

Improved versions of some Furstenberg type slicing Theorems for self-affine carpets

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Abstract

Let F be a Bedford-McMullen carpet defined by independent integer exponents. We prove that for every line $\ell \subseteq \mathbb{R}^2$ not parallel to the major axes,

$$\dim_H(\ell \cap F) \leq \max \left\{ 0, \frac{\dim_H F}{\dim^* F} \cdot (\dim^* F - 1) \right\}$$

and

$$\dim_P(\ell \cap F) \leq \max \left\{ 0, \frac{\dim_P F}{\dim^* F} \cdot (\dim^* F - 1) \right\}$$

where \dim^* is Furstenberg's star dimension (maximal dimension of microsets). This improves the state of art results on Furstenberg type slicing Theorems for affine invariant carpets.

1 Introduction

1.1 Background and main results

Let $n \geq 2$ be an integer and consider the n -fold map of the unit interval $T_n : [0, 1] \rightarrow [0, 1]$

$$T_n(x) = n \cdot x \pmod{1}. \quad (1)$$

We say that integers $m, n \geq 2$ are independent, and write $m \not\sim n$, if $\frac{\log m}{\log n} \notin \mathbb{Q}$. In the 1960's Furstenberg formulated several Conjectures aiming to capture the idea that if $m \not\sim n$ then expansions in base n and in base m should have no common structure. In 1967, Furstenberg [11] proved a landmark result of this form: If a closed subset of the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is invariant under both T_m and T_n then, assuming $m \not\sim n$, it is either finite or the entire torus. The measure theoretic analogue of this result, known as the $\times 2, \times 3$ Conjecture, remains open to this day: if μ is a Borel probability measure on \mathbb{T} , invariant under T_m and T_n , then it is a convex combination of the Lebesgue measure and a purely atomic measure.

Some of the aforementioned conjectures that Furstenberg proposed are more geometric in nature. One of them is known as the Slicing Conjecture: For $(u, t) \in \mathbb{R} \times \mathbb{R}$ let $\ell_{u,t}$ denote the planar line with slope u that intersects the y -axis at t (notice that we exclude from notation lines that are parallel to the y -axis). For a set $A \subseteq \mathbb{R}^d$ we denote its Hausdorff dimension by $\dim_H A$.

Conjecture 1.1. [12] *Let $\emptyset \neq X_1, X_2 \subseteq [0, 1]$ be closed sets that are invariant under T_m and T_n , respectively. If $m \not\sim n$ then for all $u \neq 0$ and $t \in \mathbb{R}$,*

$$\dim_H(X_1 \times X_2) \cap \ell_{u,t} \leq \max\{0, \dim_H X_1 + \dim_H X_2 - 1\}. \quad (2)$$

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This Conjecture is a geometric manifestation of the idea “if $m \not\sim n$ then expansions in base n and in base m have no common structure” in the sense that a slice of dimension larger than expected can be seen as some shared structure between X_1 and X_2 . To explain why the term on the right hand side of (2) is the expected bound, we recall the classical Marstrand slicing Theorem: For any set $X \subseteq \mathbb{R}^2$ and any fixed slope u ,

$$\dim_H X \cap \ell_{u,t} \leq \max\{0, \dim_H X - 1\} \text{ for Lebesgue almost every } t, \quad (3)$$

and this fails for any smaller value on the right hand side of (3). It is well known that for sets X_1 and X_2 as in Conjecture 1.1

$$\dim_H X_1 \times X_2 = \dim_H X_1 + \dim_H X_2.$$

So, what Furstenberg conjectured is that for $X = X_1 \times X_2$ as in Conjecture 1.1, Marstrand’s Theorem holds for *all* lines $\ell_{u,t}$ such that $u \neq 0$, that is, lines not parallel to the major axes.

Some progress towards Conjecture 1.1 was made by Furstenberg himself in [12], Wolff [21], and later by Feng, Huang and Rao [8]. In 2016 the Conjecture was proved simultaneously and independently by Shmerkin [19] and Wu [22]. In the case when $\dim_H X_1 + \dim_H X_2 \leq 1$, Yu [23] has simplified Wu’s arguments and obtained some quantitative improvement to (2). Austin [3] recently gave a new short proof of Conjecture 1.1.

The phenomenon predicted by Furstenberg was later shown to hold, in an appropriate sense, in a class of sets that strictly includes certain product sets as in Conjecture 1.1, called Bedford-McMullen carpets. These carpets are defined as follows: let m, n be integers greater than one. Let

$$\emptyset \neq D \subseteq \{0, \dots, m-1\} \times \{0, \dots, n-1\}$$

and define

$$F = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{m^k}, \sum_{k=1}^{\infty} \frac{y_k}{n^k} \right) : (x_k, y_k) \in D \right\}.$$

The set F is then called a Bedford-McMullen carpet with defining exponents m, n , and allowed digit set D . They are named after Bedford [5] and McMullen [18] who calculated their dimensions.

To recall the latest results about slicing Theorems for Bedford-McMullen carpets we need the notion¹ of star-dimension: For a set $A \subseteq [0, 1]^d$ we define

$$\dim^* A := \sup\{\dim_H M : M \text{ is a microset of } A\}$$

where microsets of A are limits in the Hausdorff metric on subsets of $[-1, 1]^2$ of “blow-up” of increasingly small balls about points in A (see e.g. [1, Section 2.2] for more details). This notion was introduced and studied by Furstenberg in [13]. Mackay [16] gave a closed combinatorial formula for $\dim^* F$ for any Bedford-McMullen carpet F in terms of m, n and D .

Returning to slicing theorems, Algom [1, Theorem 1.2] proved that for any Bedford-McMullen carpet F with independent exponents $m \not\sim n$

$$\dim^* F \cap \ell_{u,t} \leq \max\{\dim^* F - 1, 0\}, \quad \text{for all } (u, t) \in \mathbb{R}^2 \text{ such that } u \neq 0. \quad (4)$$

We remark that very recently Bárány, Käenmäki, and Yu [4], obtained similar results about slices through some non-carpet planar self-affine sets. Now, it is known [6, Chapter 4] that for any Bedford-McMullen carpet F , writing $\dim_B F$ for its box dimension and $\dim_P F$ for its packing dimension,

$$\dim_H F \leq \dim_P F = \dim_B F \leq \dim^* F \quad (5)$$

and that these inequalities are strict unless F is Ahlfors regular. So, for Ahlfors regular carpets the results of [1] are optimal for all notions of dimension previously discussed. However, in some

¹This is the same notion as Assouad dimension (see e.g. [9]), but for consistency with previous papers on the subject we work here with $*$ -dimension.

sense “most” Bedford-McMullen carpets are not Ahlfors regular [6, Chapter 4]. It is thus the main purpose of this paper to improve (4) for both the Hausdorff and the packing dimension of slices in the non-Ahlfors regular setting, and to relate them to the corresponding dimensions of the underlying carpet. Here is our main result:

Theorem 1.2. *Let F be a Bedford-McMullen carpet with exponents (m, n) . If $m \not\sim n$ then for all $u \neq 0$ and $t \in \mathbb{R}$,*

1. $\dim_H(\ell \cap F) \leq \max \left\{ 0, \frac{\dim_H F}{\dim^* F} \cdot (\dim^* F - 1) \right\}$.
2. $\dim_P(\ell \cap F) \leq \max \left\{ 0, \frac{\dim_P F}{\dim^* F} \cdot (\dim^* F - 1) \right\}$.

Some remarks are in order: First, since for non Ahlfors regular carpets the inequalities in (5) become strict, Theorem 1.2 does indeed improve (4). Secondly, it is a natural question (see e.g. [10, Question 8.3]) if Theorem 1.2 may be upgraded to a Marstrand-like result of the form

$$\dim_H(\ell_{u,t} \cap F) \leq \max\{0, \dim_H F - 1\}, \quad \text{for all } (u, t) \in \mathbb{R}^2 \text{ such that } u \neq 0.$$

Our methods currently fall short of proving such a strong statement. This will be explained in the next Section, where we outline the proof of Theorem 1.2.

1.2 Sketch of the proof of Theorem 1.2

Fix a Bedford-McMullen carpet F with digit set D and exponents $m \not\sim n$. We always assume, without loss of generality, that $m > n$. This implies that $\theta := \frac{\log n}{\log m} \notin \mathbb{Q}$ is in $(0, 1)$. Let $\ell_0 \subseteq \mathbb{R}^2$ be an affine line with slope m^{u_0} where $u_0 \in [0, 1)$, which may be assumed without any loss of generality. We want to bound $\dim_H F \cap \ell_0$ - the bound for $\dim_P F \cap \ell_0$ is obtained in a similar manner.

For every $u \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, we define a map $\Phi_u : [0, 1]^2 \rightarrow [0, 1]^2$ via

$$\Phi_u(x, y) = \begin{cases} (T_m(x), T_n(y)) & \text{if } u \in [1 - \theta, 1) \\ (x, T_n(y)) & \text{if } u \in [0, 1 - \theta). \end{cases}$$

Let $R_\theta : \mathbb{T} \rightarrow \mathbb{T}$ denote the translation by θ map

$$R_\theta(t) = t + \theta \pmod{1}.$$

For a measure μ on $[0, 1]^2$, a point $z = (x, y) \in \text{supp}(\mu)$, and $u \in \mathbb{T}$ we define a “magnifying” map via

$$M(\mu, (x, y), u) = \begin{cases} (\mu^{\mathcal{D}_m(x) \times \mathcal{D}_n(y)}, \Phi_u(z), R_\theta(u)) & \text{if } u \in [1 - \theta, 1) \\ (\mu^{[0, 1] \times \mathcal{D}_n(y)}, \Phi_u(z), R_\theta(u)) & \text{if } u \in [0, 1 - \theta) \end{cases}$$

where $\mathcal{D}_p(w)$ is the unique cell of the partition of \mathbb{R}

$$\mathcal{D}_p = \left\{ \left[\frac{i}{p}, \frac{i+1}{p} \right), \quad i \in \mathbb{Z} \right\}$$

that contains w , and the measure $\mu^{\mathcal{D}_m(x) \times \mathcal{D}_n(y)}$ is the push-forward via $T_m \times T_n$ of the conditional measure of μ on $\mathcal{D}_m(x) \times \mathcal{D}_n(y)$. The measure $\mu^{[0, 1] \times \mathcal{D}_n(y)}$ is defined similarly.

By Frostman’s Lemma we may find a Borel probability measure μ_0 supported on $\ell_0 \cap F$ such that $\dim \mu_0 = \dim_H \ell_0 \cap F - o(1)$ (see Section 2.1 for a discussion on dimension theory for measures). Roughly speaking, we pick a μ_0 typical point (x_0, y_0) and find a sequence N_j such that:

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} \delta_{M^k(\mu_0, (x_0, y_0), u_0)}$$

converges to an M -invariant distribution Q , such that for Q a.e.- ω the measure μ_ω is supported on a product set X_ω (which is a microset of F), and:

- (i) For every ω we have $\dim_H X_\omega \leq \dim^* F$.
- (ii) $\dim \mu_0 \leq \int \dim \mu_\omega dQ(\omega)$.
- (iii) $\int \dim_H X_\omega dQ(\omega) \leq \dim_H F$.
- (iv) Let $Q = \int Q_\xi d\tau(\xi)$ denote the ergodic decomposition of Q . For τ -a.e. ξ , if Q_ξ is supported on measures with strictly positive dimension, then for Q_ξ a.e. ω we have

$$\dim \mu_\omega \leq \dim_H X_\omega - 1$$

Property (i) is an easy consequence of the fact that X_ω is a microset of F . Property (ii) is a general feature of CP distributions (see Section 2.5) - and Q is such a distribution. Properties (iii)-(iv) are the main innovations of this paper: For Property (iii), we first note that it is not a trivial consequence of our construction, since in general X_ω is not a subset of F . Thus, for property (iii) we rely, among other things, on a concise choice of the sequence N_j and general properties of entropy. Part (iv) relies on a geometric consequence of Sinai's factor Theorem proved by Wu [22, Theorem 6.1]. However, new ideas are required since Wu's original argument for deducing (iv) from this result as in [22] does not apply in our setting, as the measures arising from M -orbits are not all supported on the same set X_ω . We also remark that in practice we will prove our required bounds by studying the entropy of certain measures on X_ω rather than considering X_ω itself. This can be seen as another reason why our approach gives more refined results than (4): In [1] Algom worked directly with the microset X_ω that usually satisfies $\Pi_2(X_\omega) = \Pi_2(F)$ where $\Pi_2(x, y) = y$, which resulted with the loss of some information.

Once a distribution Q satisfying the properties above has been produced, an elementary optimization argument yields the inequality as in Theorem 1.2 part (1).

Finally, we remark that if for Q a.e.- ω the measure μ_ω has strictly positive dimension, then (i)-(iv) above would yield the strong Marstrand-type inequality

$$\dim_H F \cap \ell_0 \leq \max\{0, \dim_H F - 1\}$$

We do not know, however, if it is possible to construct such a distribution.

Organization In Section 2 we survey some tools we shall use from dimension theory, entropy, and the theory of CP distributions. We proceed to prove Theorem 1.2 part (2) in Section 3 and then, using a similar scheme, Theorem 1.2 part (1) in Section 4.

Notation We use the notation $o_\epsilon(1)$ to indicate a quantity going to 0 as the positive $\epsilon \rightarrow 0$, and similarly $o_l(1)$ stands for a quantity going to 0 as the integer $l \rightarrow \infty$.

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2 Preliminaries

2.1 Hausdorff and packing dimensions of sets and measures

Recall that for a set A in a compact metric space X , we denote its Hausdorff dimension by $\dim_H A$ and its packing dimension by $\dim_P A$. For an exposition on these notions see Mattila's book [17]. Also, let $\mathcal{P}(X)$ denote the collection of all Borel probability measures on X .

Next, let $\mu \in \mathcal{P}(X)$. For every $x \in \text{supp}(\mu)$ we define the lower pointwise dimension of μ at x as

$$\dim(\mu, x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}$$

where $B(x, r)$ denotes the closed ball of radius r about x . We also define the upper pointwise dimension of μ at x as

$$\text{Dim}(\mu, x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

The measure μ is called exact dimensional if the lower and upper pointwise dimensions of μ coincide and it is constant almost surely. In this case we denote this quantity by $\dim \mu$.

Frostman's Lemma [7, Chapter 10] allows one to find measures on a set A that approximate its dimension:

$$\dim_H A = \sup\{s : \exists \mu \in \mathcal{P}(A) \text{ such that } \dim(\mu, x) \geq s \text{ almost surely}\};$$

$$\dim_P A = \sup\{s : \exists \mu \in \mathcal{P}(A) \text{ such that } \text{Dim}(\mu, x) \geq s \text{ almost surely}\}.$$

Finally, let A be a bounded set. For every $r > 0$ let $\mathcal{N}_r(A)$ denote the minimal number of sets of diameter less than r required to cover the set A . Then

$$\dim_B(A) = \lim_{r \rightarrow 0} \frac{\mathcal{N}_r(A)}{-\log r}$$

provided the limit exists. Otherwise, the upper box dimension $\overline{\dim}_B(A)$ is defined as the corresponding lim sup.

2.2 Entropy, partitions, and approximate squares

Let X be a compact metric space, $\mu \in \mathcal{P}(X)$, and let \mathcal{A} denote a finite measurable partition of X . Recall that the Shannon entropy of μ with respect to \mathcal{A} is defined as

$$H(\mu, \mathcal{A}) = - \sum_{A \in \mathcal{A}} \mu(A) \cdot \log \mu(A)$$

with the convention $0 \log 0 = 0$. The following facts about Shannon entropy are standard:

Proposition 2.1. *Let $\mu \in \mathcal{P}(X)$ and let \mathcal{A} be a finite measurable partition.*

1. *General upper bound: $H(\mu, \mathcal{A}) \leq \log |\{A \in \mathcal{A} : A \cap \text{supp}(\mu) \neq \emptyset\}|$.*
2. *Entropy is concave: Suppose we have a disintegration of μ , given by $\mu = \int \mu_\omega dQ(\omega)$. Then*

$$H(\mu, \mathcal{A}) \geq \int H(\mu_\omega, \mathcal{A}) dQ(\omega).$$

3. *The Gibbs inequality: Let $n \in \mathbb{N}$ and let (p_1, \dots, p_n) and (q_1, \dots, q_n) be two probability vectors. Then*

$$- \sum_{i=1}^n p_i \log p_i \leq - \sum_{i=1}^n p_i \log q_i$$

with equality if and only if $p_i = q_i$.

Next, let $m \geq 2$. For every integer $p \geq 0$ let \mathcal{D}_p^d denote the m^p -adic partition of \mathbb{R}^d , that is,

$$\mathcal{D}_p^d = \left\{ \prod_{i=1}^d \left[\frac{z_i}{m^p}, \frac{z_i+1}{m^p} \right) : (z_1, \dots, z_d) \in \mathbb{Z}^d \right\}.$$

We shall omit the superscript d from our notation when its value is clear from context.

Also, given integers $m > n$, let $\theta := \frac{\log n}{\log m}$. Recall that $R_\theta : \mathbb{T} \rightarrow \mathbb{T}$ is the group rotation

$$R_\theta(u) = u + \theta \pmod{1}. \tag{6}$$

For every $k \in \mathbb{N}$ and $u \in \mathbb{T}$ we define the integer

$$\mathcal{R}(k, u) := |\{0 \leq i \leq k : R_\theta^i(u) \in [1 - \theta, 1)\}|. \tag{7}$$

Now, for every $u \in \mathbb{T}$ we define a sequence of partitions of $[0, 1]^2$, called approximate squares, as follows: for every $k \in \mathbb{N}$ we let

$$\mathcal{A}_k^u := \mathcal{D}_{m^{\mathcal{R}(k, u)}} \times \mathcal{D}_{n^k}.$$

The following Lemma is standard:

Lemma 2.2. [1, Claim 4.2] *There exists some $C > 1$ such that for every $u \in \mathbb{T}$ and every $z \in [0, 1]^2$:*

1. $[\theta \cdot k] - C \leq \mathcal{R}(k, u) \leq [\theta \cdot k] + C$.
2. *For every z*

$$C^{-1} \frac{1}{n^k} \leq \text{diam}(\mathcal{A}_k^u(x)) \leq C \frac{1}{n^k}$$

where $\mathcal{A}_k^u(z)$ is the atom of the partition \mathcal{A}_k^u that contains z .

2.3 Bedford-McMullen carpets

Recall the definition of a Bedford-McMullen carpet F with defining exponents m, n and allowed digit set D from Section 1.1. We will always assume, without loss of generality, that $m > n$. We remark that F is a self-affine set generated by an IFS consisting of maps whose linear parts are diagonal matrices. Let $\Pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the projection to the second coordinate, that is, $\Pi_2(x, y) = y$. For every $j \in \Pi_2(F)$, let

$$D_j = \{0 \leq i \leq m-1 : (i, j) \in D\}$$

and

$$a(j) := |D_j|. \tag{8}$$

Recall that we have denoted

$$\theta = \frac{\log n}{\log m} \in (0, 1).$$

The following Theorem, due to Bedford and McMullen independently, describes the various dimensions of F .

Theorem 2.3. [5, 18] *Let F be a Bedford-McMullen carpet. Then:*

1. $\dim_H F = \frac{\log(\sum_{j \in \Pi_2(D)} a(j)^\theta)}{\log n}$.
2. $\dim_B F = \dim_P F = \frac{\log |\Pi_2(D)|}{\log n} + \frac{\log \frac{|D|}{|\Pi_2(D)|}}{\log m}$. *In particular, $\dim_B F$ exists as a limit.*

The following Lemma describes what happens as we zoom into F via the approximate squares \mathcal{A}_k^u defined in Section 2.2. For $(x, y) \in F$, write

$$(x, y) = \left(\sum_{k=1}^{\infty} \frac{x_k}{m^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right), \quad \text{where } (x_k, y_k) \in D$$

for the corresponding base m and base n expansions of x and y , respectively, that bare witness to $(x, y) \in F$. Notice that these expansions may not be unique in general. In this case, if say x has two such base m expansions, we choose the one that ends with 0's (the lexicographically larger one). Let $\Pi_1(x, y) = x$ denote the projection to the first coordinate.

Lemma 2.4. [2, Section 7] *Let F be a Bedford-McMullen carpet and let $(x, y) \in F$. Writing*

$$(x, y) = \left(\sum_{k=1}^{\infty} \frac{x_k}{m^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right), \quad \text{where } (x_k, y_k) \in D,$$

we have, for every $k \in \mathbb{N}$ and $u \in \mathbb{T}$, that the set

$$T_m^{\mathcal{R}(k, u)} \circ \Pi_1(\mathcal{A}_k^u(x, y) \cap F)$$

is contained in

$$\left\{ \sum_{i=1}^{\infty} \frac{b_i}{m^i} : b_i \in \{0, \dots, m-1\} \text{ and for } 1 \leq i \leq \frac{(1-\theta)k}{\theta} - o_k(C), \quad b_i \in D_{y_{i+\mathcal{R}(k, u)}} \right\}$$

where C and $\mathcal{R}(k, u)$ are as in Lemma 2.2.

Lemma 2.4 can also be recovered from the analysis of Käenmäki, Ojala, and Rossi [15].

2.4 Dynamical systems

In this paper a measure preserving system is a quadruple (X, \mathcal{B}, T, μ) , where X is a compact metric space, \mathcal{B} is the Borel sigma algebra, and $T : X \rightarrow X$ is a measure preserving map: T is Borel measurable and $T\mu = \mu$. Since we always work with the Borel sigma-algebra, we shall usually just write (X, T, μ) . When the space X is clear from context we shall sometimes just write (T, μ) . We also recall that a dynamical system is ergodic if and only if the only invariant sets are trivial. That is, if $B \in \mathcal{B}$ satisfies $T^{-1}(B) = B$ then $\mu(B) = 0$ or $\mu(B) = 1$.

A class of examples is given by symbolic dynamical systems: For $n \in \mathbb{N}$ at least 2, let $X = \{0, \dots, n-1\}^{\mathbb{N}}$ and $T = \sigma$ be the shift map $\sigma : [n]^{\mathbb{N}} \rightarrow [n]^{\mathbb{N}}$ defined by $\sigma(\omega) = \xi$ where $\xi(k) = \omega(k+1)$ for every k . We equip this space with the compatible compact metric d defined by

$$d(\omega, \xi) = \left(\frac{1}{n}\right)^{\min\{k: \omega_k \neq \xi_k\}}. \quad (9)$$

We will have occasion to use the ergodic decomposition Theorem: Let (X, T, μ) be a dynamical system. Then there is a map $X \rightarrow \mathcal{P}(X)$, denoted by $\mu \mapsto \mu_x$, such that:

1. The map $x \mapsto \mu_x$ is measurable with respect to the sub σ -algebra \mathcal{I} of T invariant sets.
2. $\mu = \int \mu_x d\mu(x)$.
3. For μ almost every x , μ_x is T invariant and ergodic. The measure μ_x is called the ergodic component of x .

Recall that if $\mu \in \mathcal{P}(X)$ is a T invariant measure we may define its metric entropy with respect to T , a quantity that we shall denote by $h(\mu, T)$. As there is an abundance of excellent texts on entropy theory (e.g. [20]), we omit a discussion on entropy here. We do recall that entropy is affine in the sense that if μ, ν, η are T invariant measures such that for some $p \in (0, 1)$ we have $p \cdot \nu + (1-p) \cdot \eta = \mu$ then

$$h(\mu, T) = p \cdot h(\nu, T) + (1-p)h(\eta, T).$$

Finally, we will consider dynamical systems of the form $([0, 1], \mu, T_n)$ (recall (1)). In this case we have the following useful result, which is an immediate consequence of the Shannon-McMillan-Breiman theorem and Billingsley's lemma:

Theorem 2.5. *Let $\mu \in \mathcal{P}([0, 1])$ be a T_n invariant and ergodic measure. Then μ is exact dimensional and*

$$\dim \mu = \frac{h(\mu, T_n)}{\log n}.$$

2.5 CP distributions with respect to approximate squares

The theory of CP distributions that we discuss in this section originated implicitly with Furstenberg in [12]. It was then reintroduced by Furstenberg in [13], and has since been used by many authors, notably by Hochman and Shmerkin in [14]. In particular, CP distributions played a crucial role in both author's works [1, 22] about slicing Theorems. Here we will only discuss a special case of this machinery, using the approximate squares $\mathcal{A}_k^u(x)$ from Section 2.2 as our partitions.

As is standard in this context, for a compact metric space X the elements of $\mathcal{P}(X)$ are called measures, and the elements of $\mathcal{P}(\mathcal{P}(X))$, measures on the space of measures, are called distributions.

Fix integers $m > n > 1$ and recall that $\theta := \frac{\log n}{\log m}$. Recall the definition of the approximate squares \mathcal{A}_k^u from Section 2.2. For $\mu \in \mathcal{P}([0, 1]^2)$, $u \in \mathbb{T}$, $k \in \mathbb{N}$, and $x \in \text{supp}(\mu)$, recalling (7), let

$$\mu^{\mathcal{A}_k^u(x)} = \left(T_m^{\mathcal{R}(k, u)} \times T_n^k\right) \left(\mu_{\mathcal{A}_k^u(x)}\right), \quad \text{where } \mu_{\mathcal{A}_k^u(x)}(B) = \frac{\mu(\mathcal{A}_k^u(x) \cap B)}{\mu(\mathcal{A}_k^u(x))}.$$

That is, $\mu^{\mathcal{A}_k^u(x)}$ is the push-forward of the conditional measure of μ on $\mathcal{A}_k^u(x)$ via the map

$$T_m^{\mathcal{R}(k,u)} \times T_n^k.$$

Now, for every $u \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, we define a map $\Phi_u : [0, 1]^2 \rightarrow [0, 1]^2$, by

$$\Phi_u(x, y) = \begin{cases} (T_m(x), T_n(y)) & \text{if } u \in [1 - \theta, 1) \\ (x, T_n(y)) & \text{if } u \in [0, 1 - \theta). \end{cases} \quad (10)$$

We also define

$$\Omega = \{(\mu, z) : \mu \in \mathcal{P}([0, 1]^2), z \in \text{supp}(\mu)\}. \quad (11)$$

Finally, we define a ‘‘magnification’’ map $M : \Omega \times \mathbb{T} \rightarrow \Omega \times \mathbb{T}$ via

$$M(\mu, (x, y), u) = \begin{cases} (\mu^{\mathcal{D}_m(x) \times \mathcal{D}_n(y)}, \Phi_u(x, y), R_\theta(u)) & \text{if } u \in [1 - \theta, 1) \\ (\mu^{[0,1] \times \mathcal{D}_n(y)}, \Phi_u(x, y), R_\theta(u)) & \text{if } u \in [0, 1 - \theta). \end{cases}$$

Recall that R_θ is the rotation by θ map (see (6)). Notice that for every $k \in \mathbb{N}$ and every $(\mu, z, u) \in \Omega \times \mathbb{T}$, the first coordinate of $M^k(\mu, z, u)$ is exactly $\mu^{\mathcal{A}_k^u(z)}$.

Definition 2.6. *A CP distribution Q with respect to the partition into approximate squares is a distribution $Q \in \mathcal{P}(\Omega \times \mathbb{T})$ such that:*

1. Q is M invariant.
2. The marginal distribution $Q_{1,2}$ of Q on the first two coordinates (μ, x) of Ω is given by choosing first μ according to Q_1 (the marginal of Q on $\mathcal{P}([0, 1]^2)$) and then choosing x according to μ .

An ergodic CP distribution is a CP distribution Q that is M -ergodic.

Note that a distribution $Q \in \mathcal{P}(\Omega \times \mathbb{T})$ is a CP-distribution in the sense of the above definition if and only if its marginal $Q_{1,2}$ is a CP-distribution in the sense of [13, 14]. We shall sometimes abuse notation by referring to Q_1 , the marginal of Q on $\mathcal{P}([0, 1]^2)$, as Q . In the following Theorem we group a few facts about CP distributions. The first three were proved by Furstenberg [13]. The last one can be found in [22, Proposition 3.7, Lemma 7.3]. For a CP distribution Q we define its dimension by

$$\dim Q := \int \dim \mu dQ(\mu, x, u).$$

Theorem 2.7. [13, 22] *The following statements hold true:*

1. The ergodic components of a CP distribution are, almost surely, themselves ergodic CP distributions.
2. Let Q be an ergodic CP distribution. Then Q almost every measure μ is exact dimensional, and $\dim \mu = \dim Q$.
3. Let Q be a CP distribution. Then $Q_2 = \int \mu dQ(\mu, x, u)$, where Q_2 is the marginal of Q on $[0, 1]^2$ (its second coordinate).
4. Let Q be an ergodic CP distribution. For every $\epsilon > 0$ there exists some $r_0 = r_0(\epsilon) > 0$ such that:

For every $r < r_0$, $u \in \mathbb{T}$, $k \in \mathbb{N}$, and $l \in \mathbb{N}$ large enough, we have for Q almost every (μ, z, u)

$$\inf_{y \in \mathbb{R}^2} H(\mu^{\mathcal{A}_k^u(z)}|_{\mathbb{R}^2 \setminus B(y,r)}, \mathcal{D}_{n^l}) \geq H(\mu^{\mathcal{A}_k^u(z)}, \mathcal{D}_{n^l}) - o_\epsilon(1) \cdot l \cdot \log n = l \cdot \log n \cdot (\dim \mu - o_\epsilon(1)).$$

Finally, we shall require a Theorem of Hochman and Shmerkin from [14]. Though it is not stated this way in [14], it nonetheless follows directly from their local entropy averages machinery [14, Section 4.2], and their discussion in [14, Section 7.5]:

Theorem 2.8. [14] Let $\mu \in \mathcal{P}([0, 1]^2)$ be a measure such that for all $k \in \mathbb{N}$ and $u \in \mathbb{T}$,

$$\mu(\partial A) = 0 \text{ for every } A \in \mathcal{A}_k^u.$$

- (1) Suppose that $\text{Dim}(\mu, x) \geq s$ for μ -a.e. x . Then for μ -a.e. x and every $u \in \mathbb{T}$ there exists a subsequence N_j such that

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} \delta_{M^k(\mu, x, u)} \rightarrow Q$$

where Q is a CP distribution with $\dim Q \geq s$.

- (2) Suppose that $\dim(\mu, x) \geq s$ for μ -a.e. x . Then for μ -a.e. x , for every $u \in \mathbb{T}$ and every subsequence N_j , there is a further subsequence $N_{j'}$ such that

$$\frac{1}{N_{j'}} \sum_{k=0}^{N_{j'}-1} \delta_{M^k(\mu, x, u)} \rightarrow Q$$

where Q is a CP distribution with $\dim Q \geq s$.

Remark 2.9. (1) Notice that the difference between (1) and (2) in Theorem 2.8 is that in part (1) we have to follow a specific subsequence to get Q , whereas in part (2) every subsequence will have a further subsequence that will yield such a distribution Q .

- (2) To see that the distributions Q as in Theorem 2.8 are M -invariant, note that their marginal on the u coordinate must be the Lebesgue measure on $[0, 1]$, so M acts continuously on their support.

3 On the proof of Theorem 1.2 part (2)

Let F be a Bedford-McMullen carpet with exponents $m > n$ and digits D such that $m \not\sim n$. Write $\theta := \frac{\log n}{\log m}$. Let ℓ_0 be a line not parallel to the major axes. Then the slope of ℓ_0 can be written as $C \cdot m^{u_0} \neq 0$ for certain $u_0 \in [0, 1)$ and some $C \neq 0$. We assume without loss of generality that $C = 1$.

From this point forward We work in \mathbb{T} and \mathbb{T}^2 , so that the maps T_m, T_n from (10) become continuous. This means that we think of F and $\ell_0 \cap F$ as subsets of \mathbb{T}^2 rather than $[0, 1]^2$. Note that since \mathbb{T}^2 and $[0, 1]^2$ are locally bi-Lipschitz equivalent, the dimension of $\ell_0 \cap F$ as a subset of \mathbb{T}^2 is equal to its dimension as a subset of $[0, 1]^2$.

Let

$$\gamma_0 := \dim_P \ell_0 \cap F$$

and let $\gamma < \gamma_0$. We will show that

$$\gamma \leq \max \left\{ 0, \frac{\dim_P F}{\dim^* F} \cdot (\dim^* F - 1) \right\}.$$

It is clear that we may assume $\gamma > 0$.

By Frostman's Lemma we may find a probability measure $\mu_0 \in \mathcal{P}(\ell_0 \cap F)$ such that

$$\text{Dim}(\mu_0, z) \geq \gamma, \quad \text{for } \mu_0 \text{ almost every } z.$$

In particular, μ_0 is continuous (has no atoms). By Theorem 2.8 part (1) there is a point $z_0 \in \ell_0 \cap F$ and a subsequence N_j such that

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} \delta_{M^k(\mu_0, z_0, u_0)} \rightarrow Q \tag{12}$$

where Q is a CP distribution with

$$\dim Q \geq \gamma. \quad (13)$$

Next, write

$$z_0 = (x_0, y_0) = \left(\sum_{k=1}^{\infty} \frac{x_k}{m^k}, \sum_{k=1}^{\infty} \frac{y_k}{n^k} \right), \quad (x_k, y_k) \in D.$$

Notice that since μ_0 is continuous, we may assume both $x_0, y_0 \notin \mathbb{Q}$, so that this representation is unique. Now, let

$$\omega_0 = (y_1, y_2, \dots) \in (\Pi_2 D)^{\mathbb{N}} \subseteq \{0, \dots, n-1\}^{\mathbb{N}}.$$

Then, by perhaps moving to a further subsequence, we assume that there are σ invariant measures $\nu, \eta, \rho \in \mathcal{P}((\Pi_2 D)^{\mathbb{N}}) \subseteq \mathcal{P}(\{0, \dots, n-1\}^{\mathbb{N}})$ such that:

$$\frac{1}{[N_j \cdot \theta]} \sum_{k=1}^{[N_j \cdot \theta]} \delta_{\sigma^k(\omega_0)} \rightarrow \nu, \quad (14)$$

$$\frac{1}{N_j - [N_j \cdot \theta]} \sum_{k=[N_j \cdot \theta]+1}^{N_j} \delta_{\sigma^k(\omega_0)} \rightarrow \eta, \quad (15)$$

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \delta_{\sigma^k(\omega_0)} \rightarrow \rho. \quad (16)$$

Using (14), (15) and (16), it is readily checked that $\rho = \theta \cdot \nu + (1 - \theta) \cdot \eta$.

The following Theorem is the key to the proof of Theorem 1.2 part (2). Recall the definition of $a(j)$ for $j \in \Pi_2(D)$ from (8).

Theorem 3.1. *Let $\lambda = Q(\{\mu : \dim \mu > 0\})$. Then:*

- (1) $\gamma \leq \lambda \cdot (\dim^* F - 1)$;
- (2)

$$\gamma + \lambda \leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n},$$

where for $j \in \{0, \dots, n-1\}$ we write $[j] = \{\omega \in \{0, \dots, n-1\}^{\mathbb{N}} : \omega_1 = j\}$;

- (3) $\gamma \leq \dim_P F - \lambda$.

Theorem 3.1 implies Theorem 1.2 part (2): Indeed, combining parts (1) and (3) we get

$$\gamma \leq \min\{\lambda \cdot (\dim^* F - 1), \dim_P F - \lambda\}.$$

An elementary optimization argument shows that for each $0 \leq \lambda \leq 1$, the right hand term of the above inequality is always bounded by the following quantity

$$\max \left\{ 0, \frac{\dim_P F}{\dim^* F} \cdot (\dim^* F - 1) \right\}.$$

Hence we obtain the desired conclusion of Theorem 1.2 part (2).

We thus proceed to prove Theorem 3.1: First, we will establish parts (1) and (2). We will then show that part (3) follows from part (2).

3.1 On the proof of Theorem 3.1

3.1.1 Preliminaries

First, we extend the definition of the map M from Definition 2.6: For every $u \in \mathbb{T}$ define a map $\sigma_u : \{0, \dots, n-1\}^{\mathbb{N}} \rightarrow \{0, \dots, n-1\}^{\mathbb{N}}$ via

$$\sigma_u(\omega) = \begin{cases} \sigma(\omega) & \text{if } u \in [1-\theta, 1) \\ \omega & \text{if } u \in [0, 1-\theta). \end{cases}$$

Recall (11) for the definition of the space Ω . We define a new map

$$T : \Omega \times \mathbb{T} \times \{0, \dots, n-1\}^{\mathbb{N}} \rightarrow \Omega \times \mathbb{T} \times \{0, \dots, n-1\}^{\mathbb{N}}$$

via

$$T(\mu, z, u, \omega) = (M(\mu, z, u), \sigma_u(\omega)).$$

Recall that the sequence $\{N_j\}$ was chosen such that (14), (15) and (16) hold. By perhaps moving to a further subsequence of $\{N_j\}$ and using the irrationality of θ , we may assume that

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} \delta_{T^k(\mu_0, z_0, u_0, \omega_0)} \rightarrow R, \text{ and } R \text{ is } T \text{ invariant.} \quad (17)$$

To see why we may assume R is T invariant, we recall Remark 2.9 part (2). Notice that by (12), we have $R_{1,2,3} = Q$, where we recall that $R_{1,2,3}$ denotes the marginal of R on the first 3 coordinates. Also, recall that for every $k \in \mathbb{N}$ and $u \in \mathbb{T}$ we write

$$\mathcal{R}(k, u) := |\{0 \leq i \leq k : R_\theta^i(u) \in [1-\theta, 1)\}|$$

and by Lemma 2.2 there is some uniform constant $C > 0$ such that

$$[\theta \cdot k] - C \leq \mathcal{R}(k, u_0) \leq [\theta \cdot k] + C, \quad \forall k \in \mathbb{N}, u \in \mathbb{T}. \quad (18)$$

Next, recall that for $j \in \Pi_2(D)$ we defined

$$D_j = \{i : (i, j) \in D\}.$$

For $\omega \in (\Pi_2 D)^{\mathbb{N}}$ we denote

$$A(\omega) = \left\{ \sum_{k=1}^{\infty} \frac{b_k}{m^k} : b_k \in D_{\omega_k} \right\}. \quad (19)$$

The following Lemma gives a description of R typical points.

Lemma 3.2. *For R almost every (μ, z, u, ω) we have:*

1. *The measure μ is supported on a line with slope m^u .*
2. $\Pi_1(\text{supp}(\mu)) \subseteq A(\omega)$.

Proof. Fix $(\mu, z, u, \omega) \in \text{supp}(R)$. By (17), there exists a sequence k_p such that

$$T^{k_p}(\mu_0, z_0, u_0, \omega_0) \rightarrow (\mu, z, u, \omega)$$

By the definition of T the first coordinate of $T^{k_p}(\mu_0, z_0, u_0, \omega_0)$ is $\mu_0^{\mathcal{A}_{k_p}^{u_0}(z_0)}$, and the fourth coordinate is $\sigma^{\mathcal{R}(u_0, k_p)}(\omega_0)$. So, applying Lemma 2.4 we have that

$$\Pi_1 \left(\text{supp} \left(\mu_0^{\mathcal{A}_{k_p}^{u_0}(z_0)} \right) \right)$$

is contained in

$$\left\{ \sum_{i=1}^{\infty} \frac{b_i}{m^i} : b_i \in \Pi_2 D, \quad \text{and for } 1 \leq i \leq \frac{(1-\theta)k_p}{\theta} - o_{k_p}(C), \quad b_i \in D_{\sigma^{\mathcal{R}(k,u_0)}(\omega_0)(i)} \right\}$$

where by $\sigma^{\mathcal{R}(k,u_0)}(\omega_0)(i)$ we mean the i -th coordinate of $\sigma^{\mathcal{R}(k,u_0)}(\omega_0)$, and C is the constant from Lemma 2.2. Since $\sigma^{\mathcal{R}(k_p,u_0)}(\omega_0) \rightarrow \omega$, taking $p \rightarrow \infty$ yields part (2) of the Lemma.

Part (1) is a consequence of the fact that for every $p \in \mathbb{N}$ the measure $\mu_0^{\mathcal{A}_{k_p}^{u_0}(z_0)}$ is supported on a line with slope $m^{R_{\theta}^{k_p}(u_0)}$, and since $R_{\theta}^{k_p}(u_0) \rightarrow u$. \square

We also have the following estimate. Recall that for $r > 0$, $\mathcal{N}_r(A)$ denotes the minimal number of sets of diameter $\leq r$ required to cover the bounded set A .

Lemma 3.3. *Let $q \in \mathbb{N}$ be large. Then*

$$\int \frac{\log \mathcal{N}_{m^{-q}}(A(\omega))}{q \log m} dR(\mu, z, u, \omega) = \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + o_q(1).$$

Proof. First, notice that by definition of A_{ω} (recall (19)), and since $a(j) = |D_j|$ for all $j \in \Pi_2(D)$,

$$\prod_{k=1}^q a(\omega_k) \leq \mathcal{N}_{m^{-q}}(A(\omega)) \leq \prod_{k=1}^q a(\omega_k) \cdot 5$$

where the 5 factor arises from the possible presence of elements with multiple base m representation in $A(\omega)$. Therefore,

$$\frac{\log \mathcal{N}_{m^{-q}}(A(\omega))}{q \log m} = \frac{\sum_{k=1}^q \log a(\omega_k)}{q \log m} + o_q(1).$$

Then by (17) and the previous equation it suffices to show that

$$\lim_{j \rightarrow \infty} \frac{1}{N_j} \sum_{k=1}^{N_j} \frac{\sum_{i=1}^q \log a(\omega_{\mathcal{R}(k,u_0)+i})}{q \log m} = \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + o_q(1). \quad (20)$$

To this end, we first notice that by the definition (14) of ν , we have

$$\lim_{j \rightarrow \infty} \frac{1}{[\theta \cdot N_j]} \sum_{k=1}^{[N_j \cdot \theta]} \frac{\log a(\omega_k)}{\log m} = \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m}. \quad (21)$$

Also, assuming $q \in \mathbb{N}$ is large and $p > q$ we have, by (18),

$$|\{k \in \mathbb{N} : \mathcal{R}(k, u_0) + 1 \leq p \leq \mathcal{R}(k, u_0) + q\}| = \frac{q}{\theta}(1 + o_q(C))$$

and consequently,

$$\sum_{k=1}^{N_j} \sum_{i=1}^q \log a(\omega_{\mathcal{R}(k,u_0)+i}) = \frac{q}{\theta}(1 + o_q(C)) \sum_{k=1}^{[N_j \cdot \theta]} \log a(\omega_k) + o_{N_j}(1).$$

Dividing the latter equation by $N_j \cdot q \cdot \log m$ and taking $j \rightarrow \infty$, we see via (21) that (20) holds true. This implies the Lemma. \square

Finally, let $\Xi : \{0, \dots, n-1\}^{\mathbb{N}} \rightarrow \mathbb{T}$ be the base n coding map

$$\Xi(\omega) = \sum_{k=1}^{\infty} \frac{\omega_k}{n^k}.$$

Recall the definition of the measure ρ from (16).

Lemma 3.4. *The measure $\Xi(\rho)$ is T_n invariant and satisfies*

$$\Xi(\rho) = \int \Pi_2(\mu) dR(\mu, z, t, \omega),$$

here $\Pi_2(x, y) = y$ is the coordinate projection in \mathbb{T}^2 .

Proof. Recall from (16) that

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \delta_{\sigma^k(\omega_0)} \rightarrow \rho.$$

Also, Ξ is a factor map in the sense that $\Xi \circ \sigma = T_n \circ \Xi$. So, since Ξ is a continuous factor map, applying it to both sides of this equation yields

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \delta_{T_n^k(y_0)} \rightarrow \Xi(\rho).$$

We also have

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} \delta_{T^k(\mu_0, z_0, u_0, \omega_0)} \rightarrow R.$$

Combining the two last displayed equations, we see that $\Xi(\rho)$ equals the marginal of R on the second coordinate y of its projection to \mathbb{T}^2 with coordinates (x, y) . Now, by Theorem 2.7, (x, y) is distributed according to

$$\int \mu dR(\mu, z, t, \omega).$$

So, the marginal of R on the y coordinate is given by

$$\int \Pi_2(\mu) dR(\mu, z, t, \omega).$$

This proves the Lemma. □

3.1.2 The skew product S

For any T invariant distribution R' we denote

$$\dim R' = \int \dim \mu dR'(\mu, z, u, \omega)$$

which is equal to $\dim Q$ for our distribution R . Now, consider the ergodic decomposition of R ,

$$R = \int R_\xi d\tau(\xi).$$

By Theorem 2.7, almost every R_ξ satisfies that its marginal on the first three coordinates $(R_\xi)_{1,2,3}$ is a CP distribution in the sense of Definition 2.6.

From this point forward

Fix an ergodic component R_ξ such that $\dim R_\xi > 0$. (22)

Then for an R_ξ typical (μ, z, u, ω) we have by ergodicity

$$\frac{1}{N} \sum_{k=1}^N \mu^{A_k^u(z)} \rightarrow \int \nu dR_\xi(\nu, z, u, \omega). (23)$$

Also, by Lemma 3.2 for every k we have

$$\Pi_1(\text{supp}(\mu^{A_k^u(z)})) \subseteq A(\sigma_{R_\theta^k(u)} \circ \dots \circ \sigma_{R_\theta(u)} \circ \sigma_u(\omega)). \quad (24)$$

Now, consider the measure $\kappa \in \mathcal{P}(\mathbb{T}^2 \times \mathbb{T} \times (\Pi_2 D)^\mathbb{N})$ defined by

$$\kappa = \int \mu \times \delta_{\{(u,\omega)\}} dR_\xi(\mu, z, u, \omega)$$

and let $S : \mathbb{T}^2 \times \mathbb{T} \times (\Pi_2 D)^\mathbb{N} \rightarrow \mathbb{T}^2 \times \mathbb{T} \times (\Pi_2 D)^\mathbb{N}$ be the map

$$S(z, u, \omega) = (\Phi_u(z), R_\theta(u), \sigma_u(\omega))$$

where we recall that Φ_u was defined in (10).

Lemma 3.5. *The measure κ is S invariant and ergodic. Moreover, for κ almost every (z, u, ω) there is an R_ξ typical measure μ such that (23) and (24) hold true.*

Proof. Recall that $\Pi_{2,3,4} : \mathcal{P}(\mathbb{T}^2) \times \mathbb{T}^2 \times \mathbb{T} \times \{0, \dots, n-1\}^\mathbb{N} \rightarrow \mathbb{T}^2 \times \mathbb{T} \times \{0, \dots, n-1\}^\mathbb{N}$ is the projection

$$(\mu, z, u, \omega) \mapsto (z, u, \omega).$$

Then $\Pi_{2,3,4} \circ T = S \circ \Pi_{2,3,4}$. By Theorem 2.7 part (3) we have that $\kappa = \Pi_{2,3,4} R_\xi$. In particular, κ is S invariant. Moreover, (S, κ) is a factor of the ergodic system (T, R_ξ) , and therefore it is ergodic.

The last assertion is an immediate consequence of the definition of κ , and since (23) and (24) are R_ξ generic properties. \square

Let us now introduce a generator for the system (S, κ) . We first recall the definition of generators: Let (X, U) be a dynamical system, and let \mathcal{D} be a finite partition of X . Let $\mathcal{D}_k = \bigvee_{i=0}^{k-1} U^{-i} \mathcal{D}$ denote the coarsest common refinement of $\mathcal{D}, U^{-1} \mathcal{D}, \dots, U^{-k+1} \mathcal{D}$. The sequence \mathcal{D}_k is called the filtration generated by \mathcal{D} with respect to U . Now, if the smallest sigma algebra that contains \mathcal{D}_k for all k is the Borel sigma algebra, we say that \mathcal{D} is an S -generating partition for (X, U) .

Back to our system (S, κ) , let

$$\mathcal{C} = (\mathcal{D}_m \times \mathcal{D}_n) \times \{[0, 1 - \theta], [1 - \theta, 1]\} \times \{[j] : j \in \Pi_2 D\}$$

be a partition of the space

$$\mathbb{T}^2 \times \mathbb{T} \times (\Pi_2 D)^\mathbb{N}.$$

Write $\mathcal{W} = \{[0, 1 - \theta], [1 - \theta, 1]\}$.

Lemma 3.6. *The partition \mathcal{C} is an S -generating partition. Moreover, $\kappa(\partial \mathcal{C}) = 0$ for every $k \in \mathbb{N}$ and every $C \in \mathcal{C}_k$.*

Proof. The first assertion is an easy consequence of the fact that, as k grows to infinity, the maximal diameter (working, say, with the sup metric) of an element in the partition \mathcal{C}_k converges to 0. For the second part, let $k \in \mathbb{N}$ and fix and element in \mathcal{C}_k . This element is of the form $A \times W \times I$ where $W \in \mathcal{W}_k$, and for some $u \in W$ we have that I is a cylinder set in $(\Pi_2 D)^\mathbb{N}$ of length $\mathcal{R}(k, u) \approx [k \cdot \theta]$, and $A \in \mathcal{A}_k^u$ (note that the latter sets are independent of the choice of $u \in W$). Notice that $\partial(I) = \emptyset$. Therefore, by two application of the "product rule" for the boundary of product sets

$$\partial(A \times W \times I) \subseteq \partial(A \times W) \times (\Pi_2 D)^\mathbb{N} \subseteq (\partial A \times W \times (\Pi_2 D)^\mathbb{N}) \bigcup (A \times \partial W \times (\Pi_2 D)^\mathbb{N}).$$

Thus,

$$\kappa(\partial(A \times W \times I)) \leq \kappa(\partial A \times W \times (\Pi_2 D)^\mathbb{N}) + \kappa(A \times \partial W \times (\Pi_2 D)^\mathbb{N}). \quad (25)$$

Now, the first summoned on the right hand side of equation (25) is 0. This is because R_ξ typical μ has positive dimension by our choice of R_ξ and Theorem 2.7. In particular, they are not atomic. Also, R_ξ almost every (μ, z, u, ω) satisfies that μ is supported on a line with slope m^u by Lemma 3.2. On the other hand, ∂A is a union of four lines that are parallel to the major axes. To sum up, R_ξ almost every μ is continuous and $\text{supp}(\mu)$ intersects ∂A in at most 2 points, so $\mu(\partial A) = 0$. Thus, the result follows from the definition of κ .

The second summoned is trivially 0 since the marginal on the second coordinate of κ is the Lebesgue measure \mathcal{L} , as this is the unique R_θ invariant measure, and ∂W consists of two points. \square

3.1.3 A geometric consequence of Sinai's factor Theorem

We say that a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{T}$ is uniformly distributed (UD) if for every sub-interval $J \subseteq \mathbb{T}$ we have

$$\frac{1}{N} |\{0 \leq k \leq N-1 : x_k \in J\}| \rightarrow \mathcal{L}(J), \quad \text{where } \mathcal{L} \text{ is the Lebesgue measure on } \mathbb{T}.$$

In [22], Wu proved following result by appealing to the Sinai factor Theorem:

Theorem 3.7. [22, Theorem 6.1] *Let (X, T, μ) be an ergodic measure preserving system. Let \mathcal{A} be a generator with finite cardinality, and let $\{\mathcal{A}_k\}_k$ denote the filtration generated by \mathcal{A} and T . Suppose that $\mu(\partial A) = 0$ for every $k \in \mathbb{N}$ and every $A \in \mathcal{A}_k$. Let $\beta \notin \mathbb{Q}$.*

Then for any $\epsilon > 0$ and for all $l \geq l(\epsilon)$ large enough there exists a disjoint family of measurable sets $\{C_i\}_{i=1}^{N(l, \epsilon)}, C_i \subset X$, such that:

1. $\mu(\bigcup C_i) > 1 - \epsilon$.
2. For every $1 \leq i \leq N(l, \epsilon)$, $|\{A \in \mathcal{A}_l : C_i \cap A\}| \leq e^{l-\epsilon}$.
3. There exists another disjoint family of measurable sets $\{\tilde{C}_i\}_{i=1}^{N(l, \epsilon)}, \tilde{C}_i \subset X$, such that for every $1 \leq i \leq N(l, \epsilon)$ we have:
 - $C_i \subseteq \tilde{C}_i$,
 - $\mu(C_i) \geq (1 - \epsilon)\mu(\tilde{C}_i)$,
 - for μ a.e. x we have that the sequence

$$\{R_\beta^k(0) \in \mathbb{T} : k \in \mathbb{N} \text{ and } T^k(x) \in \tilde{C}_i\}$$

is UD.

3.1.4 Three key estimates

We begin by establishing two bounds via Theorem 3.7. Recall the definition of the sets $A(\omega)$ as in (19). Fix $\epsilon > 0$, and note that in the construction below we use the same ϵ for all ergodic components R_ξ with positive dimension. The parameter l below will depend on both ξ and ϵ , with the dependence on ξ being measurable.

Proposition 3.8. *Fix a κ typical (z, u, ω) and a corresponding R_ξ typical measure μ satisfying (23) and (24). Then, for our small $\epsilon > 0$ and all large $l \geq l(\epsilon, \xi)$, there exists a set $\mathcal{N} = \mathcal{N}_\xi \subseteq \mathbb{N}$ such that*

$$\mathcal{N}_{n-l} \left(\bigcup_{k \in \mathcal{N}} \text{supp}(\mu^{\mathcal{A}_k^u(z)}) \right) \geq n^{l \cdot (\dim \mu + 1 - o_\epsilon(1))} \quad (26)$$

and for some uniform constant C_1 , for any $k' \in \mathcal{N}$

$$\mathcal{N}_{n-l} \left(\Pi_1 \left(\bigcup_{k \in \mathcal{N}} \text{supp}(\mu^{\mathcal{A}_k^u(z)}) \right) \right) \leq C_1 \cdot \mathcal{N}_{n-l} \left(A \left(\Pi_3 \circ S^{k'}(z, u, \omega) \right) \right). \quad (27)$$

We remark that $\Pi_3 \circ S^{k'}(z, u, \omega)$ means the third coordinate of $S^{k'}(z, u, \omega)$.

Proof. By Lemmas 3.5 and 3.6 we may apply Theorem 3.7 to

$$(\mathbb{T}^2 \times \mathbb{T} \times (\Pi_2 D)^\mathbb{N}, S, \kappa) \text{ with the generator } \mathcal{C}. \quad (28)$$

Thus, for our small $\epsilon > 0$ there exists $l(\epsilon)$ such that for all $l \geq l(\epsilon)$, we have a disjoint family $\{C_i\}_{i=1}^{N(l, \epsilon)}$ such that

$$\kappa\left(\bigcup_{i=1}^{N(l, \epsilon)} C_i\right) > 1 - \epsilon$$

and for every $1 \leq i \leq N(l, \epsilon)$,

$$C_i \subseteq \mathbb{T}^2 \times \mathbb{T} \times (\Pi_2 D)^\mathbb{N}$$

and

$$\mathcal{N}_{n-l}(\Pi_{1,3} C_i) < e^{l\epsilon}. \quad (29)$$

Furthermore, for κ almost every (z, u, ω) ,

$$\overline{\mathcal{L}(\{R_\theta^k(u) : k \in \mathbb{N}, S^k(z, u, \omega) \in C_i\})} \geq 1 - \epsilon. \quad (30)$$

To indicate the dependence of $l(\epsilon)$ on R_ξ , in the following we will write $l(\epsilon, \xi)$ for $l(\epsilon)$. Notice that the measurable dependence of $l(\epsilon, \xi)$ on ξ arises from the fact that our system (28), specifically the measure κ , depends measurably on R_ξ .

Now, fix a κ typical (z, u, ω) and a measure μ satisfying (23) and (24) (such a measure exists by Lemma 3.5). We have the following estimate, which is a consequence of Theorem 2.7 part (4):

Lemma 3.9. *There exists some $r_0 = r_0(\epsilon) > 0$ such that for every $r < r_0$, $y \in \mathbb{T}^2$, $k \in \mathbb{N}$, and $l \in \mathbb{N}$ large enough, we have*

$$\mathcal{N}_{n-l}\left(\text{supp}(\mu^{\mathcal{A}_k^u(z)}) \setminus B(y, r)\right) \geq n^{l \cdot (\dim \mu - o_\epsilon(1))}.$$

We also have the following estimate:

Claim 3.10. *For every $1 \leq i \leq N(l, \epsilon)$ there is a set $C'_i \subseteq C_i$ such that:*

1. $\text{diam}(\Pi_{1,3}(C'_i)) \leq n^{-l}$,
2. $\mathcal{N}_{n-l}(\{R_\theta^k(u) : k \in \mathbb{N}, S^k(z, u, \omega) \in C'_i\}) \geq n^{(1-o_\epsilon(1)) \cdot l}$.

Proof. This is a consequence of the properties (29) and (30) of $\{C_i\}_{i=1}^{N(l, \epsilon)}$, and the pigeon hole principle applied to the family of sets $\{C_i \cap D : D \in \mathcal{C}_l\}$. \square

Fix some i and C'_i as in Claim 3.10. Define

$$\mathcal{N} := \{k \in \mathbb{N} : S^k(z, u, \omega) \in C'_i\}.$$

Then, by combining Lemma 3.9, (29), and Claim 3.10, we can prove the inequality (26): Indeed, let $X = \Pi_1(C'_i)$. Since $C'_i \subseteq C_i$, we have by (29) that

$$\mathcal{N}_{n-l}(X) \leq e^{l\epsilon}.$$

Also, writing $\mathcal{F} = \{R_\theta^k(u) : k \in \mathcal{N}\}$, for every $t \in \mathcal{F}$ there exists a line with slope m^t intersecting X that supports a measure that satisfies Lemma 3.9. This follows by our choice of \mathcal{N} and since $\Pi_1 \circ S^k(z, u, \omega) \in \text{supp}(\mu^{\mathcal{A}_k^u(z)})$. Finally, consider the set

$$K = \left(\bigcup_{k \in \mathcal{N}} \text{supp}(\mu^{\mathcal{A}_k^u(z)}) \right) - X.$$

Then for any $t \in \mathcal{F}$, we can find some line ℓ with slope m^t that supports a measure on K that satisfies Lemma 3.9, and passes through an l -th level n -adic cube containing the origin. From this and Claim 3.10 part (2), one sees that

$$\mathcal{N}_{n^{-l}}(K) \geq n^{l \cdot (1 + \dim \mu - o_{\epsilon, l}(1))}.$$

It is well known that for every bounded sets $A, B \subseteq \mathbb{R}^2$ there is a constant C_1 such that

$$\mathcal{N}_{n^{-l}}(A + B) \leq C_1 \cdot \mathcal{N}_{n^{-l}}(A) \cdot \mathcal{N}_{n^{-l}}(B).$$

Thus, since $\mathcal{N}_{n^{-l}}(X) \leq e^{l \cdot \epsilon}$, by the definition of K and the last two displayed equations, the inequality (26) is proved.

As for the inequality (27), by Claim 3.10 we have

$$\text{diam}(\Pi_3(C'_i)) \leq n^{-l}.$$

Therefore, for any $k, k' \in \mathcal{N}$ we have that (recalling our metric on the symbolic space (9))

$$d(\Pi_3 \circ S^k(z, u, \omega), \Pi_3 \circ S^{k'}(z, u, \omega)) \leq n^{-l}.$$

So, since we have (24) at our disposal, for every $k' \in \mathcal{N}$ we have

$$\Pi_1 \left(\bigcup_{k \in \mathcal{N}} \text{supp}(\mu^{A_k^u(z)}) \right) \subseteq A(\Pi_3 \circ S^{k'}(z, u, \omega))^{(m^{-l})} \subseteq A(\Pi_3 \circ S^{k'}(z, u, \omega))^{(n^{-l})}$$

where $B^{(n^{-l})}$ is the n^{-l} -neighbourhood of a set B . Notice that we have used that $n < m$. From this, the inequality (27) readily follows. \square

Remark 3.11. *In the proof above it was also established that since the mapping $\xi \rightarrow R_\xi$ is measurable, $l(\epsilon, \xi)$ is also a measurable function of ξ .*

Next, we estimate the covering number of the Π_2 projection of the set $\bigcup_{k \in \mathcal{N}} \text{supp}(\mu^{A_k^u(z)})$ from Proposition 3.8. Recall that the measure ρ was defined in (16), and that by Lemma 3.4 its image under the base n coding map $\Xi(\rho) \in \mathcal{P}(\mathbb{T})$ is T_n invariant and

$$\Xi(\rho) = \int \Pi_2(\nu) dR(\nu, z, t, \omega).$$

From now on, we denote $\tilde{\rho} := \Xi(\rho)$. Recall that the ergodic decomposition of R is given by

$$R = \int R_{\xi'} d\tau(\xi').$$

It follows that for τ almost every ξ' , the measure

$$\tilde{\rho}_{\xi'} = \int \Pi_2(\nu) dR_{\xi'}(\nu, z, t, \omega)$$

is T_n invariant and ergodic. Thus,

$$\tilde{\rho} = \int \tilde{\rho}_{\xi'} d\tau(\xi')$$

is the ergodic decomposition of $\tilde{\rho}$.

Fix $\tilde{\rho}_{\xi'}$ for the ergodic component $R_{\xi'}$ (recall (22)) we have been working with so far. Recall that $X^{(n^{-l})}$ denotes the n^{-l} neighbourhood of a set X .

Proposition 3.12. *Let (z, u, ω) , μ , and \mathcal{N} be as in Proposition 3.8. Then, for our small $\epsilon > 0$ and all large $l \geq l(\epsilon, \xi)$, there exists a subset $\mathcal{N}' = \mathcal{N}'_l \subseteq \mathcal{N}$ and a set $A = A_{\xi, \epsilon} \subseteq \mathbb{T}$ such that for every $k \in \mathcal{N}'$,*

$$\Pi_2 \mu^{A_k^u(z)} \left(A^{(n^{-l})} \right) \geq 1 - o_\epsilon(1)$$

such that a modified version of inequality (26) holds with

$$\mathcal{N}_{n^{-l}} \left(\bigcup_{k \in \mathcal{N}'} \text{supp}(\mu^{A_k^u(z)}|_{[0,1] \times A^{(n^{-l})}}) \right) \geq n^{l \cdot (\dim \mu + 1 - o_\epsilon(1))} \quad (31)$$

and we also have for some global constant C_2 ,

$$\mathcal{N}_{n^{-l}} \left(\Pi_2 \left(\bigcup_{k \in \mathcal{N}'} \text{supp}(\mu^{A_k^u(z)}|_{[0,1] \times A^{(n^{-l})}}) \right) \right) \leq C_2 \cdot n^{l \cdot (\dim \tilde{\rho}_\xi + o_\epsilon(1))}. \quad (32)$$

Proof. By Theorem 2.5, since $\tilde{\rho}_\xi$ is T_n invariant and ergodic, it is exact dimensional. By Egorov's Theorem there exists a compact set $A = A_{\xi, \epsilon}$, with $\dim_B A = \dim_H A$, that varies measurably in ξ , such that

$$\dim_B A = \dim \tilde{\rho}_\xi, \quad \text{and} \quad \tilde{\rho}_\xi(A) = 1 - o_\epsilon(1).$$

Also, since we have (23) at our disposal,

$$\frac{1}{N} \sum_{k=1}^N \Pi_2 \mu^{A_k^u(z)} \rightarrow \int \Pi_2 \nu \, dR_\xi(\nu, z, t, \omega) = \tilde{\rho}_\xi.$$

Therefore, since for every l the set $A^{(n^{-l})}$ is open, there is a set $\mathcal{N}'' \subset \mathbb{N}$ such that the density of \mathcal{N}'' in \mathbb{N} is at least $1 - o_\epsilon(1)$, and for every $k \in \mathcal{N}''$ we have

$$\Pi_2 \mu^{A_k^u(z)} \left(A^{(n^{-l})} \right) \geq 1 - o_\epsilon(1).$$

Now, define

$$\mathcal{N}' = \mathcal{N} \cap \mathcal{N}''.$$

Since the density of \mathcal{N}'' in \mathbb{N} is at least $1 - o_\epsilon(1)$, the density of \mathcal{N}' in \mathcal{N} is also at least $1 - o_\epsilon(1)$. Then we arrive at the inequality (32) since

$$\mathcal{N}_{n^{-l}} \left(\Pi_2 \left(\bigcup_{k \in \mathcal{N}'} \text{supp}(\mu^{A_k^u(z)}|_{[0,1] \times A^{(n^{-l})}}) \right) \right) \leq \mathcal{N}_{n^{-l}} \left(A^{(n^{-l})} \right) \leq C_2 \cdot n^{l \cdot (\dim \tilde{\rho}_\xi + o_\epsilon(1))}.$$

Notice that the large l we choose here depends on our set $A = A_{\xi, \epsilon}$, so $l = l(\epsilon, \xi)$.

Finally, we need to justify the modified version of (26) given by (31). To see this, notice that the outcome of Claim 3.10 is unchanged when we move to \mathcal{N}' , since the density of \mathcal{N}' in \mathcal{N} is at least $1 - o_\epsilon(1)$. Thus, in order to run the same argument as at the end of Proposition 3.8, we need to study what happens in the setting of Lemma 3.9 with our extra conditioning on $[0, 1] \times (A)^{n^{-l}}$.

To this end, for every $k \in \mathbb{N}$, by Proposition 2.1

$$\log \mathcal{N}_{n^{-l}} \left(\text{supp}(\mu^{A_k^u(z)}|_{[0,1] \times A^{(n^{-l})}}) \right) \geq H(\mu^{A_k^u(z)}|_{[0,1] \times A^{(n^{-l})}}, \mathcal{D}_{n^l})$$

and since

$$\mu^{A_k^u(z)} \left([0, 1] \times A^{(n^{-l})} \right) \geq 1 - o_\epsilon(1),$$

we have (by [22, Lemma 7.3])

$$H(\mu^{A_k^u(z)}|_{[0,1] \times A^{(n^{-l})}}, \mathcal{D}_{n^l}) \geq H(\mu^{A_k^u(z)}, \mathcal{D}_{n^l}) - l \cdot \log n \cdot o_\epsilon(1).$$

Finally, by another application of Theorem 2.7 part (4) we have

$$H(\mu^{\mathcal{A}_k^u(z)}, \mathcal{D}_{n^l}) \geq l \cdot \log n \cdot (\dim \mu - o_\epsilon(1)).$$

Combining the last four equations shows that indeed an analogue of Lemma 3.9 holds in this modified situation as well, and we complete the proof of (31) in the same manner as in Proposition 3.8. \square

Remark 3.13. Recall that we use the same $\epsilon > 0$ for every component R_ξ with positive dimension. As we already noted in Remark 3.11, the number $l(\epsilon, \xi)$ in Proposition 3.8 depends measurably on ξ . Similarly, the dependence of $l(\epsilon, \xi)$ in Proposition 3.12 is also measurable in ξ . Note that the error terms $o_\epsilon(1)$ appearing in the inequalities (26) and (27) of Proposition 3.8, and (31) and (32) of Proposition 3.12 go to zero as $\epsilon \rightarrow 0$ in a manner dependent on both ϵ and ξ .

Recall that $R = \int R_\xi d\tau(\xi)$ is the ergodic decomposition of R . Now, let Θ be the set of all ergodic components of R that have positive dimension. Since R has positive dimension, $\tau(\Theta) > 0$. By an application of Egorov's Theorem, we may produce a subset $\Psi = \Psi(\epsilon) \subseteq \Theta$ of ergodic components of R such that:

- $\tau(\Psi) > (1 - \epsilon)\tau(\Theta)$.
- l can be chosen uniformly in both Proposition 3.8 and Proposition 3.12 for all R_ξ when $\xi \in \Psi$.

Thus, for ergodic components R_ξ with $\xi \in \Psi$, the error terms $o_\epsilon(1)$ appearing in the inequalities (26) and (27) of Proposition 3.8, and (31) and (32) of Proposition 3.12 go to zero as $\epsilon \rightarrow 0$ in a manner dependent only on ϵ (and not on ξ).

3.1.5 Proof of Theorem 3.1

Proof of Part (1) Let R_ξ be an ergodic component such that $\xi \in \Psi$ (recall Remark 3.13), and (z, u, ω) , μ , $\tilde{\rho}_\xi$, \mathcal{N} , \mathcal{N}' and A be as in Proposition 3.8 and Proposition 3.12. By these Propositions, for an error term $o_\epsilon(1)$ that is independent of ξ , for $k' = \min \mathcal{N}$:

$$\begin{aligned} n^{l(\dim \mu + 1 - o_\epsilon(1))} &\leq \mathcal{N}_{n-l} \left(\bigcup_{k \in \mathcal{N}'} \text{supp}(\mu^{\mathcal{A}_k^u(z)}|_{[0,1] \times A^{(n-l)}}) \right) \\ &\leq \mathcal{N}_{n-l} \left(\prod_1 \bigcup_{k \in \mathcal{N}'} \text{supp}(\mu^{\mathcal{A}_k^u(z)}|_{[0,1] \times A^{(n-l)}}) \right) \\ &\times \mathcal{N}_{n-l} \left(\prod_2 \bigcup_{k \in \mathcal{N}'} \text{supp}(\mu^{\mathcal{A}_k^u(z)}|_{[0,1] \times A^{(n-l)}}) \right) \\ &\leq \mathcal{N}_{n-l} \left(\prod_1 \left(\bigcup_{k \in \mathcal{N}} \text{supp}(\mu^{\mathcal{A}_k^u(z)}) \right) \right) \cdot C_2 \cdot n^{l \cdot (\dim \tilde{\rho}_\xi + o_\epsilon(1))} \\ &\leq C_1 \cdot \mathcal{N}_{n-l} \left(A(\prod_3 \circ S^{k'}(z, u, \omega)) \right) \cdot C_2 \cdot n^{l \cdot (\dim \tilde{\rho}_\xi + o_\epsilon(1))}. \end{aligned}$$

Taking log and dividing by $l \log n$ we arrive at

$$\dim \mu + 1 - o_\epsilon(1) \leq \frac{\log \mathcal{N}_{n-l} \left(A(\prod_3 \circ S^{k'}(z, u, \omega)) \right)}{l \log n} + \dim \tilde{\rho}_\xi. \quad (33)$$

We remark that in equation (33) and the following calculations, we can absorb the $o_\epsilon(1)$ factors that we encounter into each other, which is possible since they are all uniform as ϵ goes to 0, in a manner dependent only on ϵ . Also, notice that k' is a measurable function of ξ .

Next, applying Theorem 2.5 we obtain

$$\dim \mu + 1 - o_\epsilon(1) \leq \frac{\log \mathcal{N}_{n-l} \left(A(\Pi_3 \circ S^{k'}(z, u, \omega)) \right)}{l \log n} + \frac{h(\tilde{\rho}_\xi, T_n)}{\log n}. \quad (34)$$

Now, equation (34) holds as long as we are working with an ergodic component such that $\xi \in \Psi$. Recall that Θ denotes the set of ergodic components of R that have positive dimension. So, by the definition of $\dim R$, since $\tau(\Psi) > \tau(\Theta) - \epsilon$ and for $\xi \notin \Theta$ we have $\dim \mu = 0$ for R_ξ almost every μ , via (34) we see that

$$\begin{aligned} \dim R &= \int_{\Theta} \int \dim \mu dR_\xi(\mu, z, u, \omega) d\tau(\xi) \\ &\leq \int_{\Psi} \int \dim \mu dR_\xi(\mu, z, u, \omega) d\tau(\xi) + \epsilon \\ &\leq \int_{\Psi} \int \left(\frac{\log \mathcal{N}_{n-l} \left(A(\Pi_3 \circ S^{k'}(z, u, \omega)) \right)}{l \log n} + \frac{h(\tilde{\rho}_\xi, T_n)}{\log n} - 1 \right) dR_\xi(\mu, z, u, \omega) d\tau(\xi) \\ &+ o_\epsilon(1) \\ &\leq \int_{\Theta} \int \left(\frac{\log \mathcal{N}_{n-l} \left(A(\Pi_3 \circ S^{k'}(z, u, \omega)) \right)}{l \log n} + \frac{h(\tilde{\rho}_\xi, T_n)}{\log n} \right) dR_\xi(\mu, z, u, \omega) d\tau(\xi) \\ &- \tau(\Theta) + o_\epsilon(1) \\ &\leq \int_{\Theta} \int \frac{\log \mathcal{N}_{n-l} \left(A(\Pi_3 \circ S^{k'}(z, u, \omega)) \right)}{l \log n} dR_\xi(\mu, z, u, \omega) d\tau(\xi) + \int_{\Theta} \frac{h(\tilde{\rho}_\xi, T_n)}{\log n} d\tau(\xi) \\ &- \tau(\Theta) + o_\epsilon(1). \end{aligned}$$

where we have used that $\tilde{\rho}_\xi$ is constant when integrated against R_ξ . Next, applying (13),

$$\dim R = \dim Q \geq \gamma$$

We thus arrive at the inequality

$$\begin{aligned} \gamma &\leq \int_{\Theta} \int \frac{\log \mathcal{N}_{n-l} \left(A(\Pi_3 \circ S^{k'}(z, u, \omega)) \right)}{l \log n} dR_\xi(\mu, z, u, \omega) d\tau(\xi) + \int_{\Theta} \frac{h(\tilde{\rho}_\xi, T_n)}{\log n} d\tau(\xi) \quad (35) \\ &- \tau(\Theta) + o_\epsilon(1) \quad (36) \end{aligned}$$

This implies Part (1) of Theorem 3.1: Indeed, taking $\epsilon \rightarrow 0$, using that for every ξ the measure $\tilde{\rho}_\xi$ is supported on $\Pi_2(F)$ and that $\frac{h(\tilde{\rho}_\xi, T_n)}{\log n} = \dim \tilde{\rho}$, we obtain

$$\gamma \leq \tau(\Theta) \cdot \left(\sup_{\omega \in (\Pi_2 D)^{\mathbb{N}}} \overline{\dim}_B A(\omega) + \dim_H \Pi_2(F) - 1 \right) = \tau(\Theta) \cdot (\dim^* F - 1)$$

where in the last inequality we made use of Mackay's formula [16] for $\dim^* F$.

Proof of Part (2) By the T invariance of R_ξ , writing $k' = k'(\xi)$ as before and recalling (18)

$$\begin{aligned}
& \int_{\Theta} \int \left(\frac{\log \mathcal{N}_{n^{-l}}(A(\omega))}{l \log n} \right) dR_\xi(\mu, z, u, \omega) d\tau(\xi) = \\
& \int_{\Theta} \int \left(\frac{\log \mathcal{N}_{n^{-l}}(A(\omega))}{l \log n} \right) dR_\xi T^{k'}(\mu, z, u, \omega) d\tau(\xi) = \\
& \int_{\Theta} \int \left(\frac{\log \mathcal{N}_{n^{-l}}(A(\sigma^{\mathcal{R}(k', u)} \omega))}{l \log n} \right) dR_\xi(\mu, z, u, \omega) d\tau(\xi) = \\
& \int_{\Theta} \int \left(\frac{\log \mathcal{N}_{n^{-l}}(A(\Pi_3 \circ S^{k'}(z, u, \omega)))}{l \log n} \right) dR_\xi(\mu, z, u, \omega) d\tau(\xi).
\end{aligned}$$

Combining this with (35) we obtain

$$\gamma \leq \int \int \left(\frac{\log \mathcal{N}_{n^{-l}}(A(\omega))}{l \log n} \right) dR_\xi(\mu, z, u, \omega) d\tau(\xi) + \int \frac{h(\tilde{\rho}_\xi, T_n)}{\log n} d\tau(\xi) - \tau(\Theta) + o_\epsilon(1).$$

Notice that we have also removed the conditioning on the set Θ on the right hand side. Since R_ξ is a disintegration of R , we obtain

$$\gamma \leq \int \left(\frac{\log \mathcal{N}_{n^{-l}}(A(\omega))}{l \log n} \right) dR(\mu, z, u, \omega) + \int \frac{h(\tilde{\rho}_\xi, T_n)}{\log n} d\tau(\xi) - \tau(\Theta) + o_\epsilon(1).$$

Notice that up to an $o_l(1)$ factor,

$$\frac{\log \mathcal{N}_{n^{-l}}(A(\omega))}{l \log n} = \frac{\log \mathcal{N}_{m^{-[l \cdot \theta]}}(A(\omega))}{[l \cdot \theta] \log m} + o_l(1).$$

So, combining this with Lemma 3.3, using the affinity of entropy and letting $l \rightarrow \infty$, we get

$$\gamma \leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\int \tilde{\rho}_\xi d\tau(\xi), T_n)}{\log n} - \tau(\Theta) + o_\epsilon(1).$$

Finally, $\int \tilde{\rho}_\xi d\tau(\xi) = \tilde{\rho}$, and $(\tilde{\rho}, T_n)$ is a factor of (ρ, σ) , so we arrive at

$$\gamma \leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n} - \tau(\Theta) + o_\epsilon(1).$$

Taking $\epsilon \rightarrow 0$, this is Part (2) of Theorem 3.1.

Proof of Part (3) By Part (2) we have the following inequality:

$$\gamma + \tau(\Theta) \leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n}. \quad (37)$$

Recall that by (14), (15) and (16), the measures $\nu, \rho, \eta \in \mathcal{P}((\Pi_2 D)^\mathbb{N})$ are σ invariant and we have

$$\rho = \theta \cdot \nu + (1 - \theta) \cdot \eta.$$

We now show, via the equation above, that the right hand side of (37) is bounded above by

$$\dim_P F = \dim_B F = \frac{\log |\Pi_2(D)|}{\log n} + \frac{\log \frac{|D|}{|\Pi_2(D)|}}{\log m}.$$

To this end, by affinity of entropy, we have

$$h(\rho, \sigma) = h(\theta \cdot \nu + (1 - \theta) \cdot \eta, \sigma) = \theta \cdot h(\nu, \sigma) + (1 - \theta) \cdot h(\eta, \sigma).$$

Now, by the Kolmogorov-Sinai Theorem and Proposition 2.1,

$$h(\eta, \sigma) \leq H(\eta, \mathcal{D}) \leq \log |\Pi_2(D)|,$$

where \mathcal{D} is the first generation cylinder partition of $(\Pi_2 D)^\mathbb{N}$. By another application of the Kolmogorov-Sinai Theorem,

$$h(\nu, \sigma) \leq H(\nu, \mathcal{D}) = \sum_{j \in \Pi_2(D)} \nu([j]) \cdot \log \frac{1}{\nu([j])}.$$

So, by the last two displayed inequalities and recalling that $\theta = \frac{\log n}{\log m}$, we can bound

$$\begin{aligned} & \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n} \\ = & \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \theta \cdot \frac{h(\nu, \sigma)}{\log n} + (1 - \theta) \cdot \frac{h(\eta, \sigma)}{\log n} \\ \leq & \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \theta \cdot \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \cdot \log \frac{1}{\nu([j])}}{\log n} + (1 - \theta) \cdot \frac{\log |\Pi_2(D)|}{\log n} \\ = & \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \cdot \left(\log \left(\frac{a(j)}{\sum_j a(j)} \right) + \log \frac{1}{\nu([j])} \right) + \log \left(\sum_j a(j) \right)}{\log m} + \left(1 - \frac{\log n}{\log m} \right) \frac{\log |\Pi_2(D)|}{\log n} \\ \leq & \frac{\log \left(\sum_j a(j) \right)}{\log m} + \left(1 - \frac{\log n}{\log m} \right) \frac{\log |\Pi_2(D)|}{\log n} \\ = & \frac{\log |\Pi_2(D)|}{\log n} + \frac{\log \frac{|D|}{|\Pi_2(D)|}}{\log m} = \dim_P F, \end{aligned}$$

where in the fourth inequality we used the Gibbs inequality (see Proposition 2.1). Combining this with (37) we see that

$$\gamma + \tau(\Theta) \leq \dim_P F,$$

thus Part (3) of Theorem 3.1 is proved. \square

4 On the proof of Theorem 1.2 part (1)

4.1 A Hausdorff dimension version of Theorem 3.1

The idea for the proof of Theorem 1.2 part (1) is similar to that of Theorem part (2), with some modifications. Let F be a Bedford-McMullen carpet with exponents $m > n$ and digits D , such that $m \not\sim n$. Write $\theta := \frac{\log n}{\log m}$. Let ℓ_0 be a line. We may assume, as in the proof of Theorem 1.2 part (2), that the slope of ℓ_0 is $m^{u_0} \neq 0$ for $u_0 \in [0, 1)$, and that our ambient space is \mathbb{T}^2 rather than $[0, 1]^2$.

Let

$$\gamma_1 := \dim_H \ell_0 \cap F$$

and fix some $\gamma < \gamma_1$. We will show that

$$\gamma \leq \max \left\{ 0, \frac{\dim_H F}{\dim^* F} \cdot (\dim^* F - 1) \right\}.$$

It is clear that we may assume $\gamma > 0$.

By Frostman's Lemma we may find a probability measure $\mu_0 \in \mathcal{P}(\ell_0 \cap F)$ such that

$$\dim(\mu_0, z) \geq \gamma, \quad \text{for } \mu_0 \text{ almost every } z.$$

In particular, μ_0 is continuous (has no atoms). Fix a point $z_0 \in \ell_0 \cap F$ in the support of μ_0 that satisfies the conclusion of Theorem 2.8 part (2).

Write

$$z_0 = (x_0, y_0) = \left(\sum_{k=1}^{\infty} \frac{x_k}{m^k}, \sum_{k=1}^{\infty} \frac{y_k}{n^k} \right), \quad (x_k, y_k) \in D.$$

Notice that since μ_0 is continuous, we may assume both $x_0, y_0 \notin \mathbb{Q}$, so that this representation is unique. Now, consider the sequence

$$\omega_0 = (y_1, y_2, \dots) \in (\Pi_2 D)^{\mathbb{N}} \subseteq \{0, \dots, n-1\}^{\mathbb{N}}.$$

For every $k \in \mathbb{N}$, we define a sequence of measures on $(\Pi_2 D)^{\mathbb{N}}$:

$$\nu_k = \frac{1}{[\theta^{-k+1}]} \sum_{k=1}^{[\theta^{-k+1}]} \delta_{\sigma^k(\omega_0)}, \quad (38)$$

$$\eta_k = \frac{1}{[\theta^{-k}] - [\theta^{-k+1}]} \sum_{k=[\theta^{-k+1}]+1}^{[\theta^{-k}]} \delta_{\sigma^k(\omega_0)}. \quad (39)$$

We shall require the following Claim to choose a subsequence of the scenery. Let \mathcal{D} be the first generation partition of $(\Pi_2 D)^{\mathbb{N}}$.

Claim 4.1. *There exists a subsequence N_j such that*

$$\limsup_{j \rightarrow \infty} (H(\eta_{N_j}, \mathcal{D}) - H(\nu_{N_j}, \mathcal{D})) \leq 0.$$

Proof. Suppose towards a contradiction that the Claim is not true. This means that for some $c > 0$

$$\liminf_{k \rightarrow \infty} (H(\nu_k, \mathcal{D}) - H(\eta_k, \mathcal{D})) \leq -c < 0.$$

So, for all large enough k we have

$$H(\eta_k, \mathcal{D}) - H(\nu_k, \mathcal{D}) > \frac{c}{2}. \quad (40)$$

The crucial observation here is that

$$\nu_{k+1} = \frac{[\theta^{-k+1}]}{[\theta^{-k}]} \nu_k + \frac{[\theta^{-k}] - [\theta^{-k+1}]}{[\theta^{-k}]} \cdot \eta_k.$$

So by concavity of entropy (Proposition 2.1) we have

$$H(\nu_{k+1}, \mathcal{D}) \geq \frac{[\theta^{-k+1}]}{[\theta^{-k}]} \cdot H(\nu_k, \mathcal{D}) + \frac{[\theta^{-k}] - [\theta^{-k+1}]}{[\theta^{-k}]} \cdot H(\eta_k, \mathcal{D}).$$

Combining this with (40) we find that for all large enough k

$$H(\nu_{k+1}, \mathcal{D}) \geq H(\nu_k, \mathcal{D}) + \frac{[\theta^{-k}] - [\theta^{-k+1}]}{[\theta^{-k}]} \cdot \frac{c}{2}.$$

The latter equation implies that $\lim_{k \rightarrow \infty} H(\nu_k, \mathcal{D}) = \infty$, which is a contradiction since for all k ,

$$H(\nu_k, \mathcal{D}) \leq \log |\Pi_2(D)|.$$

□

From now on we work with the sequence N_j from Claim 4.1. By Theorem 2.8 part (2), by perhaps passing to a further subsequence, there exists a distribution Q such that

$$\frac{1}{N_j} \sum_{k=0}^{N_j-1} \delta_{M^k(\mu_0, z_0, u_0)} \rightarrow Q$$

where Q is a CP distribution with

$$\dim Q \geq \gamma.$$

Next, recalling (38) and (39), by perhaps moving to yet a further subsequence, we assume that there are σ invariant measures $\nu, \eta, \rho \in \mathcal{P}((\Pi_2 D)^{\mathbb{N}}) \subseteq \mathcal{P}(\{0, \dots, n-1\}^{\mathbb{N}})$ such that:

$$\nu_{N_j} \rightarrow \nu, \tag{41}$$

$$\eta_{N_j} \rightarrow \eta, \tag{42}$$

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \delta_{\sigma^k(\omega_0)} \rightarrow \rho. \tag{43}$$

It follows from (39) and (38) that $\rho = \theta \cdot \nu + (1-\theta) \cdot \eta$. We also have, by Claim 4.1, the important inequality

$$H(\eta, \mathcal{D}) \leq H(\nu, \mathcal{D}). \tag{44}$$

We can now formulate our required analogue of Theorem 3.1:

Theorem 4.2. *Let $\lambda = Q(\{\mu : \dim \mu > 0\})$. Then:*

1. $\gamma \leq \lambda \cdot (\dim^* F - 1)$.

2.

$$\gamma + \lambda \leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n}.$$

3. $\gamma \leq \dim_H F - \lambda$.

Theorem 4.2 implies Theorem 1.2 part (1), and this is completely analogous to the implication Theorem 3.1 \Rightarrow Theorem 1.2 part (2). The proof of parts (1) and (2) of Theorem 4.2 are the same as the proof of the corresponding parts of Theorem 3.1 detailed in Section 3.1. We thus omit the details. It remains to show that Part (2) implies Part (3), and this is the content of the next Section.

4.2 Proof that part (2) implies part (3) in Theorem 4.2

Recall that $\gamma_1 = \dim_H F \cap \ell_0$. By Theorem 4.2 part (2) we have the following inequality:

$$\gamma + \lambda \leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n} \tag{45}$$

where $\nu, \rho, \eta \in \mathcal{P}(\Pi_2(D)^{\mathbb{N}})$ are σ invariant and we have

$$\rho = \theta \cdot \nu + (1-\theta) \cdot \eta.$$

We now show, via the equation above and (44), that the right hand side of (45) is bounded above by

$$\dim_H F = \frac{\log \left(\sum_{j \in \Pi_2(D)} a(j)^\theta \right)}{\log n},$$

where we recall that $\theta = \frac{\log n}{\log m}$.

To this end, by affinity of entropy, the Kolmogorov-Sinai Theorem, and (44)

$$\begin{aligned}
h(\rho, \sigma) &= h(\theta \cdot \nu + (1 - \theta) \cdot \eta, \sigma) \\
&= \theta \cdot h(\nu, \sigma) + (1 - \theta) \cdot h(\eta, \sigma) \\
&\leq \theta \cdot H(\nu, \mathcal{D}) + (1 - \theta) \cdot H(\eta, \mathcal{D}) \\
&\leq \theta \cdot H(\nu, \mathcal{D}) + (1 - \theta) \cdot H(\nu, \mathcal{D}) \\
&= H(\nu, \mathcal{D}).
\end{aligned}$$

So, we can bound

$$\begin{aligned}
&\frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{h(\rho, \sigma)}{\log n} \\
&\leq \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{H(\nu, \mathcal{D})}{\log n} \\
&= \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \log a(j)}{\log m} + \frac{\sum_{j \in \Pi_2(D)} \nu([j]) \cdot \log \frac{1}{\nu([j])}}{\log n} \\
&= \frac{1}{\log n} \left(\sum_{j \in \Pi_2(D)} \nu([j]) \left(\log \frac{1}{\nu([j])} + \log \left(\frac{a(j)^\theta}{\sum_{j \in \Pi_2(D)} a(j)^\theta} \right) \right) \right) + \log \left(\sum_{j \in \Pi_2(D)} a(j)^\theta \right) \\
&\leq \frac{\log \left(\sum_{j \in \Pi_2(D)} a(j)^\theta \right)}{\log n} = \dim_H F
\end{aligned}$$

where in the last inequality we used the Gibbs inequality (see Proposition 2.1). Combining this with (37) we see that

$$\gamma_1 + \lambda \leq \dim_H F.$$

□

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