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Approximation properties of primal discontinuous Petrov-Galerkin method on general quadrilateral meshes

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Dedicated to Professor Leszek Demkowicz on the occasion of his 70th birthday.

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ABSTRACT

We consider the approximation properties of primal discontinuous Petrov-Galerkin (DPG) method on quadrilateral meshes. We show how the previous convergence results as well as the Aubin-Nitsche type duality arguments can be extended to cover arbitrary convex quadrilateral elements with bilinear isomorphisms. The arguments are based on the approximation theory of quadrilateral vector finite element spaces associated to the numerical flux variable of the DPG approximation. The theoretical results are validated by a numerical experiment that features also a comparison between the primal DPG method and a conventional least squares finite element method with the same number of degrees of freedom.

1. Introduction

Minimum residual and least squares finite element method have become very popular in numerical analysis of partial differential equations during the previous decade. A major cornerstone was put in place by Demkowicz and Gopalakrishnan in 2009 by the invention of the discontinuous Petrov-Galerkin method with optimal test functions (DPG). The original publication [1] concerned the computation of non-standard test functions to provide stable approximations for the transport equation. However, the generalizability of the approach to any partial differential equation became evident soon as well as its possibilities for a posteriori error estimation [2].

Suitably designed DPG schemes have turned to be robust for various challenging problems in science and engineering that involve physical parameters that may have disproportionate influence on the accuracy of standard approximation schemes. Examples that have been addressed concern e.g. transport phenomena (Péclet number), wave propagation (frequency), structural mechanics (plate/shell thickness) and elasticity theory (Poisson ratio), see e.g. [3–7].

Another robustness issue that may concern finite element algorithms is sensitivity to mesh distortions. The present paper addresses this problem for DPG in a fundamental way by considering the approximation properties of the primal DPG method for the Poisson's equation introduced in [8] on general quadrilateral meshes. It is known that the accuracy of standard least squares methods can be very sensitive to mesh distortions away from rectangular or parallelogram shapes and convergence may even fail completely if the approximation degree is low [9,10].

We prove that, for the primal DPG method, the same convergence rates are obtained for general shape-regular quadrilateral meshes as for simplicial, or triangular meshes. The error estimates are derived in the canonical DPG trial space norm as well as in the L_2-norm for the primal field variable by using duality techniques established in [11–13].

The structure of the paper is as follows. The problem and its primal DPG formulation are presented in the next section. Section 3 is devoted to specification of the numerical DPG algorithm and its theoretical findings. In Section 4, we report computational results and comparison with a LS method that confirm the theoretical findings.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and consider the homogeneous Dirichlet problem for Poisson's equation,

$$-\nabla^2 u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

We denote by Ω_h a partitioning of Ω made of convex quadrilaterals K with diameters bounded by h such that $\overline{\Omega} = \cup_{K\in\Omega_h} \overline{K}$, and by $\partial\Omega_h$ the collection of all element boundaries ∂K in Ω_h.

The partitioning is assumed to be shape-regular in the usual sense. This means that each angle of each $K \in \Omega_h$ is assumed to be bounded away from 0 and π by an absolute, positive constant and the ratio of

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any two sides on K is assumed to be uniformly bounded. This implies an upper bound for the shape constant defined as $\sigma_K = h_K/\rho_k$ where $h_K = \text{diam}(K)$ and ρ_k is the smallest diameter of inscribed circles to the four possible triangles obtained by joining three vertices of K. These are typical assumptions in finite element theories, see e.g. [9,10,14,15], and guarantee feasible approximation properties for the finite element spaces to be defined in the next section.

Following Demkowicz & Gopalakrishnan [8], the primal discontinuous Petrov-Galerkin weak formulation associated to Ω_h is then to find $(u, \hat{q}_n) \in X$ such that

$$b((u,\hat{q}_n),v) \doteq (\nabla u, \nabla v)_{\Omega_h} - \langle \hat{q}_n, v\rangle_{\partial\Omega_h} = (f,v)_{\Omega_h} \quad \text{for all } v \in Y. \tag{2}$$

Here $(w,v)_{\Omega_h} = \sum_{K\in\Omega_h}(w,v)_K$, where $(\cdot,\cdot)_K$ stands for the element-wise $L_2(K)$-inner product of scalar- or vector-valued functions. The test space Y is defined as the non-standard "broken" Sobolev space

$$Y = H^1(\Omega_h) = \{v \in L_2(\Omega) \ : \ v\,|_K \in H^1(K)\ \forall K \in \Omega_h\}$$

with the norm

$$||v||_Y^2 = (\nabla v, \nabla v)_{\Omega_h} + (v,v)_{\Omega_h}. \tag{3}$$

Moreover, $\langle \hat{r}_n, v\rangle_{\partial\Omega_h} = \sum_{K\in\Omega_h}\langle \hat{r}_n, v\rangle_{1/2,\partial K}$, where $\langle\cdot,\cdot\rangle_{1/2,\partial K}$ is the duality pairing between $\hat{r}_n \in H^{-1/2}(\partial K)$ and $v\,|_{\partial K} \in H^{1/2}(\partial K)$, and the trial space X is defined as

$$X = H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h).$$

Here $H_0^1(\Omega)$ is the standard Sobolev space on Ω whereas the latter space can be defined as

$$H^{-1/2}(\partial\Omega_h) = \left\{ \eta \in \prod_{K\in\Omega_h} H^{-1/2}(\partial K) \ : \right.$$
$$\left. \text{there exists } q \in H(\text{div},\Omega) \text{ such that } \eta\,|_{\partial K} = q\cdot n\,|_{\partial K} \text{ for all } K \in \Omega_h \right\},$$

where n is the outward unit normal on ∂K and $H(\text{div},\Omega)$ consists of L_2 vector fields on Ω with divergence also in L_2. The space $H^{-1/2}(\partial\Omega_h)$ can then be normed by

$$||\hat{r}_n||_{H^{-1/2}(\partial\Omega_h)} = \inf\{||q||_{H(\text{div},\Omega)} \ : q \in H(\text{div},\Omega) \text{ s.t. } \hat{r}_n\,|_{\partial K} = q\cdot n\,|_{\partial K}\}$$

and the whole space X by

$$||(w,\hat{r}_n)||_X^2 = ||\nabla w||^2_{L_2(\Omega)} + ||\hat{r}_n||^2_{H^{-1/2}(\partial\Omega_h)}.$$

It was shown in [8] under more general conditions on the domain Ω and the partitioning Ω_h, that the bilinear form $b(\cdot,\cdot)$ is continuous,

$$b((w,\hat{r}_n),v) = 0 \quad \text{for all } v \in Y \quad \text{implies that} \quad (w,\hat{r}_n) = (0,0), \tag{4}$$

and that the inf-sup condition

$$\sup_{(w,\hat{r}_n)\in X} \frac{b((w,\hat{r}_n),v)}{||(w,\hat{r}_n)||_X} \geq C||v||_Y \quad \text{for all } v \in Y \tag{5}$$

holds. It follows from the Banach Closed Range theorem (see [16]) that then also the inf-sup condition

$$\sup_{v\in Y} \frac{b((w,\hat{r}_n),v)}{||v||_Y} \geq C||(w,\hat{r}_n)||_X \quad \text{for all } (w,\hat{r}_n) \in X \tag{6}$$

holds. Hence, by Babuška's theorem, the problem (2) is well-posed, for example, when $f \in L_2(\Omega)$. More generally, the right hand side of (2) can be substituted by a continuous linear functional in the dual space of Y, see [17,8,18,16].

3. The DPG approximation and convergence results

In this section we turn to numerical DPG approximation of the problem (2) and define the finite element trial space X_h associated to the assumed partitioning Ω_h as

$$X_h = \{(w,\hat{r}_n) \in X \ : \ w\,|_K \in V_k(K), \hat{r}_n \in P_{k-1}(\partial K)\},$$

where $k \geq 1$ is an integer. The space $P_k(\partial K)$ consists of functions on ∂K that are polynomial of degree k on each edge of K and $V_k(K)$ is the quadrilateral finite element space of degree k defined as

$$V_k(K) = \{w \in L_2(K), w = \hat{w}\circ F_K^{-1}, \hat{w} \in Q_k(\hat{K})\},$$

where $Q_k(\hat{K}) = P_{k,k}(\hat{K})$ denotes the space of polynomials of degree at most k in each variable separately and F_K is the bilinear bijection between the reference square $\hat{K} = [-1,1]^2$ and the general quadrilateral K.

Setting

$$Y^r = \{v \in Y \ : \ v\,|_K \in V_r(K)\},$$

the optimal DPG test space is given by $Y_h^r = T^r(X_h)$, where the trial-to-test mapping $T^r : X \to Y^r$ is defined as

$$(T^r(w,\hat{r}_n),v)_Y = b((w,\hat{r}_n),v) \quad \text{for all } v \in Y^r, \tag{7}$$

with $(\cdot,\cdot)_Y$ being the inner product corresponding to the norm (3).

The DPG approximation $(u_h,\hat{q}_{n,h}) \in X_h$ is then defined as the solution to the finite-dimensional problem

$$b((u_h,\hat{q}_{n,h}),v) = (f,v)_{\Omega_h} \quad \text{for all } v \in Y_h^r = T^r(X_h),$$

or, in other words, as the projection

$$(T^r(u_h,\hat{q}_{n,h}), T^r(w,\hat{r}_n))_Y = (T^r(u,\hat{q}_n), T^r(w,\hat{r}_n))_Y \quad \text{for all } (w,\hat{r}_n) \in X_h. \tag{8}$$

It has been verified in [19], that the there exists a linear operator $\Pi : Y \to Y^{k+2}$, which is uniformly bounded with respect to h, such that

$$(\nabla w_h, \nabla(v - \Pi v))_{\Omega_h} - \langle \hat{r}_n, v - \Pi v\rangle_{\partial\Omega_h} = 0$$

for all $v \in Y$ and $(w_h,\hat{r}_n) \in X_h$. This validates the discrete version of the inf-sup condition (6), where X and Y are replaced by X_h and Y_h^{k+2}, respectively. Babuska's theory then implies that

$$||u-u_h||_{H^1(\Omega)} + ||\hat{q}_n - \hat{q}_{n,h}||_{H^{-1/2}(\partial\Omega_h)} \leq C \min_{(w,\hat{r}_n)\in X_h} \Big(||u-w||_{H^1(\Omega)} + ||\hat{q}_n - \hat{r}_n||_{H^{-1/2}(\partial\Omega_h)} \Big), \tag{9}$$

cf. [17,16,8].

The best approximation property leads now to convergence rate estimates in terms of the mesh parameter h by applying the Bramble-Hilbert lemma [15]. To bound the approximation error of the numerical flux $\hat{q}_n$, it is critical to employ the approximation theory of quadrilateral $H(\text{div})$ finite elements developed by Arnold, Boffi and Falk in [10].

The theory is summarized in the following two theorems.

Theorem 1. *Let $(u_h,\hat{q}_{n,h}) \in X_h$ be the DPG approximation defined by (8) with degrees k and $r = k+2$ of the exact solution $(u,\hat{q}_n)$ of the problem (2). Then*

$$||u-u_h||_{H^1(\Omega)} + ||\hat{q}_n - \hat{q}_{n,h}||_{H^{-1/2}(\partial\Omega_h)} \leq Ch^s \left(|u|_{H^{s+1}(\Omega)} + |f|_{H^s(\Omega)} \right),$$

for $0 < s \leq k$.

Proof. The first term is bounded by standard approximation theory for quadrilateral elements. The second term can be bounded by noticing first that the normal components in the vector finite element space obtained via Piola transformation from the reference space

$$ABF_{k-1}(\hat{K}) = P_{k+1,k-1}(\hat{K}) \times P_{k-1,k+1}(\hat{K})$$

are polynomials of degree $k-1$ and therefore match with the numerical flux approximation in X_h. The error estimate follows by observing that $\hat{q}_n = \nabla u \cdot n$ and by employing the $H(\text{div})$-projection Π_h of the flux $q = \nabla u$ constructed in [10], which satisfies

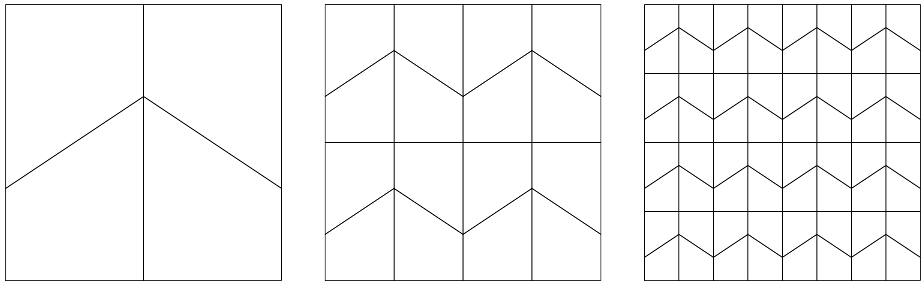

Fig. 1. Sequence of trapezoidal $2^N \times 2^N$ -meshes, $N = 1, 2, 3, \ldots$.

$$||q - \Pi_h q||_{H(\mathrm{div},\Omega)} \leq Ch^s \left(|q|_{H^s(\Omega)} + |\mathrm{div}\, q|_{H^s(\Omega)}\right) \tag{10}$$

for $0 < s \leq k$. □

Theorem 2. *If Ω is convex, then u and u_h defined in Theorem 1 satisfy*

$$||u - u_h||_{L_2(\Omega)} \leq Ch^{s+1} \left(|u|_{H^{s+1}(\Omega)} + |f|_{H^s(\Omega)}\right), \tag{11}$$

for $0 < s \leq k$.

Proof. Given $g \in L_2(\Omega)$, the duality problem for the primal DPG formulation of the Poisson's equation is to find $(z, \hat{z}_n) \in X$ and $d \in Y$ such that

$$(d, y)_Y + (\nabla z, \nabla y)_{\Omega_h} - \langle \hat{z}_n, y\rangle_{\partial\Omega_h} = 0,$$

$$(\nabla d, \nabla w)_{\Omega_h} = (g, w)_{\Omega_h},$$

$$\langle \hat{w}_n, d\rangle_{\partial\Omega_h} = 0$$

for all $y \in Y$, $(w, \hat{w}_n) \in X$. It has been verified in [11], that this problem is well-posed, the solution satisfies

$$||z||_{H^2(\Omega)} + ||d||_{H^2(\Omega)} \leq C||g||_{L_2(\Omega)}, \tag{12}$$

and that the relations

$$\begin{cases} -\Delta d = g, & \text{on } \Omega, \\ \Delta z = d + g, & \text{on } \Omega, \\ \hat{z}_n = \nabla(d+z)\cdot n, & \text{on } \partial K \text{ for all } K \in \Omega_h, \end{cases} \tag{13}$$

hold in the classical form.

Moreover, the duality calculations in [11] (cf. also [13,20,12]) reveal that we have

$$||u - u_h||_{L_2(\Omega)} \leq Ch \left(||u - u_h||_{H^1(\Omega)} + ||\hat{q}_n - \hat{q}_{n,h}||_{H^{-1/2}(\partial\Omega_h)}\right) \tag{14}$$

provided that

$$\min_{((w_h,\hat{w}_h),v_h)\in X_h\times V^r} \left(||z - w_h||_{H^1(\Omega)} + ||\hat{z}_n - \hat{w}_h||_{H^{-1/2}(\partial\Omega_h)} + ||d - v_h||_{H^1(\Omega_h)}\right) \leq Ch||g||_{L_2(\Omega)}. \tag{15}$$

The first and third term are bounded by the right hand side by standard approximation theory and (12). The bound for the second term can be established by using the last equation of (13) and utilizing the projector Π_h to the vector finite element space ABF_{k-1} element-wise. According to [10], there exists an operator $\Lambda_K : L_2(K) \to L_2(K)$ such that $\mathrm{div}(\Pi_K q) = \Lambda_K(\mathrm{div}\, q)$ and

$$||d - \Lambda_K d||_{L_2(K)} \leq Ch_K |d|_{H^1(K)}. \tag{16}$$

Observing from the first two equations of (13) that $\Delta(d+z) = d$, and using (16) and (10) locally, we get

$$||(I - \Pi_K)\nabla(d+z)||_{H(\mathrm{div},K)}$$

$$\leq ||(I - \Pi_K)\nabla(d+z)||_{L_2(K)} + ||(I - \Lambda_K)d||_{L_2(K)}$$

$$\leq Ch_K|d+z|_{H^2(K)} + Ch_K|d|_{H^1(K)}.$$

Estimate (15) is then established by squaring and summing over the elements and using (12). Finally, combination of (14) and Theorem 1 yields (11). □

4. Numerical example

In this section, we verify numerically the convergence rate predicted by (11) by solving the Poisson's equation on $\Omega = (0,1)^2$ with f and Dirichlet boundary conditions set so that the solution is $u(x_1, x_2) = \cos(\pi x_1)\cos(\pi x_2)$. We evaluate the error in the $L_2(\Omega)$-norm of the primal variable and compute

$$e = \frac{||u - u_h||_{L_2(\Omega)}}{||u||_{L_2(\Omega)}}$$

on the sequence of trapezoidal $2^N \times 2^N$-meshes visualized in Fig. 1.

We compare the performance of the primal DPG method (DPG-FEM) to the standard least squares formulation (LS-FEM) that minimizes

$$F(w, r) = ||r + \nabla w||^2_{L_2(\Omega)} + ||\mathrm{div}\, r - f||^2_{L_2(\Omega)}$$

over $\tilde{X}_h = V_h \times W_h$, where V_h represents the first component of X_h and W_h stands for a $H(\mathrm{div},\Omega)$-conforming vector finite element space. It is known (see [21]) that the minimizer $(u_h, q_h) \in \tilde{X}_h$ satisfies

$$||u - u_h||_{H^1(\Omega)} + ||q - q_h||_{H(\mathrm{div},\Omega)}$$

$$\leq C \min_{(v,\tau)\in\tilde{X}_h} \left(||u - v||_{H^1(\Omega)} + ||q - \tau||_{H(\mathrm{div},\Omega)}\right),$$

and if we use the aforementioned ABF space to construct W_h, we obtain the same convergence rates for LS-FEM as for DPG-FEM.

However, if W_h corresponds to the Raviart-Thomas space of degree $k-1$, so that X_h and $\tilde{X}_h$ have the same number of degrees of freedom, then the best known estimate for general shape-regular convex quadrilaterals for the order of approximation of $\mathrm{div}\, q$ is only $k-1$. Therefore, for $k = 1$, there is no convergence in $H(\mathrm{div},\Omega)$ in general, and it is known that the issue may propagate also to the scalar variable u [10]. This is reflected in Fig. 2 which displays a comparison between the convergence rate of DPG-FEM and LS-FEM in $L_2(\Omega)$-norm for $k = 1$. Clearly, for DPG-FEM, e behaves as predicted by Theorem 2, whereas the accuracy of LS-FEM suffers because of the non-affine mesh sequence.

Data availability

Data will be made available on request.

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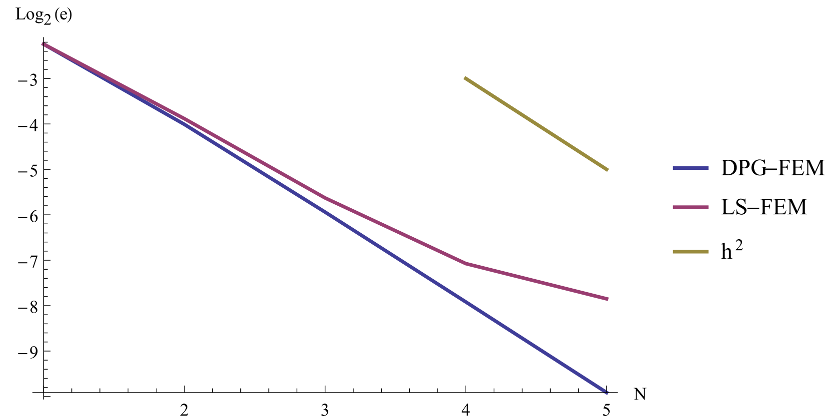

Fig. 2. Convergence rates in $L_2(\Omega)$-norm of primal discontinuous Petrov-Galerkin method (DPG-FEM) and least squares finite element method (LS-FEM) on trapezoidal mesh sequence with $h \sim 1/2^N$.

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