mathbb{Z}$ *and* $\max\{|a|, |b|\} < |d|$. *Then for all* $s, M \in \mathbb{Z}$ *with* $s \geq C$ *we have*

$$\left|\Pi_{\frac{1}{d}}(t) - \frac{M}{d^s}\right| > \frac{1}{2|d|^{s(1+\varepsilon_1(s))}}, \quad \varepsilon_1(s) = \frac{6}{\sqrt{s}} + \frac{8}{s},$$

where $C = (3\max\{|a|, |b|\} - 1)^2/4$. *Consequently,* $\upsilon_d(\Pi_{\frac{1}{d}}(t)) = 0$.

It is remarkable that (as far as we know) the only irrationality measure results for Jacobi's triple product at arbitrary rational $t \neq 0$ are outcomes of the linear independence results for (the right-hand side of) Jacobi's Theta function

$$\Theta(q, t) := \sum_{n=0}^{\infty} q^{n^2} t^n, \quad |q| < 1.$$

Namely, because $\Pi_{\frac{1}{d}}(t) = -1 + \Theta(1/d, t) + \Theta(1/d, t^{-1})$, the result of Bundschuh and Shiokawa in [5] implies the estimate

$$\mu\left(\Pi_{\frac{1}{d}}(t)\right) \leq \frac{5 + \sqrt{17}}{2} = 4.5615\ldots$$

for $d \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $t \in \mathbb{Q} \setminus \{0\}$.

We also study the restricted approximations for Euler's infinite product

$$\pi_q(t) := \prod_{n=1}^{\infty} (1 - q^n t)$$

at $t = 1$, when $q = 1/d$, $d \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $q = (1 - \sqrt{5})/(1 + \sqrt{5})$. Jacobi's triple product has a q-expansion, given by the well-known Jacobi's triple product identity

$$\prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1}t)(1 + q^{2m-1}t^{-1}) = \sum_{n=-\infty}^{\infty} t^n q^{n^2}, \tag{1}$$

see e.g. [2, p. 498]. Our proof of Theorem 1 will be based on this identity. By replacing q with $q^{3/2}$ and t with $-q^{-1/2}$ in (1), we obtain, after simplification, that

$$\prod_{m=1}^{\infty} (1 - q^{3m})(1 - q^{3m-2})(1 - q^{3m-1}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}.$$

This can be rewritten as

$$\pi_q := \prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2}\right), \tag{2}$$

which is the famous Euler's pentagonal formula, see e.g. [2, p. 500]. On the basis of the above consideration Euler's infinite product π_q seems to be a special case of Jacobi's triple product. But because of the square root substitutions we can not obtain our results for Euler's product π_q from Theorem 1. Therefore, we investigate separately the product π_q at $q = 1/d$, $d \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Theorem 2 *Let* $d \in \mathbb{Z} \setminus \{0, \pm 1\}$, $M \in \mathbb{Z}$ *and* $s \in \mathbb{Z}_+$. *Then*

$$\left|\pi_{\frac{1}{d}}(1) - \frac{M}{d^s}\right| > \frac{1}{2|d|^{s(1+\varepsilon_2(s))}}, \quad \varepsilon_2(s) = \frac{3 + \sqrt{1+24s}}{2s}.$$

Consequently, $v_d(\pi_{\frac{1}{d}}(1)) = 0$.

Theorem 2 improves considerably the earlier results concerning this special case. Recently, Leinonen et al. [7] obtained that $v_d(\pi_{\frac{1}{d}}(t)) = 1.1547\ldots$ with arbitrary $t \in \mathbb{Q} \setminus \{0\}$. It should be noted that there are more general results available which consider the irrationality exponent of Euler's product. Already in 1969 Bundschuh [4] proved that the irrationality exponent of the product $\pi_{\frac{1}{d}}(t)$ satisfies the inequality $\mu(\pi_{\frac{1}{d}}(t)) \leq 7/3$, for $|d| \in \mathbb{Z}_{\geq 2}$ and $t \in \mathbb{Q} \setminus \{0\}$. This is still the best known upper bound for $\mu(\pi_{\frac{1}{d}}(t))$. For a more extensive overview on the arithmetical properties of Euler's infinite product $\pi_q(t)$, see e.g. [7].

The next theorem is inspired by the work [8], where the authors investigated the distances between Fibonomials. Therefore we consider restricted approximations over the number field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, only. In the following, the notation $\mathbb{Z}_{\mathbb{K}}$ denotes the ring of integers of $\mathbb{K}$ and $\overline{A} := a - b\sqrt{5}$ denotes the field conjugate of $A = a + b\sqrt{5} \in \mathbb{K}$.

Theorem 3 *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, $q = (1-\sqrt{5})/(1+\sqrt{5})$, $\alpha = (1+\sqrt{5})/2$ *and* $s \in \mathbb{Z}_+$. *Let* $M \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$ *be such that* $\left|\overline{M}\right| \leq |M|$. *Then*

$$\left|\pi_q(1) - \frac{M}{\alpha^s}\right| > \frac{1}{2\alpha^{s(2+\varepsilon_3(s))}}, \quad \varepsilon_3(s) = \frac{17 + 2\sqrt{1+24s}}{s}. \tag{3}$$

The lower bound in (3) is an improvement to the result proved in [8], where the corresponding approximation exponent is $s(3 + \varepsilon(s))$ and $M = (\sqrt{5})^l$, $l \in \mathbb{Z}_+$. There are more general approximation results for Euler's infinite product and related q-series over number fields, see e.g. [9]. The results in [9] imply that there exist positive constants Γ and H_0 such that

$$\left|\pi_q(1) - \frac{M}{N}\right| > \frac{1}{H^{14/3+\varepsilon(H)}}, \quad \varepsilon(H) = \frac{\Gamma}{\sqrt{\log H}},$$

for all $M/N \in \mathbb{Q}(\sqrt{5})$, where $q = (1-\sqrt{5})/(1+\sqrt{5})$, $M, N \in \mathbb{Z}_{\mathbb{Q}(\sqrt{5})}$, $N \neq 0$ and $H = \max\{|M|, |N|, |\overline{M}|, |\overline{N}|\} \geq H_0$. On the other hand, in the Remark section of this paper we prove that for all $\tau \in \mathbb{R} \setminus \mathbb{Q}(\sqrt{5})$ there exists an infinite sequence of fractions $M/N \in \mathbb{Q}(\sqrt{5})$, where $M, N \in \mathbb{Z}_{\mathbb{Q}(\sqrt{5})}$ and $N \neq 0$, such that

$$\left|\tau - \frac{M}{N}\right| < \frac{9+4\sqrt{5}}{|N|^4}.$$

2 Proof of Theorem 1

We suppose that $N \in \mathbb{Z}_+$. By Jacobi's triple product identity (1) we have

$$\Pi_{\frac{1}{d}}(t) = \sum_{n=-\infty}^{\infty} d^{-n^2} t^n.$$

Because $t = a/b$, we obtain that

$$(ab)^N d^{N^2} \Pi_{\frac{1}{d}}(t) = A_N(d,t) + R_N(d,t), \tag{4}$$

where

$$A_N(d,t) = (ab)^N d^{N^2} + (ab)^N \sum_{n=1}^{N} d^{N^2-n^2}\left(t^n + \frac{1}{t^n}\right) \in \mathbb{Z}$$

and

$$R_N(d,t) = (ab)^N \sum_{n=N+1}^{\infty} \frac{t^n + \frac{1}{t^n}}{d^{n^2-N^2}} \in \mathbb{Q}\left[\left[t, \frac{1}{t}, \frac{1}{d}\right]\right].$$

We write $n = N+1+k$. Since $n^2 - N^2 \geq 2N+1+(2N+3)k$ for all $k \in \mathbb{Z}_{\geq 0}$, we get that

$$\begin{aligned} |R_N(d,t)| &\leq \frac{|a|^{2N+1}}{|b||d|^{2N+1}} \sum_{k=0}^{\infty}\left(\frac{|a|}{|b||d|^{2N+3}}\right)^k + \frac{|b|^{2N+1}}{|a||d|^{2N+1}} \sum_{k=0}^{\infty}\left(\frac{|b|}{|a||d|^{2N+3}}\right)^k \\ &= \frac{|a|^{2N+1}}{|b||d|^{2N+1} - |a|/|d|^2} + \frac{|b|^{2N+1}}{|a||d|^{2N+1} - |b|/|d|^2}. \end{aligned}$$

Further, our assumption $\max\{|a|, |b|\} + 1 \leq |d|$ implies that

$$\begin{aligned} |R_N(d,t)| &\leq \frac{|a|^{2N+1}}{(|a|+1)^{2N+1} - 1} + \frac{|b|^{2N+1}}{(|b|+1)^{2N+1} - 1} \\ &\leq \frac{|a|^{2N+1}}{|a|^{2N+1} + (2N+1)|a|^{2N}} + \frac{|b|^{2N+1}}{|b|^{2N+1} + (2N+1)|b|^{2N}}. \end{aligned}$$

We choose $\hat{N} := (3\max\{|a|, |b|\} - 1)/2$. Then

$$|R_N(d,t)| \le \frac{1}{2} \tag{5}$$

for all $N \ge \hat{N}$.

Let us denote

$$\Lambda := \Pi_{\frac{1}{d}}(t) - \frac{M}{d^s}.$$

By using (4) we obtain that

$$(ab)^N d^{N^2} \Lambda = A_N(d,t) - (ab)^N M d^{N^2-s} + R_N(d,t),$$

where the main term

$$\Delta_N(t) := A_N(d,t) - (ab)^N M d^{N^2-s}$$

is a rational integer, assuming that $N \ge \sqrt{s}$. Because the determinant

$$\begin{aligned}
&\begin{vmatrix} A_N(d,t) & (ab)^N d^{N^2} \\ A_{N+1}(d,t) & (ab)^{N+1} d^{(N+1)^2} \end{vmatrix} \\
&= (ab)^{N+1} d^{(N+1)^2} A_N(d,t) - (ab)^N d^{N^2} A_{N+1}(d,t) \\
&= (ab)^{2N+1} d^{N^2} d^{(N+1)^2} \left(\sum_{n=1}^{N} d^{-n^2} \left(t^n + \frac{1}{t^n} \right) - \sum_{n=1}^{N+1} d^{-n^2} \left(t^n + \frac{1}{t^n} \right) \right) \\
&= -(ab)^{2N+1} d^{N^2} \left(t^{N+1} + \frac{1}{t^{N+1}} \right) \ne 0,
\end{aligned}$$

we get that $\Delta_N(t) \ne 0$ or $\Delta_{N+1}(t) \ne 0$. We let now N be such that $\hat{N} \le \sqrt{s} \le N < \sqrt{s} + 2$ and $\Delta_N(t) \in \mathbb{Z} \setminus \{0\}$. Hence,

$$1 \le |\Delta_N(t)| = |(ab)^N d^{N^2} \Lambda - R_N(d,t)| \le |ab|^N |d|^{N^2} |\Lambda| + |R_N(d,t)|.$$

By (5) we get the approximation

$$1 \le 2|ab|^N |d|^{N^2} |\Lambda| < 2|d|^{s(1+6/\sqrt{s}+8/s)} |\Lambda|,$$

which completes the proof of Theorem 1.

3 Proof of Theorem 2

We suppose that $N \in \mathbb{Z}_+$. By Euler's pentagonal formula (2) we have

$$\pi_{1/d}(1) = 1 + \sum_{n=1}^{\infty}(-1)^n \left(d^{-n(3n-1)/2} + d^{-n(3n+1)/2}\right).$$

Hence, we can write

$$d^{N(3N+1)/2}\pi_{1/d}(1) = A_N(d) + R_N(d), \tag{6}$$

where

$$A_N(d) = d^{N(3N+1)/2} + \sum_{n=1}^{N}(-1)^n \left(d^{N(3N+1)/2-n(3n-1)/2} + d^{N(3N+1)/2-n(3n+1)/2}\right) \in \mathbb{Z}[d],$$

and

$$R_N(d) = \sum_{n=N+1}^{\infty} (-1)^n \left(\frac{1}{d^{n(3n-1)/2-N(3N+1)/2}} + \frac{1}{d^{n(3n+1)/2-N(3N+1)/2}}\right) \in \mathbb{Z}[[1/d]].$$

By noting that $(N+1)(3(N+1)-1)/2 - N(3N+1)/2 = 2N+1$ we deduce that

$$|R_N(d)| \leq \frac{1}{|d|^{2N+1}} \sum_{n=0}^{\infty} \frac{1}{|d|^n} \leq \frac{2}{|d|^{2N+1}} \leq \frac{1}{4}. \tag{7}$$

Let us denote

$$\Lambda := \pi_{1/d}(1) - \frac{M}{d^s}, \tag{8}$$

where $M \in \mathbb{Z}$, $s \in \mathbb{Z}_+$. By using (6) and (8), we get that

$$d^{N(3N+1)/2}\Lambda = A_N(d) - Md^{N(3N+1)/2-s} + R_N(d),$$

where the term

$$\Delta_N := A_N(d) - Md^{N(3N+1)/2-s}$$

is an integer if $N(3N+1)/2 - s > 0$. Additionally,

$$d \nmid \Delta_N = (-1)^N + (-1)^N d^N + \cdots + d^{N(3N+1)/2} - Md^{N(3N+1)/2-s}.$$

Hence, $\Delta_N \neq 0$ and further

$$1 \leq |\Delta_N| = |d^{N(3N+1)/2}\Lambda - R_N(d)| \leq |d|^{N(3N+1)/2}|\Lambda| + |R_N(d)|.$$

By (7), we obtain that

$$1 < 2|\Lambda||d|^{N(3N+1)/2}.$$

In particular, this lower bound holds when N is such that

$$(N-1)(3N-2)/2 \leq s < N(3N+1)/2.$$

In this case,

$$\begin{aligned} N(3N+1)/2 &= (N-1)(3N-2)/2 + 3N - 1 \\ &\leq s\left(1 + \frac{3+\sqrt{1+24s}}{2s}\right), \end{aligned}$$

and we obtain Theorem 2.

4 Proof of Theorem 3

We suppose that $N \in \mathbb{Z}_+$. By (2), we have

$$\begin{aligned} \pi_q(1) &= 1 + \sum_{n=1}^{\infty}(-1)^n\left(\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n(3n-1)/2} + \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n(3n+1)/2}\right) \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+n(3n-1)/2}}{\alpha^{n(3n-1)}} + \frac{(-1)^{n+n(3n+1)/2}}{\alpha^{n(3n+1)}}\right). \end{aligned}$$

Hence, we can write

$$\alpha^{N(3N+1)}\pi_q = A_N(\alpha) + R_N(\alpha), \tag{9}$$

where

$$\begin{aligned} A_N(\alpha) = \alpha^{N(3N+1)} + \sum_{n=1}^{N}((-1)^{n+n(3n-1)/2}\alpha^{N(3N+1)-n(3n-1)} \\ +(-1)^{n+n(3n+1)/2}\alpha^{N(3N+1)-n(3n+1)}) \in \mathbb{Z}[\alpha], \end{aligned} \tag{10}$$

and

$$R_N(\alpha) = \sum_{n=N+1}^{\infty} \left(\frac{(-1)^{n+n(3n-1)/2}}{\alpha^{n(3n-1)-N(3N+1)}} + \frac{(-1)^{n+n(3n+1)/2}}{\alpha^{n(3n+1)-N(3N+1)}} \right) \in \mathbb{Z}[[1/\alpha]].$$

By noting that $(N+1)(3(N+1)-1) - N(3N+1) = 4N+2$, we deduce that

$$|R_N(\alpha)| \leq \frac{1}{\alpha^{4N+2}} \sum_{n=0}^{\infty} \frac{1}{\alpha^n} = \frac{1}{\alpha^{4N+1}(\alpha-1)} = \frac{1}{\alpha^{4N}}. \tag{11}$$

We denote

$$\Lambda := \pi_q(1) - \frac{M}{\alpha^s}. \tag{12}$$

From Eqs. (9) and (12) we get that

$$\alpha^{N(3N+1)}\Lambda = A_N(\alpha) - M\alpha^{N(3N+1)-s} + R_N(\alpha).$$

Because $A_N(\alpha) \in \mathbb{Z}[\alpha]$ and $M, \alpha, 1/\alpha \in \mathbb{Z}_{\mathbb{K}}$, we have

$$\Delta_N := A_N(\alpha) - M\alpha^{N(3N+1)-s} \in \mathbb{Z}_{\mathbb{K}}.$$

Since the determinant

$$\begin{aligned} \begin{vmatrix} A_N(\alpha) & \alpha^{N(3N+1)} \\ A_{N+1}(\alpha) & \alpha^{(N+1)(3(N+1)+1)} \end{vmatrix} &= \alpha^{(N+1)(3(N+1)+1)} A_N(\alpha) - \alpha^{N(3N+1)} A_{N+1}(\alpha) \\ &= \alpha^{N(3N+1)}(-1)^{N+(N+1)(3N+2)/2} \left(\alpha^{2(N+1)} + (-1)^{N+1} \right) \end{aligned}$$

is non-zero. We have that $\Delta_N \neq 0$ or $\Delta_{N+1} \neq 0$. So, we can choose N such that $\Delta_N \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$. Hence, we have

$$1 \leq |N_{\mathbb{K}/\mathbb{Q}}(\Delta_N)| = |\Delta_N||\overline{\Delta_N}| = |\alpha^{N(3N+1)}\Lambda - R_N(\alpha)||\overline{\Delta_N}|. \tag{13}$$

Let us bound from above the absolute value of the conjugate $\overline{\Delta_N}$. First, we note that

$$\begin{aligned} |\overline{\Delta_N}| &= |\overline{A_N(\alpha) - M\alpha^{N(3N+1)-s}}| = |A_N(\overline{\alpha}) - \overline{M}\overline{\alpha}^{N(3N+1)-s}| \\ &\leq |A_N(\overline{\alpha})| + |\overline{M}||\overline{\alpha}|^{N(3N+1)-s}. \end{aligned}$$

By using (10), we get that

$$|A_N(\overline{\alpha})| < \sum_{n=0}^{N(3N+1)} |\overline{\alpha}|^n < \sum_{n=0}^{\infty} |\overline{\alpha}|^n = \frac{1}{1+\overline{\alpha}} = \alpha^2.$$

We can restrict the approximation to such numbers M/α^s that

$$\left|\pi_q(1) - \frac{M}{\alpha^s}\right| \le 1.$$

Because

$$\pi_q(1) \sim 1.226742\ldots,$$

it is enough to consider numbers M/α^s satisfying

$$0 < M/\alpha^s \le 2.226742\ldots < \alpha^2.$$

Now we suppose that $N(3N+1) \ge 2s$. Since $|\overline{M}| \le |M|$, by our assumption, we obtain that

$$|\overline{M}||\overline{\alpha}|^{N(3N+1)-s} \le |M|\left|\frac{-1}{\alpha}\right|^{N(3N+1)-s} = |M|\alpha^{s-N(3N+1)} \le \frac{M}{\alpha^s} < \alpha^2.$$

Hence,

$$|\overline{\Delta_N}| < 2\alpha^2. \tag{14}$$

Inequalities (13) and (14) imply now that

$$\frac{1}{2\alpha^2} < \alpha^{N(3N+1)}|\Lambda| + |R_N(\alpha)|.$$

By (11), we have

$$|R_N(\alpha)| \le \frac{1}{\alpha^4},$$

which implies

$$1 < 2|\Lambda|\alpha^5\alpha^{N(3N+1)}.$$

We fix an integer $\hat{N}$ such that $\hat{N}(3\hat{N}+1) < 2s \le (\hat{N}+1)(3(\hat{N}+1)+1)$. We can now suppose that N is $\hat{N}+1$ or $\hat{N}+2$. Hence,

$$\begin{aligned} N(3N+1) &\le (\hat{N}+2)(3(\hat{N}+2)+1) = \hat{N}(3\hat{N}+1) + 12\hat{N} + 14 \\ &< s\left(2 + \frac{12 + 2\sqrt{1+24s}}{s}\right), \end{aligned}$$

and we obtain Theorem 3.

5 Remark

When the approximations $M/N \in \mathbb{Q}(\sqrt{5})$ are not restricted, then there are better approximations for general $\tau \in \mathbb{R}\backslash\mathbb{Q}(\sqrt{5})$. The ring of integers $\mathbb{Z}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{5}}{2}$. We call the fraction

$$\frac{a+b\omega}{c+d\omega}, \quad a, b, c, d \in \mathbb{Z}$$

primitive whenever the vector (a, b, c, d) is primitive, meaning $\gcd(a, b, c, d) = 1$.

Lemma 1 *Let $\tau \in \mathbb{R} \setminus \mathbb{Q}(\sqrt{5})$. Then there exists an infinite sequence of primitive fractions $M/N \in \mathbb{Q}(\sqrt{5})$, where $M, N \in \mathbb{Z}_{\mathbb{Q}(\sqrt{5})}$, $N \neq 0$, such that*

$$\left|\tau - \frac{M}{N}\right| < \frac{9+4\sqrt{5}}{|N|^4}. \tag{15}$$

Proof Let $\tau \in \mathbb{R} \setminus \mathbb{Q}(\sqrt{5})$. Because $1, \omega, \tau$ and $\tau\omega$ are linearly independent over $\mathbb{Q}$, there exists an infinite sequence of primitive integer 4–tuples $(a, b, c, d) \in \mathbb{Z}^4 \setminus \{(0, 0, 0, 0)\}$ such that

$$|a + b\omega + c\tau + d\tau\omega| < \frac{1}{H^3}, \tag{16}$$

where $H = \max\{|b|, |c|, |d|\} \geq 1$ (see Corollary 1D in [11, p. 27]). If $c = d = 0$, then $H = |b| \geq 1$ and

$$0 \neq |a + b\omega| < \frac{1}{|b|^3}.$$

Thus,

$$1 \leq |(a + b\omega)(a + b\overline{\omega})| < \frac{|a + b\overline{\omega}|}{|b|^3} \leq \frac{\left|\sqrt{5}\right|}{|b|^2} + \frac{1}{|b|^6}$$

implying $|b| = 1$ and so $|a| \leq 2$, contradicting the fact that there are infinitely many (a, b, c, d). Hence there exists an infinite sequence of integer 4–tuples $(a, b, c, d) \in \mathbb{Z}^4 \setminus \{(0, 0, 0, 0)\}$ satisfying (16) with $(c, d) \neq (0, 0)$. We also note that $|c + d\omega| \leq (1 + \omega)H$. Consequently,

$$\left|\tau - \frac{a+b\omega}{c+d\omega}\right| < \frac{1}{H^3|c+d\omega|} \leq \frac{(1+\omega)^3}{|c+d\omega|^4} = \frac{9+4\sqrt{5}}{|c+d\omega|^4},$$

which completes the proof . □

The above bound (15) is a variation of the fundamental result presented in e.g. [11, p. 253].

Funding Open Access funding provided by University of Oulu including Oulu University Hospital.

References

1. Amou, M., Bugeaud, Y.: Exponents of Diophantine approximation and expansions in integer bases. J. Lond. Math. Soc. **81**(2), 297–316 (2010)
2. Andrews, G.E., Askey, R., Roy, R.: Special functions, encyclopedia of mathematics and its applications, vol. 71. Cambridge University Press, Cambridge (1999)
3. Bennett, M.A., Bugeaud, Y.: Effective results for restricted rational approximation to quadratic irrationals. Acta Arith. **155**(3), 259–269 (2012)
4. Bundschuh, P.: Ein Satz über ganze Funktionen und Irrationalitätsaussagen (German). Invent. Math. **9**, 175–184 (1969/1970)
5. Bundschuh, P., Shiokawa, I.: A measure for the linear independence of certain numbers. Results Math. **7**(2), 130–144 (1984)
6. Dubickas, A.K.: Approximation of some logarithms of rational numbers by rational fractions of special form. Moscow Univ. Math. Bull. **45**(2), 45–47 (1990)
7. Leinonen, L., Leinonen, M., Matala-aho, T.: On approximation measures of q-exponential function. Int. J. Number Theory **12**, 287–303 (2016)
8. Luca, F., Marques, D., Stanica, P.: On the spacings between C-nomial coeffients. J. Number Theory **130**(1), 82–100 (2010)
9. Matala-aho, T., Väänänen, K.: On Diophantine approximations of mock theta functions of third order. Ramanujan J. **4**(1), 13–28 (2000)
10. Rivoal, T.: Convergents and irrationality measures of logarithms. Rev. Mat. Iberoam **23**(3), 931–952 (2007)
11. Schmidt, W.M.: Diophantine approximation. Lecture Notes in Mathematics, vol. 785. Springer, New York (1980)