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FROM DYNAMICS TO GEOMETRY ON SELF-AFFINE SETS AND MEASURES

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ALEKSI PYÖRÄLÄ

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Abstract

This dissertation concerns the geometry of self-similar, self-affine and self-conformal sets and measures in Euclidean spaces. The aim is to determine whether certain geometric and arithmetic properties of such objects are completely or partially determined by algebraic or dynamical properties of the defining iterated function systems. Three research objectives are formulated in order to approach this theme from different perspectives. The results of this dissertation are based on the four research articles I–IV.

The first objective is to obtain more refined information on the size of self-similar sets: Which self-similar sets are tube-null, that is, can be covered with tubular neighbourhoods of affine hyperplanes of arbitrarily small total volume? We make progress towards this objective by providing a large class of new non-trivial self-similar sets that are tube-null, including the classical Sierpinski carpet and the Menger sponge. Previously, only a few such examples were known in the literature.

The second objective is to provide comprehensive descriptions of the local structures of self-affine and self-conformal measures. We make progress by describing the local sceneries of irreducible planar self-affine measures with weaker assumptions than those present in the existing literature, and obtain a complete description for the local statistics of self-conformal measures on the line without assuming any separation conditions from the defining iterated function systems.

The third objective is to show that, in general, there is no geometric resonance between two self-affine or two self-conformal measures, that is, their convolution has the largest possible Hausdorff dimension. It is conjectured that there can be geometric resonance between such measures only if there is arithmetic resonance between the defining iterated function systems. The results of this thesis verify this principle for arbitrary self-conformal measures on the line without any separation conditions, and for a large class of self-affine measures on the plane. The result for self-affine measures provides what seems to be the first evidence of this phenomenon in higher dimensions.

Keywords: iterated function systems, resonance between measures, scenery flow, tube-null sets

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Tiivistelmä

Fraktaaleiksi kutsutaan yleisesti sellaisia olioita, joiden keskinäinen vertailu ei onnistu perinteisen geometrian keinoin. Ominainen piirre fraktaalille on esimerkiksi uusien yksityiskohtien esiintyminen mikroskooppisen pienilläkin mittakaavoilla. Vaikka teoreettisia esimerkkejä fraktaaleista on tutkittu jo pitkään, sovellukset matematiikan ulkopuolelle ovat suhteellisen tuoreita: *Fraktaaligeometrian* menetelmiä käytetään poikkeuksellisen epäsäännöllisten olioiden mallintamiseen ja tutkimiseen, kuten osakemarkkinoiden vaihteluiden ennustamiseen ja syöpäsolujen erottamiseen terveistä.

Matemaattisesti fraktaalin voi usein mallintaa itsesimilaarina, -affiinina tai -konformina joukkona; esimerkkejä tällaisista on kuvattu luvun 1 kuvassa 1. Tämän väitöskirjatutkimuksen tarkoituksena on löytää uusia perustavanlaatuisia eroja tällaisten fraktaalien rakenteissa, sekä osoittaa, että yleisesti nämä rakenteet eivät tuhoudu, vaikka fraktaalia altistaisi vääristymille. Näiden teemojen lähestymiseksi eri näkökulmista muotoillaan kolme tutkimustavoitetta, joihin tämä väitöskirjatutkimus pyrkii vastaamaan tutkimusartikkelien I–IV tulosten pohjalta.

Ensimmäinen tavoite on vertailla itsesimilaarien joukkojen kokoa aiempaa tarkemmin: Mitkä itsesimilaarit joukot ovat *tuubiohuita* (tube-null), eli tehokkaasti peitettävissä suorien ympäristöillä? Väitöskirja esittää tähän kysymykseen osittaisen vastauksen esittelemällä suuren määrän uusia, epätriviaaleja tuubiohuita itsesimilaareja joukkoja, kuten Sierpinskin matto tasossa.

Toinen tavoite on kuvata tyhjentävästi itseaffiinien ja -konformien mittojen paikallista rakennetta. Väitöskirjan tulokset vastaavat tähän tavoitteeseen kuvaamalla tason itseaffiinien mittojen paikallista rakennetta aiempaa yleisemmillä säännöllisyysoletuksilla ja tarjoamalla tyhjentävän kuvauksen kaikkien itsekonformien mittojen paikallisesta rakenteesta.

Kolmas tavoite on osoittaa, että itseaffiinien ja -konformien mittojen välillä ei yleisesti ole geometrista resonanssia. Yleisesti otaksutaan, että resonanssia kyseisten mittojen välillä voi olla vain, mikäli ne määrittävät iteroidut funktiosysteemit resonoivat aritmeettisesti. Väitöskirjan tulokset vahvistavat tämän periaatteen kaikille suoran itsekonformeille ja suurelle joukolle tason itseaffiineja mittoja.

Asiasanat: dynaamiset systeemit, fraktaaligeometria, iteroidut funktiosysteemit

To Toivo

Acknowledgements

This thesis would certainly not exist if not for the many people who have guided, supported and encouraged me to keep going forward through all these years.

First of all, I thank my thesis supervisors, Professors Ville Suomala and Meng Wu, for introducing me to this exciting world that has occupied a significant part of my life for the past four years. Not only have I been able to rely on your reassuring guidance whenever I felt I needed it, in the relaxed atmosphere you have upheld it has been easy for me to take my time digging into different topics of my own choice and find out which ones I find the most interesting.

I am deeply grateful to Professor Ariel Rapaport for agreeing to act as the opponent in the public defense of this thesis, and to the pre-examiners, Dr. Natalia Jurga and Dr. Tom Kempton, who spared their time to carefully read through the manuscript of this thesis. It is at once exciting and slightly daunting to present my work to be reviewed by people whose research has been and continues to be such an inspiration for myself.

A substantial part of the results included in this thesis were obtained in collaboration with Professors Balázs Bárány, Antti Käenmäki and Pablo Shmerkin. Having been able to discuss and exchange ideas with such experts has had an irreplacable role in developing my own view of the world of math.

I also thank all the people of the research unit here at Oulu, and all my friends and colleagues with whom I have spent any amount of time in the same office. Having people around me to whom I can complain about all the difficulties of this work makes them feel a lot less difficult. In particular, I thank my colleagues and friends Markus Myllyoja and Roope Anttila, for all the long lunches and coffee breaks talking about work and life or just hanging out and making the days not boring. Roope also deserves my thanks as the man behind the MATLAB functions I have used to draw most of the pictures in Chapters 1 and 3.

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Finally, and most importantly, I thank my dear Elisa, for your support and belief in me at every moment of this journey. You and Toivo form the bedrock atop which all of this is built.

List of original publications

This dissertation consists of an introductory part and the following four original articles. We refer to these articles by Roman numerals I–IV.

I Pyörälä, A., Shmerkin, P., Suomala, V. & Wu, M. (2020). Covering the Sierpiński carpet with tubes. Accepted for publication in Israel Journal of Mathematics, available at arXiv:2006.00499

II Pyörälä, A. (2022). The scenery flow of self-similar measures with weak separation condition. *Ergodic Theory and Dynamical Systems*, 42(10), 3167–3190. http://doi.org/10.1017/etds.2021.86

III Pyörälä, A. (2023). Resonance between planar self-affine measures. Submitted manuscript, available at arXiv:2302.05240

IV Bárány, B., Käenmäki, A., Pyörälä, A. & Wu, M. (2023). Scaling limits of self-conformal measures. Submitted manuscript, available at arXiv:2308.11399

Articles II and III are indepedendent work of the author. The contribution of the author in Articles I and IV is one quarter.

Contents

1 Introduction

For a long time, objects with highly irregular geometry, such as nowhere-differentiable functions or uncountable sets with zero length, were considered to be individual theoretical curiosities without much potential for application. Over the course of the 20th century, this attitude underwent a dramatic change as it was discovered that for a surprisingly high number of natural objects and phenomena, such as fluctuations in stock exchanges and the structure of cancer cells, the most accurate mathematical models are indeed ones exhibiting such irregularity. These discoveries, together with the development of computers capable of plotting detailed images of increasingly complex systems has caused an explosion of interest in *fractal geometry*—the study of geometrically irregular objects exhibiting non-trivial details at arbitrarily small scales.

A classical way to formulate a fractal mathematically is with *iterated function systems*: If $f_1, \dots, f_m$ are contractive functions on $\mathbb{R}^d$, a classical result of Hutchinson states that there exists a unique compact set K which satisfies

$$K = \bigcup_{i=1}^{m} f_i(K). \tag{1}$$

Iterating this equality, we see that in fact,

$$K = \bigcap_{n \in \mathbb{N}} \bigcup f_{i_1} \circ \cdots \circ f_{i_n}(K) \tag{2}$$

where the union is taken over all vectors $(i_1, \dots, i_n)$ of length n. Taking n arbitrarily large, we see that K exhibits distorted copies of itself at arbitrarily small scales. In fact, on the right-hand side of (2) the set K may be replaced by any compact set mapped into itself by all of the maps f_i, whence K is usually called the *attractor* of the *iterated function system* $\{f_1, \dots, f_m\}$.

Many natural shapes in the world exhibiting fractal-like properties can be modelled as an attractor of an iterated function system in the three-dimensional space $\mathbb{R}^3$. However, it is also possible for more abstract objects, such as probability distributions, to display fractal-like qualities. Such probability distributions can sometimes be modelled mathematically as measures μ which satisfy

$$\mu = \sum_{i=1}^{m} p_i \cdot \mu \circ f_i^{-1} \tag{3}$$

for some probability vector $(p_1, \dots, p_m)$. A measure that satisfies (3) is called *stationary* and is supported on the attractor of the iterated function system $\{f_1, \dots, f_m\}$.

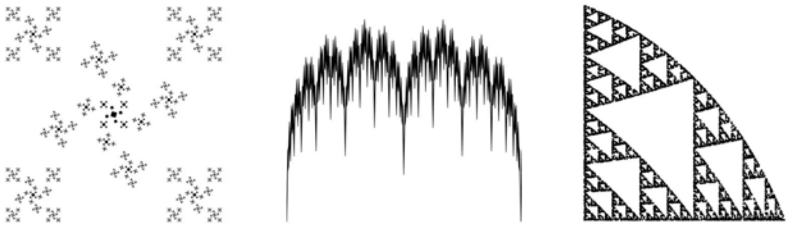

Fig. 1. From left to right: A self-similar, a self-affine, and a self-conformal set in $\mathbb{R}^2$.

Iterated function systems can be divided into different classes by imposing different regularity assumptions on the contractions f_i.

Definition 1.0.1. *Let Γ be a finite set with $\#\Gamma \geq 2$, and let $f_i : \mathbb{R}^d \to \mathbb{R}^d$ be contracting functions for $i \in \Gamma$. The iterated function system $\{f_i\}_{i\in\Gamma}$, its attractor and any stationary measure are called*

a) self-similar *if for each $i \in \Gamma$, $f_i(x) = r_i O_i x + a_i$ for some $0 < r_i < 1$, orthogonal matrix O_i and $a_i \in \mathbb{R}^d$,*

b) self-affine *if for each $i \in \Gamma$, $f_i(x) = A_i x + a_i$ for some real matrix A_i with $0 < \|A_i\| < 1$ and $a_i \in \mathbb{R}^d$,*

c) self-conformal *if there exists a bounded, convex open set U such that for each $i \in \Gamma$, $f_i : U \to \mathbb{R}^d$ is injective and differentiable, its differentiation $x \mapsto Df_i|_x$ is Hölder continuous, $\sup_{x\in U} \|Df_i|_x\| < 1$, $Df_i|_x \neq 0$ for each $x \in U$, and $\|Df_i|_x\|^{-1} Df_i|_x$ is orthogonal for each $x \in U$.*

See Figure 1. The purpose of this thesis is to study various geometric properties of attractors and stationary measures associated to iterated function systems as in Definition 1.0.1. Because of the highly irregular structure of such objects, many questions that are trivial or well-known for objects of classical geometry turn out to be subtle and deep in the context of fractals.

1.1 Research questions

In the following sections we formulate three research questions, each of which contributes to the following general topic:

To what extent is the geometry of attractors and stationary measures on $\mathbb{R}^d$ determined by algebraic and dynamical properties of the defining iterated function systems?

Perhaps the most classical question on this topic is the following: Which properties of an iterated function system determine the 'size' of an attractor as a subset of $\mathbb{R}^d$?

The size of a fractal can be measured in a multitude of different ways, and comparison between different notions of size is a way to measure regularity. Perhaps the most classical of the notions is the *Hausdorff dimension*.

Definition 1.1.1. *For $s \geq 0$, the s-*dimensional Hausdorff measure *$\mathscr{H}^s$ on $\mathbb{R}^d$ is given by*

$$\mathscr{H}^s(A) = \liminf_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} diam(U_i)^s : \begin{array}{l} U_1, U_2, \ldots \text{ are sets in } \mathbb{R}^d \text{ with} \\ diam(U_i) \leq \delta \text{ and } A \subseteq \bigcup_{i=1}^{\infty} U_i \end{array} \right\}$$

for any $A \subseteq \mathbb{R}^d$. The Hausdorff dimension *of A is defined as*

$$\dim_{\mathrm{H}} A = \inf\{s \geq 0 : \mathscr{H}^s(A) = 0\} = \sup\{s \geq 0 : \mathscr{H}^s(A) = \infty\}.$$

The Hausdorff dimension of a Borel probability measure μ is defined as

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : A \text{ is a Borel set with } \mu(A) > 0\}.$$

1.1.1 Tube-null sets

The Hausdorff dimension of self-similar sets is an active research topic in fractal geometry, and under suitable separation conditions it is completely understood. One of the objectives of this dissertation is to measure the size of self-similar sets with a notion that was relatively recently introduced by Carbery, Soria, and Vargas [7]. In the following, a *tube* T of *width* $w(T)$ is a closed $\frac{w(T)}{2}$-neighbourhood of an affine hyperplane.

Definition 1.1.2. *A set $E \subseteq \mathbb{R}^d$ is* tube-null *if for any $\varepsilon > 0$ there exists a countable family of tubes, $\{T_i\}_{i \in \mathbb{N}}$, such that $E \subseteq \bigcup_{i \in \mathbb{N}} T_i$ and*

$$\sum_{i \in \mathbb{N}} w(T_i) < \varepsilon.$$

The notion was first introduced by Carbery et. al. in a harmonic analytic setting: A tube-null set is small enough to support a function that cannot be recovered from its pointwise Fourier means. Such sets are called *sets of divergence*. Not many non-trivial examples of such sets are known, and it is a challenging open problem to find a characterisation for them in any dimension greater than one.

Research Question 1. *Which self-similar sets in $\mathbb{R}^d$ are tube-null?*

It is often a subtle question to verify whether a given set is tube-null or not. While subsets of $\mathbb{R}^d$ with finite $\mathscr{H}^{d-1}$-measure are always tube-null [7], few examples with

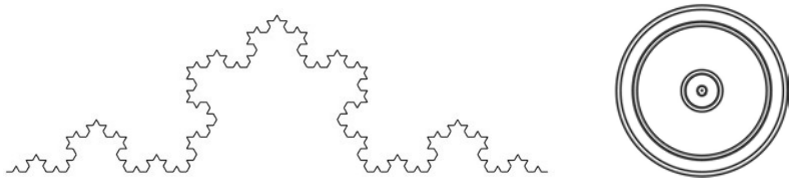

Fig. 2. Approximations of the self-similar Von Koch snowflake curve and a λ-Cantor target for λ slightly below $\frac{1}{4}$. These sets are tube-null.

Hausdorff dimension in the interval $(d-1,d]$ are known. One remarkable example is the Von Koch curve, a self-similar set of Hausdorff dimension $\frac{\log 4}{\log 3}$ shown to be tube-null by Harangi [17].

Theorem 1.1.3 (Harangi)**.** *The Von Koch curve in* $\mathbb{R}^2$ *is tube-null.*

Another non-trivial example of a tube-null set is the *Cantor target* of small enough dimension [7]. What makes this example particularly interesting is that by varying the parameter, the model can be used to produce both tube-null and non tube-null sets.

Theorem 1.1.4 (Carbery et. al.)**.** *For* $0 \leq \lambda \leq 1$*, let* C_λ *denote the attractor of the iterated function system*

$$\{x \mapsto \lambda x,\ x \mapsto \lambda x + 1 - \lambda\}$$

on $\mathbb{R}$*. The* λ-Cantor target $\{x \in \mathbb{R}^2 :\ \|x\| \in C_\lambda\}$ *is tube-null for* $\lambda \in [0, \frac{1}{4})$*, and not tube-null for* $\lambda \in (\frac{1}{4}, 1]$.

See Figure 2. Given that sets with Hausdorff dimension strictly less than $d-1$ are always tube-null, it is natural to ask whether this bound is sharp. This question was answered affirmatively by Shmerkin and Suomala [28] using a random construction.

Theorem 1.1.5 (Shmerkin & Suomala)**.** *There exists a non tube-null set* $A \subset \mathbb{R}^d$ *with Hausdorff and box counting dimension* $d-1$.

1.1.2 Local structure of measures

Taking tangents and obtaining geometric information on an object by studying its small-scale structure is a classical idea in mathematical analysis. The idea also has applications in geometric measure theory: For a Radon measure μ, a point x in its support and a number $t \geq 0$, let $\mu_{x,t}$ denote the *magnification* of μ at x by 2^{-t}:

$$\mu_{x,t}(A) = \frac{\mu((2^{-t}A + x) \cap B(x, 2^{-t}))}{\mu(B(x, 2^{-t}))}$$

for all Borel $A \subseteq B(0,1)$. Here and in the following we use $B(x,r)$ to denote the closed ball of radius r centered at x. The statistics of the one-parameter family $(\mu_{x,t})_{t\geq 0}$, called the *scenery* of μ at x, have substantial implications on the local geometry of μ at x. To capture these statistics, we study the family of *scenery distributions* of μ at x,

$$\left(\frac{1}{T}\int_0^T \delta_{\mu_{x,t}}\,dt\right)_{T\geq 1}.$$

We use δ_x to denote the Dirac measure at x. Accumulation points of this family in the weak*-topology are called *tangent distributions* of μ at x.

Definition 1.1.6. *A measure μ is called* uniformly scaling *if there exists a measure P on the space of Borel probability measures on $B(0,1)$ such that*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \delta_{\mu_{x,t}}\,dt = P$$

for μ-almost every x.

While the property of being uniformly scaling seems very restrictive, it turns out that many naturally arising, dynamically defined measures are indeed uniformly scaling, or at least admit a unique tangent distribution at almost every point.

Research Question 2. *Let μ be a self-affine or self-conformal measure on $\mathbb{R}^d$. Does μ admit a unique tangent distribution at almost every x?*

The first result towards the resolution of this question is due to Gavish [16] who answered this question in affirmative for self-similar measures with the open set condition (Definition 2.2.1). The result was later generalised for self-conformal measures with the same separation condition by Hochman and Shmerkin [22].

Theorem 1.1.7 (Hochman & Shmerkin)**.** *Any self-conformal measure on $\mathbb{R}$ satisfying the open set condition is uniformly scaling.*

For self-affine measures that are not self-similar, the state-of-the-art result is due to Ferguson, Fraser and Sahlsten [12].

Theorem 1.1.8 (Ferguson et. al.)**.** *Let μ be a self-affine measure on a Bedford-McMullen carpet on $\mathbb{R}^2$. Then μ is uniformly scaling.*

Self-affine measures based on irreducible systems (Definition 2.2.2) are not necessarily uniformly scaling as is indicated by the work of Kempton [23] (see also Figure 3). However, this does not rule out the existence of a unique tangent distribution almost

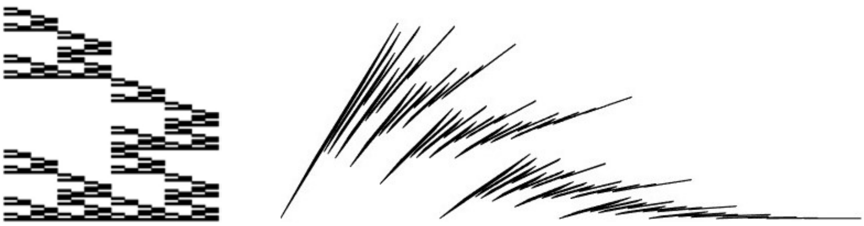

Fig. 3. A Bedford-McMullen carpet and an irreducible self-affine set. Because of the involved rotations, irreducible self-affine measures are usually not uniformly scaling.

everywhere for such measures which is something we aim to investigate as part of Research Question 2.

The idea of applying local statistics to global geometry of measures has a long history, dating back to Furstenberg's work [14] with CP-chains. During recent years, this machinery was substantially refined and developed further by Hochman and Shmerkin in their works [18, 21, 22]. For example, the *local entropy averages* [21], together with the scenery flow, has been used to prove strong versions of the classical projection theorem of Hunt-Kaloshin for self-similar and self-affine measures [9, 10, 12], and dynamical properties of the tangent distributions have found applications to normal numbers in fractals [1, 2, 22].

1.1.3 Resonance between measures

In the 1960s, Hillel Furstenberg proposed several conjectures aiming to capture the following general idea: If m and n are integers that are multiplicatively independent in the sense that $\frac{\log m}{\log n} \notin \mathbb{Q}$, then multiplication by m and n are fundamentally different operations. According to Furstenberg, one of the ways this difference would manifest is the following: If X and Y are closed subsets of the one-dimensional torus $\mathbb{T}$, invariant under multiplication by integers m and n, respectively, then the inequality

$$\dim_{\mathrm{H}}(X+Y) < \min\{1, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\} \tag{4}$$

should imply that $\frac{\log m}{\log n} \in \mathbb{Q}$. In current terminology, it is said that *geometric resonance* of the dynamical systems X and Y should imply *algebraic resonance* of the dynamics. Indeed, the right-hand side of (4) is an upper bound for $\dim_{\mathrm{H}}(X+Y)$ whenever X and Y have equal Hausdorff and box-counting dimensions, while a strict inequality implies arithmetic similarities in the structures of X and Y in the sense that the fibres $\{y \in Y : z - y \in X\}$ are large for many $z \in \mathbb{R}$. Replacing the sum by convolution in (4), one can formulate an analogous conjecture for $\times m$- and $\times n$-invariant measures μ and ν

on the torus:

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{1, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\} \qquad \text{implies that} \qquad \frac{\log m}{\log n} \in \mathbb{Q}. \tag{5}$$

One can view the conjecture of Furstenberg as a special case of a more general phenomenon: For dynamically defined sets (or measures) X and Y, their geometric resonance (4) should imply a kind of algebraic resonance between the defining dynamics. Since there is a natural dynamical structure endowed to stationary measures on iterated function systems, it is reasonable to ask whether this phenomenon extends to self-conformal and self-affine measures.

Research Question 3. *For self-affine or self-conformal measures μ and ν on $\mathbb{R}^d$, does the inequality*

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{d, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\}$$

imply a kind of algebraic resonance between the defining iterated function systems?

One of the first results contributing towards the resolution of this question is due to Moreira [8]. We say that a set $E \subseteq \mathbb{R}$ is *arithmetic* if $E \subseteq \alpha\mathbb{N} = \{\alpha n : n \in \mathbb{N}\}$ for some $\alpha \in \mathbb{R}$.

Theorem 1.1.9 (Moreira, Shmerkin)**.** *Let $\Phi = \{f_i\}_{i\in\Gamma}$ and $\Psi = \{g_j\}_{j\in\Lambda}$ be iterated function system of C^2-contractions on $\mathbb{R}$. Let X and Y denote their attractors. Suppose that*

1. $\Phi \cap \Psi = \emptyset$, and
2. for some $i, j \in \Gamma$, $(f_i \circ f_j^{-1})''$ is not identically zero on X.

Then the inequality

$$\dim_{\mathrm{H}}(X+Y) < \min\{1, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\}$$

implies that

$$\{\log|f_i'(x_i)| : \ i \in \Gamma, \ f_i(x_i) = x_i\} \cup \{\log|g_j'(y_j)| : \ j \in \Lambda, \ g_j(y_j) = y_j\}$$

is an arithmetic set.

The original proof of Moreira contained some errors which were subsequently corrected in a note of Shmerkin [26], where the above formulation of the theorem can also be found.

An analogous result was proven for self-similar sets by Peres and Shmerkin [25].

Theorem 1.1.10 (Peres & Shmerkin)**.** *Let* $\Phi = \{f_i(x) = r_i x + a_i\}_{i\in\Gamma}$ *and* $\Psi = \{g_j(x) = \rho_j x + b_j\}_{j\in\Lambda}$ *be self-similar iterated function systems on* $\mathbb{R}$*. Let* X *and* Y *denote their attractors. The inequality*

$$\dim_{\mathrm{H}}(X+Y) < \min\{1, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\}$$

implies that $\{\log r_i : \ i \in \Gamma\} \cup \{\log \rho_j : \ j \in \Lambda\}$ *is an arithmetic set.*

Shortly after, the result was generalised for Bernoulli convolution measures by Nazarov, Peres and Shmerkin [24].

Furstenberg's conjecture along with the measure-theoretic analogue (5) was verified in a celebrated paper of Hochman and Shmerkin [21]. Their method allowed them to handle not only multiplication-invariant measures but also self-similar measures with the open set condition.

Theorem 1.1.11 (Hochman & Shmerkin)**.** *Let* $\Phi = \{f_i(x) = r_i x + a_i\}_{i\in\Gamma}$ *and* $\Psi = \{g_j(x) = \rho_j x + b_j\}_{j\in\Lambda}$ *be self-similar iterated function systems on* $\mathbb{R}$ *with the open set condition. Then, for any self-similar measures* μ *and* ν *associated to* Φ *and* Ψ*, the inequality*

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{1, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\}$$

implies that $\{\log r_i : \ i \in \Gamma\} \cup \{\log \rho_j : \ j \in \Lambda\}$ *is an arithmetic set.*

Very recently, Bruce and Jin [6] obtained state-of-the-art results for very general measures on homogeneous self-similar sets on the line, notably without any separation conditions. In their work, Bruce and Jin employed a refinement of the method introduced by Hochman and Shmerkin.

Theorem 1.1.12 (Bruce & Jin)**.** *Let* $\Phi = \{x \mapsto rx + a_i\}_{i\in\Gamma}$ *and* $\Psi = \{x \mapsto \rho x + b_j\}_{j\in\Lambda}$ *be self-similar iterated function systems on* $\mathbb{R}$*. Then, if* μ *and* ν *are Mandelbrot cascades on the attractors of* Φ *and* Ψ*, for example self-similar measures, the inequality*

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{1, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\}$$

implies that $\frac{\log r}{\log \rho} \in \mathbb{Q}$*.*

To date, very little seems to be known of (5) or the set-theoretic analogue in higher dimensions: It seems natural to conjecture that given self-affine measures μ and ν on the plane with suitable irreducibility conditions, then

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{2, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\} \tag{6}$$

should imply an algebraic condition on the defining iterated function systems. The necessity of an irreducibility assumption is easy to see by considering convolutions of product measures.

2 Preliminaries

In this chapter we introduce the notation and definitions required in formulating the main results of the thesis. Most of the notation is standard, whence a reader with standard knowledge of the area is recommended to continue directly to Chapter 3.

2.1 Dynamical systems

It is often useful to be able to 'code' sets of interest using infinite sequences of characters from a finite alphabet. There are many natural ways to code points of $\mathbb{R}^d$, such as different number systems. However, these systems are designed to be able to code the entirety of $\mathbb{R}^d$, making them substantially larger than what is required to code most sets we are interested in. For attractors of iterated function systems, there are natural, more efficient codings available.

Let us begin with some notation regarding these codings. Let Γ be a finite set with $\#\Gamma \geq 2$. Elements of $\Gamma^{\mathbb{N}}$ are called infinite words. Let $\Gamma^* = \bigcup_{k=1}^{\infty} \Gamma^k$ denote the set of finite words. For $\mathtt{i} \in \Gamma^{\mathbb{N}}$ and $k \in \mathbb{N}$, let $\mathtt{i}|_k \in \Gamma^k$ denote the projection of $\mathtt{i}$ onto the first k coordinates. For $\mathtt{j} \in \Gamma^*$, let $|\mathtt{j}| \in \mathbb{N}$ denote the length of $\mathtt{j}$, that is, the unique integer such that $\mathtt{j} \in \Gamma^{|\mathtt{j}|}$. We equip $\Gamma^{\mathbb{N}}$ with the topology generated by the cylinder sets $[\mathtt{j}] = \{\mathtt{i} \in \Gamma^{\mathbb{N}} : \mathtt{i}|_{|\mathtt{j}|} = \mathtt{j}\}$, $\mathtt{j} \in \Gamma^*$.

Let σ denote the continuous surjection on $\Gamma^{\mathbb{N}}$ given by

$$\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots).$$

The pair $(\Gamma^{\mathbb{N}}, \sigma)$ is called the *shift space*. Every iterated function system has an underlying shift space.

Definition 2.1.1. *Let Γ be a finite set, and let $\{f_i\}_{i \in \Gamma}$ be an iterated function system with attractor K. Let U be a compact set such that $f_i(U) \subseteq U$ for each $i \in \Gamma$. The map $\Pi : \Gamma^{\mathbb{N}} \to K$ given by*

$$\Pi(i_1, i_2, \ldots) = \bigcap_{n \in \mathbb{N}} f_{i_1} \circ \cdots \circ f_{i_n}(U)$$

is called the natural projection.

The existence of the natural projection is a standard result in the theory of iterated function systems. Modelling the attractor of an iterated function system as an image of a shift space $(\Gamma^{\mathbb{N}}, \sigma)$ through the natural projection Π gives access to tools from another area of mathematics, *dynamical systems*.

Definition 2.1.2. *A* dynamical system *is a triplet* (X,A,μ)*, where* (X,μ) *is a probability space and A is a semigroup acting measurably on X. If a is a generator of A, we write* $(X,A,\mu) = (X,a,\mu)$*.*

The underlying shift space gives a natural way to assign probability measures on the attractor of an iterated function system. In fact, self-conformal and self-affine measures are images of Bernoulli measures on the shift space.

Proposition 2.1.3. *Let* μ *be the unique measure satisfying* $\mu = \sum_{i\in\Gamma} p_i \cdot \mu \circ f_i^{-1}$*. Then if* $\bar{\mu}$ *denotes the Bernoulli measure on* $\Gamma^{\mathbb{N}}$ *with marginal* $(p_i)_{i\in\Gamma}$*, we have*

$$\mu = \bar{\mu} \circ \Pi^{-1}.$$

The orbits $\{\sigma^n(\mathtt{i})\}_{n\in\mathbb{N}}$ of infinite words $\mathtt{i} \in \Gamma^{\mathbb{N}}$ are in a natural way connected to the process of 'magnifying' the attractor at the point $\Pi(\mathtt{i})$. The advantage of this connection is that the orbits of almost all sequences with respect to Bernoulli measures, or more generally with respect to ergodic measures, are very well understood by the classical ergodic theorems.

Definition 2.1.4. *A dynamical system* (X,A,μ) *is* measure-preserving *if* $\mu \circ a^{-1} = \mu$ *for each* $a \in A$*, and* ergodic *if it is measure-preserving and for any set* $E \subseteq X$ *with* $a^{-1}E \subseteq E$ *for every* $a \in A$*, we have* $\mu(E) \in \{0,1\}$*.*

When the set X is clear from the context, we may also say that μ is A-ergodic, or a-ergodic if a is a generator of A, to indicate that the dynamical system (X,A,μ) is ergodic.

2.2 Iterated function systems

The types of iterated function systems studied in this thesis were introduced in Definition 1.0.1. In order to make the study of their attractors more tractable, there are a number of different geometric and algebraic conditions one can impose on the iterated function systems.

Definition 2.2.1 (Separation conditions)**.** *Let* $\Phi = \{f_i\}_{i\in\Gamma}$ *be an iterated function system on* $\mathbb{R}^d$*. We say that* Φ*, its attractor K and any stationary measure satisfies the*

a) strong separation condition *if* $f_i(K) \cap f_j(K) = \emptyset$ *for any distinct* $i, j \in \Gamma$*,*
b) open set condition *if there exists an open set* $U \subseteq \mathbb{R}^d$ *such that* $f_i(U) \subseteq U$ *for every* $i \in \Gamma$ *and* $f_i(U) \cap f_j(U) = \emptyset$ *for any distinct* $i, j \in \Gamma$*,*
c) weak separation condition *if* Φ *is self-affine and the identity is not an accumulation point of the topological group generated by* $\{f_i\}_{i\in\Gamma} \cup \{f_i^{-1}\}_{i\in\Gamma}$ *under composition.*

Let $\mathbb{RP}^1$ denote the collection of one-dimensional linear subspaces of $\mathbb{R}^2$, identified with $[0,\pi)$ through the angle a line makes with the positive x-axis and equipped with the topology carried through the identification from the usual topology of $[0,\pi)$. For a 2×2-matrix A, we let A also denote the action of A on $\mathbb{RP}^1$.

Definition 2.2.2. *Let $\Phi = \{x \mapsto A_i x + a_i\}_{i\in\Gamma}$ be a self-affine iterated function system on $\mathbb{R}^2$. We say that Φ, its attractor and any stationary measure satisfies*

a) irreducibility *if for every $\theta \in \mathbb{RP}^1$ there exists $i \in \Gamma$ such that $A_i(\theta) \neq \theta$,*
b) hyperbolicity *if there exists $i \in \Gamma$ such that A_i has two real eigenvalues of different absolute values,*
c) domination *if there exist closed and connected sets $C_1,\dots,C_n \subseteq \mathbb{RP}^1$ such that $C := \bigcup_{i=1}^n C_i \neq \mathbb{RP}^1$ and $A_i(C) \subset \mathrm{int}(C)$ for each $i \in \Gamma$, where* int *denotes the interior.*

To any self-affine measure there is an associated probability measure on $\mathbb{RP}^1$, called the *Furstenberg measure*.

Theorem 2.2.3 (Furstenberg, Oseledets)**.** *Let $\Phi = \{f_i(x) = A_i x + a_i\}_{i\in\Gamma}$ be a self-affine iterated function system that satisfies hyperbolicity and irreducibility, and let μ be a Bernoulli measure on $\Gamma^{\mathbb{N}}$. Then there exists a unique measure μ_F on $\mathbb{RP}^1$, called the* Furstenberg measure*, such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{A_{i_k}^{-1}\cdots A_{i_1}^{-1}\theta} = \mu_F$$

in the weak*-topology, for μ-almost every $(i_1,i_2,\dots) \in \Gamma^{\mathbb{N}}$ and all but one $\theta \in \mathbb{RP}^1$.

3 Results

The results of this thesis provide partial answers to each of the Research Questions 1, 2 and 3. In the following, each section collects the progress made in one research question. In this thesis, log denotes the logarithm in base 2.

3.1 Self-similar sets that are tube-null

Progress towards the resolution of Research Question 1 is made by Article I, the main result of which provides a large new class of self-similar tube-null sets in $\mathbb{R}^d$. Let $T_N : [0,1]^d \to [0,1]^d$ denote the map $(x_1,\dots,x_d) \mapsto (Nx_1 \mod 1,\dots,Nx_d \mod 1)$.

Theorem 3.1.1 (Theorem 1.2 of Article I)**.** *Let $K \subset [0,1]^d$ be a closed set such that $T_N(K) \subseteq K$ for some $N \in \mathbb{N}$. Then either $K = [0,1]^d$ or K is tube-null.*

There are many well-studied self-similar sets that satisfy the hypothesis of the theorem. For example, the classical Sierpiński carpet in $\mathbb{R}^2$ and the Sierpiński sponge in $\mathbb{R}^3$.

Corollary 3.1.2. *Let $N \geq 2$ be an integer, and let $\Phi = \{f_i\}_{i\in\Gamma}$ be a self-similar iterated function system on $\mathbb{R}^d$, where $\#\Gamma < N^d$ and for each $i \in \Gamma$, $f_i(x) = \frac{1}{N}(x+a_i)$ for some $a_i \in \{0,\dots,N-1\}^d$.*

Then the attractor of Φ is a tube-null self-similar set of Hausdorff dimension $\frac{\log\#\Gamma}{\log N}$.

See Figure 4. By choosing N to be large, the Hausdorff dimension of the attractor can be taken arbitrarily close to d.

The proof of Theorem 3.1.1 can be adapted to handle certain other self-similar sets as well, as long as they contain no rotations and the contraction ratios of the defining similarities satisfy a certain upper bound.

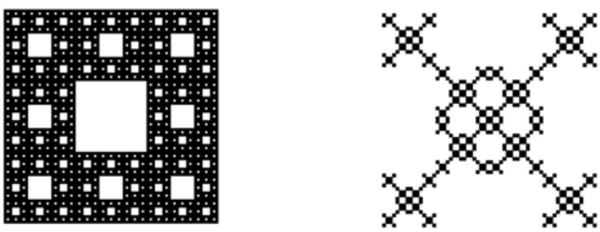

Fig. 4. The Sierpiński carpet and another self-similar set in $\mathbb{R}^2$ that is tube-null by Corollary 3.1.2.

Fig. 5. The self-similar set on the left involves no rotations, has Hausdorff dimension ≈ 1.26 and is tube-null by Proposition 3.1.3. The self-similar set on the right has Hausdorff dimension > 1 and involves an irrational rotation; it is not known whether such sets are tube-null or not.

Proposition 3.1.3 (Proposition 1.3 of Article I). *Let $\Phi = \{\varphi_i(x) = rx + \lambda_i\}_{i=0}^{m-1}$ be a self-similar iterated function system on $\mathbb{R}^d$, and suppose that $-\log r > \log m - \frac{2}{m}$. Then the attractor of Φ is tube-null.*

We remark that Proposition 3.1.3 covers many self-similar sets with Hausdorff dimension in the interval $(d-1, d)$ which have no Lebesgue-null projections and do not satisfy the hypothesis of Theorem 3.1.1: See Figure 5 for an example.

3.2 Local structure of measures

Progress towards the resolution of Research Question 2 is made by the results of Articles II, III and IV. First of all, the main result of Article II shows that self-similar measures on $\mathbb{R}^d$ that satisfy the weak separation condition are uniformly scaling. Let S denote the semigroup $\{S_t : t \geq 0\}$, where $S_t : \mu \mapsto \mu_{0,t}$ for $t \geq 0$.

Theorem 3.2.1 (Theorem 1.1 of Article II). *Let $\Phi = \{f_i\}_{i \in \Gamma}$ be a self-similar iterated function system on $\mathbb{R}^d$ satisfying the weak separation condition, and let μ be an associated self-similar measure. Then μ is uniformly scaling and generates an S-ergodic tangent distribution.*

We also obtain the following description for measures arising in the scenery of μ at almost every point. For any finite word $\mathtt{i} = (i_1, i_2, \ldots, i_n) \in \Gamma^*$, write

$$\mu^{\mathtt{i}} := (\mu \circ f_{i_n}^{-1} \circ \cdots \circ f_{i_1}^{-1})_{[-1,1]^d},$$

where we use the notation ν_A to denote the normalised restriction of the measure ν on A.

Proposition 3.2.2 (Proposition 3.3 of Article II)**.** *Let* Φ *and* μ *be as in Theorem 3.2.1. There exists a finite word* $\mathtt{a} \in \Gamma^*$ *and a constant* $0 < C < \infty$ *such that for any* $\mathtt{i} \in \Gamma^*$*,*

$$\mu^{\mathtt{ia}} = ((\zeta_{\mathtt{i}}\, d\nu) \circ R_{\mathtt{i}})_{[-1,1]^d}$$

for some integrable $\zeta_{\mathtt{i}} \geq C$ *and orthogonal matrix* $R_{\mathtt{i}}$ *depending on* $\mathtt{i}$*, where*

$$\nu = \sum_{i=1}^{n} \mu \circ g_i^{-1}$$

for some similarity maps $g_i : \mathbb{R} \to \mathbb{R}$*.*

Note the connection to scenery measures: For μ-almost every $\mathtt{i} = (i_1, i_2, \ldots) \in \Gamma^{\mathbb{N}}$ and any n such that $\sigma^{n-|\mathtt{a}|}(\mathtt{i}) \in [\mathtt{a}]$, for any large enough t we have

$$\mu_{\Pi(\mathtt{i}),t} = ((\zeta_t\, d\nu) \circ R_t)_{\Pi(\sigma^n \mathtt{i}), t + \log \|(f_{i_1} \circ \cdots \circ f_{i_n})'\|}$$

for some integrable $\zeta_t \geq C$ and orthogonal R_t. In particular, there exists a fixed 'reference measure' ν, which often appears, weighed by ζ_t and rotated by R_t, in the magnifications of μ around any typical point.

The main results of Article IV extend Theorem 3.2.1 to self-conformal measures without any separation conditions, in particular providing a complete answer to Research Question 2 for self-conformal measures on the line and for self-similar measures in any dimension. A probability measure μ on $\Gamma^{\mathbb{N}}$ is called *quasi-Bernoulli* if there exists a constant $C \geq 1$ such that for any $\mathtt{i}, \mathtt{j} \in \Gamma^*$,

$$\frac{1}{C}\mu([\mathtt{i}])\mu([\mathtt{j}]) \leq \mu([\mathtt{ij}]) \leq C\mu([\mathtt{i}])\mu([\mathtt{j}]).$$

In particular, Bernoulli measures are also quasi-Bernoulli.

Theorem 3.2.3 (Theorem 1.1 of Article IV)**.** *Let* Φ *be a self-conformal iterated function system on* $\mathbb{R}^d$*, and let* $\bar{\mu}$ *be a quasi-Bernoulli measure on* $\Gamma^{\mathbb{N}}$*. If* $d = 1$ *or* Φ *consists of similarities, then* $\bar{\mu} \circ \Pi^{-1}$ *is uniformly scaling and generates an S-ergodic tangent distribution.*

Regarding measures in higher dimensions, we can say the following.

Theorem 3.2.4 (Theorems 1.2 and 1.3 of Article IV)**.** *Let* Φ *be a self-conformal iterated function system on* $\mathbb{R}^d$*, let* $\bar{\mu}$ *be a quasi-Bernoulli measure on* $\Gamma^{\mathbb{N}}$*, and write* $\mu := \bar{\mu} \circ \Pi^{-1}$*. Then there exists an S-ergodic measure P, a sequence* $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ *and a diffeomorphism* $h : \mathbb{R}^d \to \mathbb{R}^d$ *such that*

1. *for $\mu \circ h^{-1}$-almost every x,*

$$\lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(\mu\circ h^{-1})_{x,t}}\, dt = P,$$

and

2. *for μ-almost every x, if Q is a tangent distribution of μ at x, then there exists a measure Q' on the orthogonal group of $\mathbb{R}^d$ such that*

$$Q = \int P \circ O^{-1}\, dQ'(O).$$

The results of Article III make progress in Research Question 2 by providing new information on the scenery measures of planar self-affine measures. In order to state the result, we require some more notation.

For $\theta \in \mathbb{RP}^1$, let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection onto the line perpendicular to θ identified with $\mathbb{R}$. Let $\mu = \int \mu_x^\theta \, d\mu \circ \pi_\theta^{-1}$ denote the disintegration of μ with respect to π_θ. Let $\mu_{x,\theta}$ denote the measure μ_x^θ mapped onto $\mathbb{R}$ through a linear isomorphism. For a self-affine iterated function system $\Phi = \{x \mapsto A_i x + a_i\}_{i\in\Gamma}$ and any finite word $\mathtt{i} = (i_1, \ldots, i_n) \in \Gamma^*$, let $\alpha_1(\mathtt{i}) \le \alpha_2(\mathtt{i})$ denote the singular values of the matrix $A_\mathtt{i} := A_{i_1} \cdots A_{i_n}$. For infinite words $\mathtt{i} \in \Gamma^\mathbb{N}$, let $\theta(\mathtt{i})$ denote the limiting direction of major semiaxes of the sequence of ellipses $(A_{\mathtt{i}|_k}(B(0,1)))_{k\in\mathbb{N}}$. If the iterated function system Φ is hyperbolic, the direction $\theta(\mathtt{i})$ is well-defined almost everywhere with respect to any Bernoulli measure on $\Gamma^\mathbb{N}$. Let also $M : \Gamma^\mathbb{N} \times \mathbb{RP}^1 \to \Gamma^\mathbb{N} \times \mathbb{RP}^1$ denote the map $(\mathtt{i}, \theta) \mapsto (\sigma\mathtt{i}, A_{i_0}^{-1}\theta)$, and let d denote the Lévy-Prokhorov metric on the space of probability measures.

Proposition 3.2.5 (Proposition 4.1 of Article III)**.** *Let $\Phi = \{f_i\}_{i\in\mathbb{N}}$ be a self-affine iterated function system on $\mathbb{R}^2$ with strong separation, irreducibility and hyperbolicity. Let $\bar{\mu}$ be a Bernoulli measure on $\Gamma^\mathbb{N}$ and let $\mu := \bar{\mu} \circ \Pi^{-1}$ denote the associated self-affine measure. Let $\varepsilon > 0$.*

For $\bar{\mu}$-almost every $\mathtt{i} \in \Gamma^\mathbb{N}$ and all $\theta \in \mathbb{RP}^1$ in a set of positive μ_F-measure, there exists an unbounded increasing sequence $(\ell_k)_{k\in\mathbb{N}}$ and a set $\mathcal{N}_\varepsilon \subseteq \mathbb{N}$ such that $\liminf_{n\to\infty} \frac{\#(\mathcal{N}_\varepsilon \cap [0,n])}{n} \ge 1 - \varepsilon$ *and*

$$d(\mu_{\Pi(\mathtt{i}),k} \circ \pi_{\theta(\mathtt{i})}^{-1},\ \ \mu_{M^{\ell_k}(\mathtt{i},\theta),k+\log\alpha_1(\mathtt{i}|_{\ell_k})} \circ \rho(\ell_k,(\mathtt{i},\theta))) < \varepsilon$$

for every $k \in \mathcal{N}_\varepsilon$, where $\rho : \mathbb{N} \times \Gamma^\mathbb{N} \times \mathbb{RP}^1 \to \{x \mapsto x,\ x \mapsto -x\}$ is a cocycle.

3.3 Resonance between measures

Progress in Research Question 3 is made by the results of Articles III and IV. First, we obtain a result on resonance between self-affine measures on the plane, with standard regularity and irreducibility assumptions on the defining iterated function systems.

Theorem 3.3.1 (Theorem 1.1 of Article III)**.** *Let* $\Phi := \{\varphi_i(x) = A_i x + a_i\}_{i\in\Gamma}$ *and* $\Psi := \{\psi_j(x) = B_j x + b_j\}_{j\in\Lambda}$ *be self-affine iterated function systems on* $\mathbb{R}^2$ *that satisfy strong separation, hyperbolicity and irreducibility. Let* μ *and* ν *be self-affine measures associated to* Φ *and* Ψ*, and suppose that* $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \nu$*. If*

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{2, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\}, \tag{7}$$

then $\dim_{\mathrm{H}} \mu > 1 > \dim_{\mathrm{H}} \nu$ *and, if* Φ *and* Ψ *also satisfy the domination condition, then there exists* $\alpha \in \mathbb{R}$ *such that*

$$\{\log|\lambda_1(A_i)| :\ i \in \Gamma\} \cup \{\log|\lambda_2(B_j)| :\ j \in \Lambda\} \subset \alpha\mathbb{N}. \tag{8}$$

Combined with a known variational principle for self-affine sets with the domination condition, Theorem 3.3.1 also implies the corresponding result for sums of self-affine sets.

Corollary 3.3.2 (Corollary 1.2 of Article III)**.** *Let* $\Phi := \{\varphi_i(x) = A_i x + a_i\}_{i\in\Gamma}$ *and* $\Psi := \{\psi_j(x) = B_j x + b_j\}_{j\in\Lambda}$ *be self-affine iterated function systems on* $\mathbb{R}^2$ *that satisfy strong separation, domination and irreducibility. Let* X *and* Y *be the attractors of* Φ *and* Ψ*, and suppose that* $\dim_{\mathrm{H}} X \geq \dim_{\mathrm{H}} Y$*. If*

$$\dim_{\mathrm{H}}(X + Y) < \min\{2, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\}, \tag{9}$$

then $\dim_{\mathrm{H}} X > 1 > \dim_{\mathrm{H}} Y$ *and there exist* $n \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$ *such that*

$$\{\log|\lambda_1(A_{\mathtt{i}})| :\ \mathtt{i} \in \Gamma^n\} \cup \{\log|\lambda_2(B_{\mathtt{j}})| :\ \mathtt{j} \in \Lambda^n\} \subset \alpha\mathbb{N}. \tag{10}$$

The third result deals with convolutions of measures supported on self-conformal sets on the line. For a contracting diffeomorphism f, let $\lambda(f)$ denote its derivative at the unique fixed point.

Theorem 3.3.3 (Theorem 1.5 of Article IV)**.** *Let* $\Phi = \{f_i\}_{i\in\Gamma}$ *and* $\Psi = \{g_j\}_{j\in\Lambda}$ *be self-conformal iterated function systems on* $\mathbb{R}$*. Let* $\bar{\mu}$ *and* $\bar{\nu}$ *be fully supported quasi-Bernoulli measures on* $\Gamma^{\mathbb{N}}$ *and* $\Lambda^{\mathbb{N}}$*, and write* $\mu := \bar{\mu} \circ \Pi^{-1}$ *and* $\nu := \bar{\nu} \circ \Pi^{-1}$*. If*

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{1, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\},$$

then there exists $\alpha \in \mathbb{R}$ *such that*

$$\{\log|\lambda(f_i)| :\ i \in \Gamma\} \cup \{\log|\lambda(g_j)| :\ j \in \Lambda\} \subseteq \alpha\mathbb{N}.$$

4 Discussion

In this chapter we discuss some implications of the results stated in Chapter 3.

4.1 Self-similar sets that are tube-null

Corollary 3.1.2 seems to be the first result in the literature that can be used to produce tube-null self-similar sets of Hausdorff dimension arbitrarily close to that of the ambient space and no Lebesgue-null projections. Another implication is an interesting difference between how deterministic and random sets behave under covering by tubes: Shmerkin and Suomala [28] have shown that many sets in $\mathbb{R}^d$ that are T_N-invariant *in distribution*, including the classical fractal percolation, are almost surely not tube-null, when the Hausdorff dimension lies in the interval $(d-1,d)$.

The strategy of the proofs of Theorem 3.1.1 and Proposition 3.1.3 is to decompose K into finitely many subsets, each of which admits an orthogonal projection of Hausdorff dimension strictly less than one.

Theorem 4.1.1 (Theorem 1.1 of Article I)**.** *Let $K \subset \mathbb{R}^d$ be as in Theorem 3.1.1 or Proposition 3.1.3. Then either $K = [0,1]^d$ or K can be decomposed as a finite union, $K = \bigcup_{i=1}^n E_i$, such that for each i there exists a direction $v_i \in \mathbb{RP}^1$ such that*

$$\dim_H \pi_{v_i} E_i < 1.$$

With this result, we can in fact say even more about the tube-covering behaviour. Define the *tube-dimension* of K as the number

$$\dim_T K = \inf\left\{ s \geq 0 : \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there exist tubes } T_1, T_2, \ldots \\ \text{such that } E \subseteq \bigcup_{i=1}^\infty T_i \text{ and } \sum_{i=1}^\infty w(T_i)^s < \varepsilon \end{array} \right\}$$

where $w(T_i)$ denotes the width of the tube. Note that this is just the Hausdorff dimension with arbitrary sets replaced by tubes and diameters replaced by widths.

Corollary 4.1.2. *Let $K \subseteq \mathbb{R}^d$ be as in Theorem 3.1.1 or Proposition 3.1.3. Then either $K = [0,1]^d$ or* $\dim_T K < 1$.

This follows immediately from Theorem 4.1.1. In his proof that the Von Koch curve is tube-null, Harangi [17] also obtained an analogue of Theorem 4.1.1 for the set he considered. His proof is combinatorial and does not easily extend outside the particular set. Our proof of Theorem 4.1.1 is dynamical: We first prove a result for orthogonal projections of T_N-invariant measures on the torus, and then apply a known variational principle from ergodic theory to deduce a statement on orthogonal projections of sets. This approach seems to be new in the context of tube-null sets.

Proposition 4.1.3 (Proposition 4.3 of Article I)**.** *For any integer N, there exists a finite set $\mathscr{V}_N \subseteq \mathbb{RP}^1$ and a constant $\delta > 0$ such that for any T_N-invariant measure on $\mathbb{T}^d$ other than the Lebesgue measure, there exists $v \in \mathscr{V}_N$ such that*

$$\dim_{\mathrm{H}} \mu \circ \pi_v^{-1} < 1 - \delta.$$

The proof of this proposition is where the invariance under T_N is most crucial. It seems likely that with a similar framework, Theorem 3.1.1 could be extended for other dynamically defined sets for which an analogue of Proposition 4.1.3 holds.

With elementary methods from Fourier analysis, we can obtain an upper bound for the number of directions required in covering the set with tubes.

Proposition 4.1.4 (Section 5.3 of Article I)**.** *For any $\varepsilon > 0$ and $d \in \mathbb{N}$ there exists $C_{\varepsilon,d} > 0$ such that the following holds: For any $N \in \mathbb{N}$ and T_N-invariant self-similar set $K \subset [0,1]^d$, at most $C_{\varepsilon,d} N^{1+\varepsilon}$ directions are required in Theorem 3.1.1.*

4.1.1 Further questions

Proposition 3.1.3 states that self-similar sets with no grid structure are also tube-null when the parameters of the defining similarities satisfy an additional restriction. We believe this restriction to be a by-product of the proof, however, and conjecture that the result holds for any self-similar set with no rotations and non-maximal Hausdorff dimension.

Question 4.1.5. *Let $K \subset \mathbb{R}^d$ be an attractor of a self-similar iterated function system $\{x \mapsto r_i x + a_i\}_{i=0}^{m-1}$ with $\dim_{\mathrm{H}} K < d$. Is K tube-null?*

Conversely, self-similar sets with an irrational rotation involved might have more trouble being tube-null. In particular, an analogue of Proposition 4.1.3 no longer holds by the projection theorem of Hochman-Shmerkin [21]. It also follows from the slicing theorem of [27] that self-similar sets with irrational rotations have full tube-dimension.

Proposition 4.1.6 (Proposition 5.2 of Article I)**.** *Let $K \subset \mathbb{R}^2$ be an attractor of a self-similar iterated function system $\{\varphi_i(x) = r_i O x + a_i\}_{i=0}^{m-1}$, where $\{O^n\}_{n\in\mathbb{N}}$ is dense in $O_2(\mathbb{R})$. Then $\dim_T K = 1$.*

However, it remains a challenging open problem to determine whether such sets are tube-null or not.

Question 4.1.7. *Let K be a self-similar set as in Proposition 4.1.6. Is K non tube-null?*

Another natural question is whether sets invariant under the *non-conformal* automorphism $T_{N,M} : (x,y) \mapsto (Nx \mod 1, My \mod 1)$ are also tube-null. Such sets include the classical Bedford-McMullen self-affine carpets. The following question was also posed in a survey paper of Fraser [13].

Question 4.1.8. *Let $K \subset \mathbb{R}^2$ be a closed set such that $T_{N,M}(K) \subseteq K$ for some integers N,M, and $K \neq [0,1]^2$. Is K tube-null?*

4.2 Local structure of measures

The statement of Theorem 3.2.1 was previously known for self-similar measures with the open set condition. Examples of measures that satisfy the weak separation condition but not the open set condition are given by the classical Bernoulli convolutions [15].

Corollary 4.2.1. *Let μ be a self-similar measure associated to the iterated function system $\{x \mapsto \lambda x,\ x \mapsto \lambda x + 1 - \lambda\}$ with $0 < \lambda < 1$. If λ^{-1} is a Pisot number, then μ is uniformly scaling and generates an S-ergodic tangent distribution.*

The uniform scaling property for self-similar measures with the open set condition was proven by Gavish [16]. His proof made use of the fact that such measures are *homogeneous*, in the sense that any measure in the weak*-closure of the family $\{\mu_{x,t} :\ x \in \mathbb{R}^d, t \geq 0\}$ equals a restriction of a scaled and translated copy of μ itself. On the other hand, Gavish showed that the closure of the family $\{\mu_{x,t} :\ x \in \mathbb{R}^d, t \geq 0\}$ always contains many uniformly scaling measures, when μ is *any* Radon measure. Combining this with the homogeneity of μ concluded the proof.

In the absence of the open set condition, the homogeneity property is lost due to the presence of overlaps in the associated iterated function system. This calls for Proposition 3.2.2 which provides an 'average homogeneity'—a way to relate a sufficient number of measures in $\{\mu_{x,t} :\ x \in \mathbb{R}^d, t \geq 0\}$ back to μ itself. An analogous homogeneity property for self-similar sets with the weak separation condition was observed by Feng and Lau [11]; however, for measures, Proposition 3.2.2 appears to give new information.

The proof of Theorem 3.2.1 is implicit. In particular, it does not give an explicit description of the generated tangent distribution. It would be interesting to find a proof that can be used to produce an explicit formula for the tangent distribution. This would require control of the functions ζ in Proposition 3.2.2, however.

A complete resolution to Research Question 2 for self-similar measures on $\mathbb{R}^d$ and self-conformal measures on $\mathbb{R}$ is given by Theorem 3.2.3. The result is sharp in the sense that for $d \geq 2$, self-conformal measures on $\mathbb{R}^d$ are not necessarily uniformly scaling. Indeed, the length measure on the upper semi-circle of $\mathbb{R}^2$ is a self-conformal measure, yet at any two different points, all tangent distributions are supported on measures that are supported on line segments oriented in different directions. Thus, the measure is not uniformly scaling. However, Theorem 3.2.4 provides a close alternative, in the sense that any tangent distribution of a self-conformal measure can be expressed by averaging rotations of a single S-ergodic measure P. Alternatively, a diffemorphic image of a self-conformal measure is always uniformly scaling along a subsequence, although we believe the requirement of a subsequence to be a by-product of the proof.

Although Theorem 3.2.3 contains the statement of Theorem 3.2.1 as a special case, the proof is more involved and relies on deep results of Hochman [18] on the structure of tangent distributions. The proof also does not produce a description of the measures that appear in the scenery around typical points, such as Proposition 3.2.2.

The key observation in the proof is the following. Let $\mathscr{D}_n(\mathbb{R}^d)$ denote the partition of $\mathbb{R}^d$ onto dyadic cubes of side length 2^{-n}, and $H_n(\mu)$ the Shannon entropy of a probability measure μ with respect to $\mathscr{D}_n$.

Proposition 4.2.2 (Proposition 3.4 of Article IV). *Let μ and ν be probability measures such that μ is supported on $\mathbb{R}^d$ and ν is supported on μ-measurable functions $\mathbb{R}^d \to \mathbb{R}^d$. For any $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that the following holds for all $n \geq N$: If*

1. $\mu \circ g^{-1}(\{x : \frac{1}{n}\sum_{k=1}^{n} H_1((\mu \circ g^{-1})_{x,k}) \geq \alpha\}) \geq 1 - \delta$ *for ν-a.e. g,*
2. $H_1(\int \mu \circ h^{-1}\, d\nu(h)) \leq n(\alpha + \delta)$,
3. $\dim_{\mathrm{H}} \int \mu \circ h^{-1}\, d\nu(h) > 0$

for some $\alpha \geq 0$, then

$$\iint \frac{1}{n}\int_0^n d\left(\left(\int \mu \circ h^{-1}\, d\nu(h)\right)_{y,k},\ (\mu \circ g^{-1})_{y,k}\right) dt\, d(\mu \circ g^{-1})(y)\, d\nu(g) < \varepsilon.$$

This proposition is the key technical contribution of Article IV. One application to measures μ and ν on the line is the following: If convolving μ with another measure ν on $\mathbb{R}$ does not increase the dimension, then the sceneries of μ and the convolution $\mu * \nu$ are asymptotic, on average. Since nothing is assumed of the structures of μ and ν, it seems likely that this proposition will also find applications elsewhere. The relevance of this result in the proof of Theorem 3.2.3 is indicated by the following description of the local structure of quasi-Bernoulli measures on self-conformal sets:

Proposition 4.2.3 (Lemma 3.5 of Article IV). *Let Φ be a self-conformal iterated function system on $\mathbb{R}^d$, let $\bar{\mu}$ be a quasi-Bernoulli measure on $\Gamma^{\mathbb{N}}$, and write $\mu := \bar{\mu} \circ \Pi^{-1}$. For*

μ-almost every x and any tangent distribution P at x, for P-almost every measure η, there exists a measure ν on the space of diffeomorphisms on $\mathbb{R}^d$ such that η and $(\int \mu \circ h^{-1}\, d\nu(h))_{B(0,1)}$ are comparable.

Applying the machinery of Hochman and Shmerkin [22], we obtain the following application of Theorem 3.2.3 in the prevalence of normal numbers in self-conformal sets on the line. This generalises the recent result of Algom, Baker and Shmerkin on normal numbers in self-similar sets [2] to the self-conformal setting, and extends the recent results of Baker and Sahlsten [4] and Algom, Hertz and Wang [3] on normal numbers in self-conformal sets to include non-integer bases and quasi-Bernoulli measures.

Theorem 4.2.4 (Theorem 1.4 of Article IV)**.** *Let $\Phi = \{f_i\}_{i\in\Gamma}$ be a self-conformal iterated function system on $\mathbb{R}$, and let $\bar{\mu}$ be a fully supported quasi-Bernoulli measure on $\Gamma^{\mathbb{N}}$. Then for any Pisot number β such that $\frac{\log\beta}{\log|\lambda(f_i)|} \notin \mathbb{Q}$ for some $i \in \Gamma$, $\bar{\mu}\circ\Pi^{-1}$-almost every x is normal in base β.*

The local structure of self-affine measures that are not self-similar is much less understood. While the scenery of self-affine sets has also been studied in the irreducible setting [5], the scenery of self-affine measures has previously been studied only in the context of Bedford-McMullen self-affine carpets [12] and in the presence of somewhat restrictive projection conditions [23].

Proposition 3.2.5 describes the scenery of planar self-affine measures assuming only the irreducibility and strong separation conditions. The proposition shows that self-affine measures enjoy a local 'fibre structure' in the sense that orthogonal projections of the scenery measures are related to slices of the original measure. The same phenomenon was previously observed for self-affine measures on carpets by Ferguson, Fraser and Sahlsten [12], and by Kempton [23] for self-affine measures associated to non-negative matrices with an additional projection condition. Without the non-negativity assumption, the action of the maps in the defining iterated function system on lines may contain more complicated reflections, making the geometry behind Proposition 3.2.5 more delicate. However, the conclusions of [12, 23] on the local structure are stronger than that of Proposition 3.2.5 in the sense that they also describe the conditional measures of $\mu_{\Pi(\mathtt{i}),t}$ with respect to the orthogonal projection $\pi_{\theta(\mathtt{i})}$; these are equal to orthogonal projections of the original measure in directions typical to the Furstenberg measure. While heuristic arguments suggest this also is the case in the setting of Proposition 3.2.5, it is more difficult to verify due to the lack of projection conditions.

In Article III, Proposition 3.2.5 appears in a slightly different form. Namely, the measure is magnified along dyadic squares instead of balls centred at $\Pi(\mathtt{i})$, and the domination condition is assumed in the result. However, if one chooses to magnify along balls instead of dyadic squares, the domination condition is not required and the same proof gives the statement of Proposition 3.2.5. In Article III it is also assumed that $\dim_{\mathrm{H}} \mu > 1$. However, with minor modifications the proof goes through in the case $\dim_{\mathrm{H}} \mu \leq 1$ as well, but the conclusion is less interesting since almost all slices of μ are just Dirac measures.

4.2.1 Further questions

Given an iterated function system $\Phi = \{f_i\}_{i \in \Gamma^{\mathbb{N}}}$ with attractor K, the natural projection $\Pi : \Gamma^{\mathbb{N}} \to K$ gives a way to assign probability measures on K. Since self-similar measures arise as projections of Bernoulli measures on $\Gamma^{\mathbb{N}}$ through Π, a natural generalisation is to take a σ-ergodic measure μ on $\Gamma^{\mathbb{N}}$ and consider the probability measure $\mu \circ \Pi^{-1}$ supported on K.

In the presence of open set condition, it follows from the existing methods of [18, 22] that when Φ is self-similar and satisfies the open set condition, the measure $\mu \circ \Pi^{-1}$ as above is uniformly scaling and generates an S-ergodic tangent distribution. In light of Theorem 3.2.3, it is natural to ask whether this is also the case without any separation conditions.

Question 4.2.5. *Let $\Phi = \{f_i\}_{i \in \Gamma}$ be a self-similar iterated function system on $\mathbb{R}^d$, and let μ be a shift-ergodic measure on $\Gamma^{\mathbb{N}}$. Is the measure $\mu \circ \Pi^{-1}$ uniformly scaling?*

Regarding self-affine measures on the plane, a partial description of the scenery of the measure is given by Proposition 3.2.5. It would be interesting to find a complete description as in [12, 23].

Question 4.2.6. *Let μ be a self-affine measure on $\mathbb{R}^2$ with strong separation and irreducibility conditions. Does μ admit a unique tangent distribution at almost every x?*

4.3 Resonance of measures

Theorem 3.3.3 provides a complete solution for Research Question 3 for self-conformal measures on the line. The essential part in the proof is Theorem 3.2.3; using the existing methods of Hochman-Shmerkin [21] and Hochman [18], the result on resonance between self-conformal measures follows from the uniform scaling property.

Theorem 3.3.1 provides new information on resonance between dynamically defined measures in higher dimensions. It seems to be the only result in the literature that aims to address Research Question 3 in any dimension greater than one: While there is certainly a vast amount of literature on the size of convolutions or sumsets in higher dimensions, such as the celebrated inverse theorem of Hochman [19, 20], this primarily focuses on showing that the convolution $X * Y$ is *strictly larger* than X unless X and Y have a very special structure. On the other hand, the existing one-dimensional methods of studying resonance swiftly fall apart in higher dimensions, primarily because the set of n-dimensional subspaces of $\mathbb{R}^{2n}$ is substantially larger than $\mathbb{RP}^1$.

Like most of the recent results on resonance, the proof of Theorem 3.3.1 utilises the local entropy averages of Hochman-Shmerkin [21]. Because of this, the local structure of self-affine measures described by Proposition 3.2.5 plays a substantial role in the proof. The strategy is to reduce the problem of resonance between planar self-affine measures to resonance between slices and orthogonal projections of such measures, using Proposition 3.2.5. These are measures on the line, with a kind of 'dynamically self-similar' structure which allows us to relate their resonance to mixing properties of certain underlying dynamical systems, similarly as in the self-conformal case. However, these underlying systems are much more complicated than in the actual self-conformal situation, and the study of their mixing properties also requires ideas that have not appeared before in this context.

The assumptions of Theorem 3.3.1 are likely not optimal: As in Theorem 3.3.3 we expect that the assumption of the strong separation condition can be substantially weakened or removed altogether. The assumption of hyperbolicity is crucial in the proof, since self-affine measures with hyperbolicity enjoy the very special local structure of Proposition 3.2.5. Of course, it would be interesting to find an analogous result for self-affine measures without the hyperbolicity assumption: these are essentially self-similar. As discussed at the end of Section 1.1.3, the requirement of an irreducibility assumption seems to be necessary, although it is possible that assuming irreducibility of just one of the systems Φ and Ψ could be enough. Finally, in the proof the domination condition is crucial in deducing the arithmetic condition (8) from the geometric resonance (7), but the condition is likely just a by-product of the method.

4.3.1 Further questions

Theorem 3.3.1 states that resonance of self-affine measures (7) implies the arithmetic condition (8) on the defining iterated function systems. However, it is not clear if the condition (8) is at all possible in the presence of the irreducibility and hyperbolicity assumptions.

Question 4.3.1. *Does there exist a self-affine iterated function system on $\mathbb{R}^2$ which satisfies irreducibility, hyperbolicity and the condition* (8)*?*

The proof of Theorem 3.3.1 relies heavily on the measures μ and ν being self-affine. However, in light of Theorems 1.1.12 and 3.3.3, it seems reasonable to expect the result to extend to more general measures as well. The natural first step would be to consider natural projections of quasi-Bernoulli measures instead of self-affine ones.

Question 4.3.2. *Let $\Phi = \{\varphi_i\}_{i\in\Gamma}$ and $\Psi = \{\psi_j\}_{j\in\Lambda}$ be self-affine iterated function systems as in Theorem 3.3.1. Let $\bar{\mu}$ and $\bar{\nu}$ be quasi-Bernoulli measures on $\Gamma^{\mathbb{N}}$ and $\Lambda^{\mathbb{N}}$, and write $\mu := \bar{\mu} \circ \Pi^{-1}$ and $\nu := \bar{\nu} \circ \Pi^{-1}$. Suppose also that $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \nu$. Does the inequality*

$$\dim_{\mathrm{H}}(\mu * \nu) < \min\{2, \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu\}$$

imply that

$$\{\log|\lambda_1(A_i)| : \ i \in \Gamma\} \cup \{\log|\lambda_2(B_j)| : \ j \in \Lambda\} \subset \alpha\mathbb{N}$$

for some $\alpha \in \mathbb{R}$?

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I Pyörälä, A., Shmerkin, P., Suomala, V. & Wu, M. (2020). Covering the Sierpiński carpet by tubes. Accepted for publication in Israel Journal of Mathematics, available at arXiv:2006.00499

II Pyörälä, A. (2022). The scenery flow of self-similar measures with weak separation condition. *Ergodic Theory and Dynamical Systems*, 42(10), 3167–3190. http://doi.org/10.1017/etds.2021.86

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