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Tiivistelmä

Tutkielman päämääränä on tarkastella sekä suoraa että käänteistä sirontaongelmaa neljännen kertaluvun differentiaalioperaattorille

$$L_4u = \frac{d^4}{dx^4}u + (q_1u')' + 2iq_2u' + iq_2'u + Vu,$$

missä funktioita q_1, q_2 ja V kutsutaan potentiaaleiksi. Suorassa sirontaongelmassa tutkimme differentiaaliyhtälöä $L_4u = k^4u$ parametrilla k ja oletamme, että potentiaalit ovat tunnettuja funktioita. Käyttäen differentiaalioperaattorin perusratkaisua, voimme muuntaa differentiaaliyhtälön $L_4u = k^4u$ integraaliyhtälöksi ja ratkaista sen.

Kun integraaliyhtälön ratkaisu on löydetty, tutkimme sen asymptoottista käyttäytymistä sekä suurilla muuttujan x että parametrin k arvoilla. Päätämme suoran sirontaongelman tarkastelun määrittelemällä näiden asymptoottien avulla ns. välitys- ja heijastuskertoimet $a(k)$ ja $b(k)$, joita tarvitaan myöhemmin käänteisen sirontaongelman ratkaisussa.

Käänteisessä sirontaongelmassa oletamme yksinkertaisuuden vuoksi, että potentiaali $q_1 = 0$ ja että potentiaali q_2 kuuluu sopivaan Sobolevin avaruuteen. Tällöin kiinnostavaksi potentiaaliksi jää funktio V. Ongelman asettelu on seuraava: etsi potentiaalin V mahdolliset hyppyepäjatkuvuudet ja singulariteetit, kun heijastuskerroin $b(k)$ on tunnettu. Kertoimen $b(k)$ avulla voimme määritellä ns. käänteisen Bornin approksimaation V_B. Osoitamme, että itse asiassa erotus $V - V_B$ koostuu kahdesta osasta, joista toinen kuuluu Hölderin avaruuteen $C^\alpha(\mathbb{R})$ parametrilla $0 < \alpha < 1$ ja toinen on sileä funktio. Tällöin erotus $V - V_B$ on jatkuva funktio, mikä tarkoittaa, että potentiaalin V hypyt ja singulariteetit voidaan löytää laskemalla V_B.

Muita tietoja

Direct and inverse scattering problems for operator of order 4 on the line

Master's thesis
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Contents

Introduction

Let us consider the one-dimensional differential operator of order four

$$L_4 u := \frac{d^4}{dx^4}u + (q_1 u')' + 2iq_2 u' + iq_2' u + Vu \tag{1}$$

with real-valued potentials q_1, q_2 and V from spaces that are specified later. The $2i$ and i multipliers in front of q_2 and q_2' prove to be useful because they simplify certain calculations considerably. They also provide a self-adjoint operator, but this information is not needed. In this thesis we will consider both direct and inverse scattering problems for this operator on the real line. In scattering theory a function u is called the scattering solution to equation

$$L_4 u = k^4 u, \tag{2}$$

where $k \in \mathbb{R}$ is a parameter, if it is of form $u = u_0 + u_{sc}$, where $u_0 = e^{ikx}$ is an incoming plane wave and u_{sc} is the outgoing scattered wave. The direct scattering problem concerns finding the properties of the scattered wave when the scatterers i.e. potentials q_1, q_2 and V are known. Conversely, in broad sense, the inverse scattering problem is about finding the properties of the potentials when we have data about how the scattered wave behaves far away.

In 2008 Aktosun and Papanicolaou [1] studied operator (1) with $q_2 = 0$. It is noted that in this case equation (2) is the canonical equivalent of the Euler–Bernoulli equation. While the second order differential operators such as the Schrödinger operator

$$L_2 u := -\frac{d^2}{dx^2}u + qu \tag{3}$$

can be used in the study of scattering of particles and vibrations of strings, [1, 6] fourth order differential operators can be applied to modeling vibrations of beams.

This thesis will follow the guidelines laid in work of Serov and Harju in 2006 [6] where they studied the inverse scattering problem for the Schrödinger operator in one-dimensional case. The method used is to apply the fundamental solution of operator (3) to form an integral equation and solve it. Having obtained the solution, they proceeded to study the asymptotic behaviour of the solution to find so called transmission and reflection coefficients $a(k)$ and $b(k)$. The asymptotics happen to have certain nice properties that help defining the inverse Born approximation q_B for potential q. It is then shown that the difference $q - q_B$ is actually a continuous function, meaning that we can gather essential information about q from q_B. Namely, this information is the jumps and singularities of q. The objective of this thesis is to apply the method of [6] to operator (1) and define the inverse Born approximation V_B of potential V and show that $V - V_B$ is a continuous function.

The rest of the thesis is organized as follows. In Section 1 we give definitions of certain function spaces, the Fourier transform and tempered distributions. We also give several properties of Fourier transform without proofs. In Section 2 we solve the direct scattering problem for operator (1) on the real line for the incident wave $u_0 = e^{ikx}$. After that in Section 3 we proceed to study the asymptotic behaviour of the solution obtained in the previous section. Section 4 concerns the inverse scattering problem of operator (1). We define the inverse Born approximation of potential V and carefully study the behaviour of the first non-linear term appearing in the Born series. Finally in Section 5 we discuss the results.

1 Preliminaries

1.1 Function spaces

First we define some function spaces that are needed in the sequel. We denote the Lebesgue spaces by $L^p(\mathbb{R})$ for $1 \leq p < \infty$ and define

$$L^p(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \text{ is measurable} : \|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p} < \infty \right\}$$

and

$$L^\infty(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \text{ is measurable} : \|f\|_\infty := \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty \right\}.$$

If $\Omega \subset \mathbb{R}$ is a bounded domain, we define the space of k times differentiable functions on $\overline{\Omega}$ by

$$C^k(\overline{\Omega}) := \left\{ f : \overline{\Omega} \to \mathbb{C} : \|f\|_{C^k(\overline{\Omega})} := \max_{x \in \overline{\Omega}} \sum_{i=0}^{k} \left| \frac{d^i}{dx^i} f(x) \right| < \infty \right\},$$

where $\overline{\Omega}$ is the closure of Ω. We say that a function f is smooth, that is, $f \in C^\infty(\mathbb{R})$ if $f \in C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$ and all bounded $\Omega \subset \mathbb{R}$. Furthermore we define the space of compactly supported smooth functions by

$$C_0^\infty(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) : \operatorname{supp}(f) \text{ is compact in } \mathbb{R} \},$$

where $\operatorname{supp}(f) := \overline{\{x : f(x) \neq 0\}}$. Closely related but different function space, the Hölder space, is defined as

$$C^\alpha(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \exists C > 0 \text{ such that } \sup_{x \in \mathbb{R}} |f(x+h) - f(x)| \leq C|h|^\alpha \right\}$$

for $0 < \alpha \leq 1$. The Schwartz space $S(\mathbb{R})$ of rapidly decaying functions is defined by

$$S(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : |f|_{\alpha,\beta} := \sup_{x \in \mathbb{R}} \left| x^\alpha \frac{d^\beta}{dx^\beta} f(x) \right| < \infty \text{ for any } \alpha, \beta \in \mathbb{N} \right\}$$

and the Sobolev space $W_p^k(\mathbb{R})$ is defined by

$$W_p^k(\mathbb{R}) := \left\{ f \in L^p(\mathbb{R}) : \|f\|_{W_p^k(\mathbb{R})} := \left(\sum_{i=0}^{k} \| \frac{d^i}{dx^i} f(x) \|_p^p \right)^{1/p} < \infty \right\}$$

for $k \in \mathbb{N}$. Even though we have not defined the Fourier transform or the tempered distributions yet, we give the definition of the generalized Sobolev space $W_p^s(\mathbb{R})$ for real number $s > 0$ and $1 \leq p \leq \infty$ as

$$W_p^s(\mathbb{R}) := \left\{ f \in S' : F^{-1}((1+|\xi|^2)^{\frac{s}{2}} \widehat{f}) \in L^p(\mathbb{R}) \right\}.$$

Here S' denotes the space of tempered distributions, $F^{-1}(f)$ the inverse Fourier transform and $\widehat{f}$ the Fourier transform of a distribution f.

We are now ready to define the Fourier transform and showcase its basic properties.

1.2 Fourier transform

In this section we list some definitions and well-known results about the Fourier transform. Proofs to these results are commonly found in literature and we refer to [4] and [5].

Definition 1.1. The Fourier transform in $S(\mathbb{R})$ is defined by

$$\widehat{f}(\xi) = F(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx, \quad \xi \in \mathbb{R}. \tag{4}$$

Actually, the Fourier transform maps Schwartz functions back to Schwartz functions, that is, $F : S \to S$. One can find multiple different definitions for the Fourier transform in literature and we chose to use this definition, because in our case this particular choice of the normalization constants provides convenient symmetry between formulas.

Theorem 1.2. *Let $f \in S(\mathbb{R})$. The inverse Fourier transform is defined by*

$$F^{-1}(f)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} f(\xi)d\xi, \quad x \in \mathbb{R},$$

and it satisfies $F^{-1}(F(f)) = F(F^{-1}(f)) = f$.

Proof. See [5]. □

We note that $F^{-1}(f)(\xi) = F(f)(-\xi)$. Beside the Fourier transform we will also apply convolution of two functions (or distributions) to find a particular solution to a differential equation.

Definition 1.3. The convolution $\phi * \psi$ is defined as

$$(\phi * \psi)(x) = \int_{-\infty}^{\infty} \phi(x-y)\psi(y)dy,$$

when ψ and ϕ are such functions that this integral converges.

Convolution is symmetric, i.e. $\phi * \psi = \psi * \phi$. Furthermore the Fourier transform of convolution satisfies

$$F(\phi * \psi)(\xi) = \sqrt{2\pi} F(\phi)(\xi) F(\psi)(\xi)$$

and by applying the inverse Fourier transform

$$F(\phi\psi)(\xi) = \frac{1}{\sqrt{2\pi}} (F(\phi) * F(\psi))(\xi).$$

Finally we provide some properties of the Fourier transform under differentiation.

Theorem 1.4. *Let $f \in S(\mathbb{R})$ and $k \in \mathbb{N}$. Then*

1. $F\left(\frac{d^k f}{dx^k}\right)(\xi) = (i\xi)^k F(f)(\xi)$,
2. $\frac{d^k F(f)}{d\xi^k}(\xi) = F((-ix)^k f(x))(\xi)$,
3. *F is a linear and continuous map from S into S.*

Most of these properties can be generalized into wider function spaces than S is, but we do not require more right now. What is important is that similar properties can be generalized to tempered distributions.

1.3 Tempered distributions and the fundamental solution

In this section we are working towards defining the fundamental solution to a differential operator, but before that we define two different distribution spaces, the Schwartz distributions and the tempered distributions. Schwartz distributions are certain functionals acting on the space of compactly supported smooth functions.

Definition 1.5. Let $\{\phi_j\}_{j=1}^{\infty}$ be a sequence in $C_0^{\infty}(\mathbb{R})$. We call $\{\phi_j\}_{j=1}^{\infty}$ a null-sequence if

1. There exists a compact set $K \subset \mathbb{R}$ such that $\text{supp}(\phi_j) \subset K$ for all $j \in \mathbb{N}$.
2. For any $k \in \mathbb{N}$ there holds $\sup_{x \in K} \left| \frac{d^k}{dx^k} \phi_j(x) \right| \to 0, \quad j \to \infty$.

If these properties are satisfied, we write $\phi_j \to 0$ as $j \to \infty$.

Null-sequence allows us to define the continuity of functionals. We denote the application of a functional T to a function f by $\langle T, f \rangle$ and define the Schwartz distributions as follows.

Definition 1.6. A functional $T : C_0^\infty(\mathbb{R}) \to \mathbb{C}$ is a Schwartz distribution if it is linear and continuous, that is,

1. $\langle T, a\phi + b\psi \rangle = a\langle T, \phi \rangle + b\langle T, \psi \rangle$
2. $\langle T, \phi_j \rangle \to 0, \quad j \to \infty,$

for any $a, b \in \mathbb{C}$ and $\phi, \psi \in C_0^\infty(\mathbb{R})$ and null-sequence $\{\phi_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R})$.

When T is a locally integrable function, we may define the application of T to function $\phi \in C_0^\infty(\mathbb{R})$ by

$$\langle T, \phi \rangle = \int_{-\infty}^{\infty} T(x)\phi(x)dx.$$

We can define the derivative of functionals in the sense of distributions as follows.

Definition 1.7. Let T be a Schwartz distribution. The derivative $\frac{dT}{dx}$ is defined by

$$\left\langle \frac{dT}{dx}, \phi \right\rangle = \left\langle T, -\frac{d\phi}{dx} \right\rangle,$$

and it is a Schwartz distribution.

Similarly as the Schwartz distributions, the tempered distributions are functionals, but they operate on a wider function space, the Schwartz space.

Definition 1.8. A functional $T : S(\mathbb{R}) \to \mathbb{C}$ is a tempered distribution if it is linear and continuous on $S(\mathbb{R})$, i.e.

1. $\langle T, a\phi + b\psi \rangle = a\langle T, \phi \rangle + b\langle T, \psi \rangle$ for all $a, b \in \mathbb{C}$ and $\phi, \psi \in S(\mathbb{R})$.
2. There exists $n_0 \in \mathbb{N}$ and $C_0 > 0$ such that

   $$|\langle T, \phi \rangle| \le C_0 \sum_{|\alpha|, |\beta| \le n_0} |\phi|_{\alpha,\beta}$$

   for any $\phi \in S(\mathbb{R})$.

Because $F : S \to S$, we can define the Fourier transform of a tempered distribution by formula

$$\langle F(T), \phi \rangle = \langle T, F(\phi) \rangle,$$

and it is again a tempered distribution.

There is one specifically interesting and useful distribution, so-called δ-distribution, which is defined by

$$\langle \delta, \phi \rangle = \phi(0).$$

This distribution will play an important role in the definition of a fundamental solution. Let us consider a differential operator of form

$$Lu(x) = \sum_{j=0}^{n} a_j(x) \frac{d^j}{dx^j} u(x).$$

Definition 1.9. We say that a distribution E is the fundamental solution to L if

$$L_x E(x, y) = \delta(x - y),$$

with parameter $y \in \mathbb{R}$, i.e.

$$\langle L_x E(\cdot\,, y), \phi \rangle = \phi(y).$$

Fundamental solutions are useful in studying partial differential equations since, for example, when the coefficients $a_j(x)$ are constant functions we can find a particular solution to inhomogeneous equation

$$Lu(x) = f(x)$$

as convolution $E * f = f * E = u$, see [5].

2 Direct scattering problem

Let us consider the differential equation

$$\frac{d^4}{dx^4}u = k^4 u. \tag{5}$$

In this section we first show that

$$G_k^+(|x|) := \frac{1}{4|k|^3}\left(ie^{i|k||x|} - e^{-|k||x|}\right) \tag{6}$$

is the fundamental solution to (5). We then proceed to solve the integral equations that can be obtained from formula (1) by applying G_k^+ to it via convolution. One way to find the formula (6) for fundamental solution in our case is provided in Appendix A.

In what follows we assume that the potentials $q_1 = q_1(x), q_2 = q_2(x)$ and $V = V(x)$ are real-valued and q_1 and q_2 belong to the Sobolev space $W_1^1(\mathbb{R})$ while V is integrable, that is, $V \in L^1(\mathbb{R})$. Furthermore, we assume that the unknown $u = u(x,k)$ and its first derivative are bounded functions i.e. $u, u' \in L^\infty(\mathbb{R})$.

2.1 The fundamental solution and integral equation

Theorem 2.1. *Let $k \in \mathbb{R}$. Then G_k^+ is the fundamental solution to equation* (5) *in the sense of distributions.*

Proof. Let $\phi \in C_0^\infty(\mathbb{R})$. Since

$$\left\langle \frac{d^4}{dx^4}G_k^+, \phi \right\rangle = \left\langle G_k^+, \frac{d^4}{dx^4}\phi \right\rangle$$

in the sense of distributions, we will calculate

$$\left\langle G_k^+, \frac{d^4}{dx^4}\phi \right\rangle = \frac{1}{4|k|^3}\int_{-\infty}^{\infty}\left(ie^{i|k||x|} - e^{-|k||x|}\right)\frac{d^4}{dx^4}\phi(x)dx.$$

To simplify the notations we multiply both sides by $4|k|^3$ and denote $a := |k| > 0$. Then the integral above can be divided into two parts as

$$\int_{-\infty}^{\infty}\left(ie^{i|k||x|} - e^{-|k||x|}\right)\frac{d^4}{dx^4}\phi(x)dx = \int_{-\infty}^{0}\left(ie^{-iax} - e^{ax}\right)\frac{d^4}{dx^4}\phi(x)dx$$
$$+ \int_{0}^{\infty}\left(ie^{iax} - e^{-ax}\right)\frac{d^4}{dx^4}\phi(x)dx.$$

Now integrating by parts four times yields

$$\begin{aligned}
&\int_{-\infty}^{0} \left(ie^{-iax} - e^{ax}\right) \frac{d^4}{dx^4}\phi(x)dx + \int_{0}^{\infty} \left(ie^{iax} - e^{-ax}\right) \frac{d^4}{dx^4}\phi(x)dx \\
&= -\left(-a^3 e^{-iax} - a^3 e^{ax}\right)\phi(x)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} \left(ia^4 e^{-iax} - a^4 e^{ax}\right)\phi(x)dx \\
&\quad - \left(a^3 e^{iax} + a^3 e^{-ax}\right)\phi(x)\Big|_{0}^{\infty} + \int_{0}^{\infty} \left(ia^4 e^{iax} - a^4 e^{-ax}\right)\phi(x)dx \\
&= 4a^3\phi(0) + \int_{-\infty}^{\infty} \left(ia^4 e^{ia|x|} - a^4 e^{-a|x|}\right)\phi(x)dx \\
&= 4a^3 \langle \delta, \phi \rangle + 4a^7 \left\langle G_k^+, \phi \right\rangle .
\end{aligned}$$

Finally by dividing both sides by $4a^3$ and rearranging the terms we obtain

$$\left\langle \frac{d^4}{dx^4} G_k^+ - a^4 G_k^+, \phi \right\rangle = \langle \delta, \phi \rangle ,$$

which means that G_k^+ is the fundamental solution to (5). □

Before going further into application of the fundamental solution, we prove the following lemma.

Lemma 2.2. *If $f \in W_1^1(\mathbb{R})$, then f is uniformly continuous and $f \to 0$ when $x \to \pm\infty$.*

Proof. Let us denote

$$h(x) := \int_{-\infty}^{x} f'(y)dy$$

and let $\phi \in C_0^\infty(\mathbb{R})$. We show that h is uniformly continuous and $h = f$ almost everywhere. Let $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ be real sequences such that $|x_n - y_n| \to 0$ as $n \to \infty$. Then

$$|h(x_n) - h(y_n)| = \left| \int_{y_n}^{x_n} f'(z)dz \right| \to 0$$

as $n \to \infty$, since $f' \in L^1(\mathbb{R})$. This means that h is uniformly continuous, because if it was not uniformly continuous, then there would exist $\epsilon > 0$ such that for any $\delta > 0$ we could find points x and y such that $|x - y| < \delta$, but $|h(x) - h(y)| \geq \epsilon$. In this case we could choose sequences $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ so that $|x_n - y_n| < \frac{1}{n}$ and $|h(x_n) - h(y_n)| \geq \epsilon$, but this contradicts the fact that $|h(x_n) - h(y_n)| \to 0$ as $n \to \infty$.

Moreover

$$\int_{-\infty}^{\infty} f(x)\phi'(x)dx = -\int_{-\infty}^{\infty} f'(x)\phi(x)dx$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} h(x)\phi'(x)dx &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f'(y)dy\right)\phi'(x)dx \\ &= \int_{-\infty}^{\infty} f'(y)\left(\int_{y}^{\infty}\phi'(x)dx\right)dy \\ &= -\int_{-\infty}^{\infty} f'(y)\phi(y)dy.\end{aligned}$$

This means that $h = f$ at least almost everywhere.

To obtain $f(x) \to 0$ as $x \to \infty$ we argue by contradiction that there exists $c > 0$ such that for any $n \in \mathbb{N}$ we find $x_n > n$ such that $|f(x_n)| > c$. Then because f is uniformly continuous we find $\delta > 0$, that only depends on c, such that $|f(x_n) - f(x)| > \frac{c}{2}$ when $|x_n - x| < \delta$. This means that

$$\int_{x_n-\delta}^{x_n+\delta} |f(x)|dx > c\delta$$

for any $n \in \mathbb{N}$. Since $x_n \to \infty$ as $n \to \infty$, f can not be integrable which contradicts our assumption. Thus $f(x) \to 0$ as $x \to \infty$. The same argument shows that also $f(x) \to 0$ as $x \to -\infty$. This concludes the proof. □

Having obtained the fundamental solution to operator

$$L_0 u := \frac{d^4}{dx^4}u - k^4 u$$

we can find any particular solution to equation

$$\frac{d^4}{dx^4}u - k^4 u = f$$

by convolution. Namely, any particular solution can be written in terms of the fundamental solution G_k^+ as

$$u = G_k^+ * f.$$

We consider first the case when k is real and positive. In our considerations we are interested in equation

$$L_4 u = \frac{d^4}{dx^4}u + (q_1 u')' + 2iq_2 u' + iq_2' u + Vu = k^4 u.$$

By rearranging the terms we obtain

$$\frac{d^4}{dx^4}u - k^4 u = -(q_1 u')' - 2iq_2 u' - iq_2' u - Vu,$$

which means that this differential equation can be written as an integral equation in a form

$$u(x,k) = -\int_{-\infty}^{\infty} G_k^+(|x-y|)\left((q_1 u')' + 2iq_2 u' + iq_2' u + Vu\right) dy, \qquad (7)$$

where we have dropped the parameter k and integration variable for simplicity.

Since we are working on a scattering problem we choose our solution to be of the form

$$u = u_0 + u_{sc}, \qquad (8)$$

where u_0 is defined as

$$u_0(x,k) = e^{ikx}.$$

Clearly

$$\frac{d^4}{dx^4}u_0 - k^4 u_0 = 0$$

so that we can write

$$u_{sc}(x,k) = -\int_{-\infty}^{\infty} G_k^+(|x-y|)\left((q_1 u')' + 2iq_2 u' + iq_2' u + Vu\right) dy. \qquad (9)$$

We show that this integral equation can be solved by iterations

$$u_j(x,k) = -\int_{-\infty}^{\infty} G_k^+(|x-y|)\left((q_1 u_{j-1}')' + 2iq_2 u_{j-1}' + iq_2' u_{j-1} + Vu_{j-1}\right) dy,$$

for $j = 1, 2, \ldots,$ $k > 0$ and u_0 as before.

Remark 2.3. By combining the derivatives and integrating by parts we get

$$\begin{aligned}
u_{sc}(x,k) &= -\int_{-\infty}^{\infty} G_k^+(|x-y|)\left((q_1u')' + i(q_2u)' + iq_2u' + Vu\right)dy \\
&= \int_{-\infty}^{x} \frac{1}{4k^2}\left(e^{ik(x-y)} - e^{-k(x-y)}\right)[q_1u' + iq_2u]\,dy \\
&\quad + \int_{x}^{\infty} \frac{1}{4k^2}\left(-e^{ik(y-x)} + e^{-k(y-x)}\right)[q_1u' + iq_2u]\,dy \\
&\quad - \int_{-\infty}^{\infty} G_k^+(|x-y|)\left(iq_2u' + Vu\right)dy \\
&= \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(e^{ik|x-y|} - e^{-k|x-y|}\right)[q_1u' + iq_2u]\,dy \\
&\quad - \int_{-\infty}^{\infty} G_k^+(|x-y|)\left(iq_2u' + Vu\right)dy,
\end{aligned}$$

where $\operatorname{sgn}(x)$ is the signum function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{when } x > 0, \\ 0, & \text{when } x = 0, \\ -1, & \text{when } x < 0. \end{cases}$$

This form is better for our considerations since we don't have to deal with the second order derivatives.

2.2 Solving the integral equation

Next we find estimates for the iterations defined in previous section.

Lemma 2.4. *We have estimates*

$$\|u_j\|_\infty \le \left(\frac{C_0}{k}\right)^j$$

and

$$\|u_j'\|_\infty \le k\left(\frac{C_0}{k}\right)^j$$

for $j = 0, 1, \ldots,$ *where* $C_0 = \|q_1\|_1 + \|q_2\|_1 + \|V\|_1,$ *when* $k \ge 1.$

Proof. Let us prove this statement by induction. It is clear that the claim holds for u_0 and u_0'. Assume it is true that

$$\|u_{j-1}\|_\infty \le \left(\frac{C_0}{k}\right)^{j-1}$$

and

$$\|u'_{j-1}\|_\infty \le k\left(\frac{C_0}{k}\right)^{j-1}.$$

Then we can write the formula of u_j and estimate it. Since

$$\begin{aligned}u_j(x,k) = &\int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(e^{ik|x-y|} - e^{-k|x-y|}\right)\left[q_1 u'_{j-1} + iq_2 u_{j-1}\right] dy \\ &- \int_{-\infty}^{\infty} G_k^+(|x-y|)\left(iq_2 u'_{j-1} + V u_{j-1}\right) dy,\end{aligned}$$

we have

$$\begin{aligned}|u_j(x,k)| &\le \frac{1}{2k^2}\left(\|q_1\|_1\|u'_{j-1}\|_\infty + \|q_2\|_1\|u_{j-1}\|_\infty\right) \\ &\quad + \frac{1}{2k^3}\left(\|q_2\|_1\|u'_{j-1}\|_\infty + \|V\|_1\|u_{j-1}\|_\infty\right) \\ &\le \frac{1}{2k^2}\left(\|q_1\|_1 k\left(\frac{C_0}{k}\right)^{j-1} + \|q_2\|_1\left(\frac{C_0}{k}\right)^{j-1}\right) \\ &\quad + \frac{1}{2k^3}\left(\|q_2\|_1 k\left(\frac{C_0}{k}\right)^{j-1} + \|V\|_1\left(\frac{C_0}{k}\right)^{j-1}\right) \\ &\le \left(\frac{C_0}{k}\right)^{j},\end{aligned}$$

but we still need to prove the second claim to conclude the induction proof.

The estimate for u'_j can be obtained by a straight-forward differentiation and an application of the previous result. Since

$$\begin{aligned}u'_j(x,k) = &\frac{d}{dx}\int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2}\left(e^{ik|x-y|} - e^{-k|x-y|}\right)\left[q_1 u' + iq_2 u\right] dy \\ &- \frac{d}{dx}\int_{-\infty}^{\infty} G_k^+(|x-y|)\left(iq_2 u' + V u\right) dy,\end{aligned}$$

we can split the integrals with respect to the signum function. Then the integrand and its first derivative are continuous with respect to x and we can calculate these derivatives by the Leibniz rule to obtain

$$\begin{aligned}
u_j'(x,k) = &\int_{-\infty}^{x} \frac{1}{4k^2}\left(ike^{ik(x-y)} + ke^{-k(x-y)}\right)(q_1 u_{j-1}' + iq_2 u_{j-1})dy \\
&- \int_{x}^{\infty} \frac{1}{4k^2}\left(-ike^{ik(y-x)} - ke^{-k(y-x)}\right)(q_1 u_{j-1}' + iq_2 u_{j-1})dy \\
&- \int_{-\infty}^{x} \frac{1}{4k^3}\left(-ke^{ik(x-y)} + ke^{-k(x-y)}\right)(iq_2 u_{j-1}' + V u_{j-1})dy \\
&- \int_{x}^{\infty} \frac{1}{4k^3}\left(ke^{ik(y-x)} - ke^{-k(y-x)}\right)(iq_2 u_{j-1}' + V u_{j-1})dy.
\end{aligned}$$

This calculation is only formal for the moment, but we will see that these integrals converge uniformly. Using this we get the estimate

$$\begin{aligned}
|u_j'(x,k)| \le &\frac{1}{2k}(\|q_1\|_1 \|u_{j-1}'\|_\infty + \|q_2\|_1 \|u_{j-1}\|_\infty) \\
&+ \frac{1}{2k^2}(\|q_2\|_1 \|u_{j-1}'\|_\infty + \|V\|_1 \|u_{j-1}\|_\infty)
\end{aligned}$$

and further

$$\begin{aligned}
|u_j'(x,k)| &\le \frac{1}{2k}\left(\|q_1\|_1 k\left(\frac{C_0}{k}\right)^{j-1} + \|q_2\|_1\left(\frac{C_0}{k}\right)^{j-1}\right) \\
&\quad + \frac{1}{2k^2}\left(\|q_2\|_1 k\left(\frac{C_0}{k}\right)^{j-1} + \|V\|_1\left(\frac{C_0}{k}\right)^{j-1}\right) \\
&\le k\left(\frac{C_0}{k}\right)^{j},
\end{aligned}$$

when $k \ge 1$. By induction this is what we wanted. □

Remark 2.5. If we set $q_1 = 0$ at the beginning, we get considerably better estimates, namely

$$|u_j(x,k)| \le \left(\frac{\widetilde{C_0}}{k^2}\right)^{j}$$

and

$$|u_j'(x,k)| \le k\left(\frac{\widetilde{C_0}}{k^2}\right)^{j},$$

for $j = 0, 1, \ldots$, where $\widetilde{C_0} = \|q_2\|_1 + \|q_2'\|_1 + \|V\|_1$ and the iterations are defined as

$$u_j(x,k) = -\int_{-\infty}^{\infty} G_k^+(|x-y|) \left(2iq_2 u_{j-1}' + iq_2' u_{j-1} + V u_{j-1}\right) dy, \quad j = 1, 2, \ldots$$

These estimates will play an important role in Section 4.

Lemma 2.6. *The function $u(x,k)$ defined by the series*

$$u(x,k) = \sum_{j=0}^{\infty} u_j(x,k)$$

converges uniformly when $k \geq 1$ and $k > C_0$ and is the unique solution to the equation (9), when $k > C := C_0 + \|q_1'\|_1 + \|q_2'\|_1$.

Proof. By Lemma 2.4 we have estimates

$$\sum_{j=0}^{\infty} |u_j(x,k)| \leq \sum_{j=0}^{\infty} \left(\frac{C_0}{k}\right)^j$$

and

$$\sum_{j=0}^{\infty} |u_j'(x,k)| \leq k \sum_{j=0}^{\infty} \left(\frac{C_0}{k}\right)^j.$$

When k is as claimed, these series are simply geometric progressions that converge, so the series above converge uniformly. Moreover substituting $u(x,k)$ into the right-hand side of the form obtained in Remark 2.3 yields

$$\int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2} \left(e^{ik|x-y|} - e^{-k|x-y|}\right) \left[q_1 \left(\sum_{j=0}^{\infty} u_j\right)' + iq_2 \sum_{j=0}^{\infty} u_j\right] dy$$
$$- \int_{-\infty}^{\infty} G_k^+(|x-y|) \left(iq_2 \left(\sum_{j=0}^{\infty} u_j\right)' + V \sum_{j=0}^{\infty} u_j\right) dy =: \widetilde{u_{sc}}(x,k)$$

By uniform convergence we can differentiate the series term by term and change the order of integration and summation to obtain

$$\widetilde{u_{sc}}(x,k) = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2} \left(e^{ik|x-y|} - e^{-k|x-y|}\right) \left[q_1 u_j' + iq_2 u_j\right] dy$$
$$- \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} G_k^+(|x-y|) \left(iq_2 u_j' + V u_j\right) dy$$
$$= \sum_{j=0}^{\infty} u_{j+1}(x,k) = \sum_{j=1}^{\infty} u_j(x,k).$$

This series satisfies the form $u = u_0 + \widetilde{u_{sc}}$.

This solution is also unique when $k > C$. To see this, we integrate by parts the form for u_{sc}, that was obtained in Remark 2.3, to get rid of derivatives of u. Then, we find an estimate

$$|u_{sc}(x,k)| \leq \frac{C}{k}\|u\|_\infty.$$

This means that if v is another solution to equation (9) with the property that $v = u_0 + v_{sc}$, we have

$$|u(x,k) - v(x,k)| = |u_{sc}(x,k) - v_{sc}(x,k)| \leq \frac{C}{k}\|u - v\|_\infty.$$

This inequality holds for all $x \in \mathbb{R}$ when $k > C$ and $k \geq 1$, so

$$\|u - v\|_\infty \leq \frac{C}{k}\|u - v\|_\infty.$$

Since the constant $\frac{C}{k} < 1$, this inequality is only possible when $u = v$. $\square$

Remark 2.7. Since the parameter k is in our hands, we can choose it to be large enough in absolute value to provide the existence and the uniqueness of the solution. Moreover, while scattering problems can arise, for example, in physics, we focus only on the mathematics of the integral equation and do not require more special properties from u_{sc}.

We have now found the solution to equation (9) for $k > C_0$. For $k < 0$ we define $u(x,k) = \overline{u(x,-k)}$ and $u'(x,k) = \overline{u'(x,-k)}$, where $\overline{z}$ denotes the complex conjugate of z, and obtain a new integral equation of the form

$$u(x,k) = -\frac{1}{4k^3}\int_{-\infty}^{\infty}\left(ie^{ik|x-y|} + e^{k|x-y|}\right)\left((q_1 u')' - 2iq_2 u' - iq_2' u + Vu\right)dy.$$

Similar calculation shows that this equation can also be solved by iterations

$$u_j(x,k) = -\frac{1}{4k^3}\int_{-\infty}^{\infty}\left(ie^{ik|x-y|} + e^{k|x-y|}\right)\left((q_1 u_{j-1}')' - 2iqu_{j-1}' - iq'u_{j-1} + Vu_{j-1}\right)dy$$

for $j = 1, 2, \ldots$ with same u_0 and estimates as before.

3 Asymptotic behaviour of the solution

Having obtained solutions to our integral equations we will consider their asymptotic behaviour for large $|x|$. For $k > 0$ we have equation

$$u(x,k) = e^{ikx} + \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2} \left(e^{ik|x-y|} - e^{-k|x-y|}\right) [q_1 u' + i q_2 u] \, dy$$
$$- \int_{-\infty}^{\infty} G_k^+(|x-y|) \, (i q_2 u' + V u) \, dy.$$

By defining

$$\Psi_1(x,k) := q_1(x) u'(x,k) + i q_2(x) u(x,k)$$

and

$$\Psi_2(x,k) := i q_2(x) u'(x,k) + V(x) u(x,k)$$

we can write out this integral as

$$\begin{aligned} u(x,k) = e^{ikx} &+ \frac{1}{4k^2} e^{ikx} \int_{-\infty}^{x} e^{-iky} \Psi_1 dy - \frac{1}{4k^2} e^{-kx} \int_{-\infty}^{x} e^{ky} \Psi_1 dy \\ &- \frac{1}{4k^2} e^{-ikx} \int_{x}^{\infty} e^{iky} \Psi_1 dy + \frac{1}{4k^2} e^{kx} \int_{x}^{\infty} e^{-ky} \Psi_1 dy \\ &- \frac{1}{4k^3} e^{ikx} \int_{-\infty}^{x} i e^{-iky} \Psi_2 dy + \frac{1}{4k^3} e^{-kx} \int_{-\infty}^{x} e^{ky} \Psi_2 dy \\ &- \frac{1}{4k^3} e^{-ikx} \int_{x}^{\infty} i e^{iky} \Psi_2 dy + \frac{1}{4k^3} e^{kx} \int_{x}^{\infty} e^{-ky} \Psi_2 dy. \end{aligned}$$

Further, splitting the first and fifth integral into two parts as $\int_{-\infty}^{x} = \int_{-\infty}^{\infty} - \int_{x}^{\infty}$ and combining all the integrals suitably yields

$$\begin{aligned} u(x,k) = e^{ikx} &+ \frac{1}{4k^3} e^{ikx} \int_{-\infty}^{\infty} e^{-iky} (k\Psi_1 - i\Psi_2) dy \\ &- \frac{1}{4k^3} e^{ikx} \int_{x}^{\infty} e^{-iky} (k\Psi_1 - i\Psi_2) dy \\ &- \frac{1}{4k^3} e^{-kx} \int_{-\infty}^{x} e^{ky} (k\Psi_1 - \Psi_2) dy \\ &- \frac{1}{4k^3} e^{-ikx} \int_{x}^{\infty} e^{iky} (k\Psi_1 + i\Psi_2) dy \\ &+ \frac{1}{4k^3} e^{kx} \int_{x}^{\infty} e^{-ky} (k\Psi_1 + \Psi_2) dy. \end{aligned} \tag{10}$$

We note that Ψ_1 and Ψ_2 are integrable functions.

Lemma 3.1. *Let $\Psi \in L^1(\mathbb{R})$ and $k > 0$. Then*

$$e^{-kx} \int_{-\infty}^{x} e^{ky} \Psi(y) dy \to 0 \ \textit{as} \ x \to \pm\infty$$

and

$$e^{kx} \int_{x}^{\infty} e^{-ky} \Psi(y) dy \to 0 \ \textit{as} \ x \to \pm\infty.$$

Proof. We have

$$\begin{aligned}\left| e^{-kx} \int_{-\infty}^{x} e^{ky} \Psi(y) dy \right| &= \left| e^{-kx} \int_{-\infty}^{\frac{1}{2}x} e^{ky} \Psi(y) dy + e^{-kx} \int_{\frac{1}{2}x}^{x} e^{ky} \Psi(y) dy \right| \\ &\leq e^{-\frac{kx}{2}} \int_{-\infty}^{\frac{1}{2}x} |\Psi(y)| dy + \int_{\frac{1}{2}x}^{x} |\Psi(y)| dy.\end{aligned}$$

Since $\Psi \in L^1(\mathbb{R})$, both terms tend to zero as $x \to +\infty$. On the other hand

$$\left| e^{-kx} \int_{-\infty}^{x} e^{ky} \Psi(y) dy \right| \leq \int_{-\infty}^{x} |\Psi(y)| dy \to 0,$$

when $x \to -\infty$. Similarly

$$\begin{aligned}\left| e^{kx} \int_{x}^{\infty} e^{-ky} \Psi(y) dy \right| &= \left| -e^{kx} \int_{\frac{1}{2}x}^{x} e^{-ky} \Psi(y) dy + e^{kx} \int_{\frac{1}{2}x}^{\infty} e^{-ky} \Psi(y) dy \right| \\ &\leq \int_{x}^{\frac{1}{2}x} |\Psi(y)| dy + e^{\frac{kx}{2}} \int_{\frac{1}{2}x}^{\infty} |\Psi(y)| dy,\end{aligned}$$

where both terms tend to zero as $x \to -\infty$. Correspondingly,

$$\left| e^{kx} \int_{x}^{\infty} e^{-ky} \Psi(y) dy \right| \leq \int_{x}^{\infty} |\Psi(y)| dy \to 0,$$

when $x \to \infty$. □

Now, by letting $x \to \pm\infty$ and applying Lemma 3.1 to (10) we obtain the asymptotics

$$u(x,k) = e^{ikx} \left(1 + \frac{1}{4k^3} \int_{-\infty}^{\infty} e^{-iky} (k\Psi_1 - i\Psi_2) dy \right) + o(1), \quad x \to \infty$$

and

$$u(x,k) = e^{ikx} - \frac{e^{-ikx}}{4k^3} \int_{-\infty}^{\infty} e^{iky}(k\Psi_1 + i\Psi_2)dy + o(1), \quad x \to -\infty.$$

Earlier we defined $u(x,k) = \overline{u(x,-k)}$ for $k < 0$ and the integral equation for negative k can be written as

$$\begin{aligned} u_{sc}(x,k) = & \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{4k^2} \left(e^{ik|x-y|} - e^{-k|x-y|}\right)[q_1 u' - iq_2 u]\, dy \\ & - \int_{-\infty}^{\infty} \frac{1}{4k^3} \left(ie^{ik|x-y|} + e^{k|x-y|}\right)(-iq_2 u' + Vu)\, dy. \end{aligned}$$

Defining

$$\Phi_1(x,k) := q_1(x)u'(x,k) - iq_2(x)u(x,k)$$

and

$$\Phi_2(x,k) := -iq_2(x)u'(x,k) + V(x)u(x,k)$$

and doing similar calculations as above we find that the asymptotics are

$$u(x,k) = e^{ikx}\left(1 + \frac{1}{4k^3}\int_{-\infty}^{\infty} e^{-iky}(k\Phi_1 - i\Phi_2)dy\right) + o(1), \quad x \to \infty$$

and

$$u(x,k) = e^{ikx} - \frac{e^{-ikx}}{4k^3} \int_{-\infty}^{\infty} e^{iky}(k\Phi_1 + i\Phi_2)dy + o(1), \quad x \to -\infty.$$

These asymptotics are usually written in the form

$$u(x,k) = e^{ikx}a(k) + o(1), \quad x \to \infty$$

and

$$u(x,k) = e^{ikx} + e^{-ikx}b(k) + o(1), \quad x \to -\infty.$$

Analogously to the Schrödinger operator in [6], we may define the transmission and the reflection coefficients by

$$a(k) = \begin{cases} 1 + \frac{1}{4k^3}\int_{-\infty}^{\infty} e^{-iky}\left(k\Psi_1 - i\Psi_2\right) dy & \text{when } k > 0, \\ 1 + \frac{1}{4k^3}\int_{-\infty}^{\infty} e^{-iky}\left(k\Phi_1 - i\Phi_2\right) dy & \text{when } k < 0 \end{cases}$$

and

$$b(k) = \begin{cases} -\frac{1}{4k^3}\int_{-\infty}^{\infty} e^{iky}\left(k\Psi_1 + i\Psi_2\right)dy & \text{when } k > 0, \\ -\frac{1}{4k^3}\int_{-\infty}^{\infty} e^{iky}\left(k\Phi_1 + i\Phi_2\right)dy & \text{when } k < 0. \end{cases}$$

By definition $\Psi_i(x,k) = \overline{\Phi_i(x,-k)}$, $i = 1,2$ so that $a(-k) = \overline{a(k)}$ and $b(-k) = \overline{b(k)}$. This means that it is enough to consider $k > 0$ since we can extend the results for $k < 0$ with these formulas. These asymptotics were for large $|x|$. Next we study the asymptotics for large $|k|$.

Lemma 3.2. *If $|k| > 2C_0$, where $C_0 = \|q_1\|_1 + \|q_2\|_1 + \|V\|_1$ and $k \geq 1$, then the tails of the series representations of u and u' satisfy*

$$\sum_{j=m}^{\infty} |u_j(x,k)| \leq \frac{C_m}{|k|^m}$$

and

$$\sum_{j=m}^{\infty} |u_j'(x,k)| \leq \frac{C_m}{|k|^{m-1}}$$

for $m = 0, 1, \ldots$, with $C_m = 2C_0^m$.

Proof. By Lemma 2.4 we have

$$\begin{aligned}\sum_{j=m}^{\infty} |u_j(x,k)| &\leq \sum_{j=m}^{\infty}\left(\frac{C_0}{|k|}\right)^j = \left(\frac{C_0}{|k|}\right)^m \sum_{j=0}^{\infty}\left(\frac{C_0}{|k|}\right)^j \\ &= \left(\frac{C_0}{|k|}\right)^m \frac{1}{1-\frac{C_0}{|k|}} = \left(\frac{C_0}{|k|}\right)^m \frac{|k|}{|k|-C_0} \leq \frac{2C_0^m}{|k|^m}.\end{aligned}$$

For u' the only difference is additional k in the numerator. □

Remark 3.3. Again, if we set $q_1 = 0$, define $\widetilde{C}_m = 2\widetilde{C}_0^m = 2(\|q_2\|_1 + \|q_2'\|_1 + \|V\|_1)^m$ and use the iterations defined in Remark 2.5 we find that the tails of the series representations behave as

$$\sum_{j=m}^{\infty} |u_j(x,k)| \leq \frac{\widetilde{C}_m}{|k|^{2m}}$$

and

$$\sum_{j=m}^{\infty} |u_j'(x,k)| \leq \frac{\widetilde{C}_m}{|k|^{2m-1}},$$

for $m = 0, 1, \ldots$, when $|k| \geq \sqrt{2\widetilde{C}_0}$.

By Lemma 3.2 we may conclude that u and u' behave asymptotically like

$$u(x,k) = e^{ikx} + O\left(\frac{1}{k}\right), \quad k \to \infty$$
$$u'(x,k) = ike^{ikx} + O(1), \quad k \to \infty.$$

Using these formulas we see that

$$\begin{aligned} a(k) &\approx 1 + \frac{1}{4k^3}\int_{-\infty}^{\infty} e^{-iky}\left(kq_1ike^{iky} + ikq_2e^{iky} + ikq_2e^{iky} - iVe^{iky}\right)dy \\ &= 1 + \frac{1}{4k^3}\int_{-\infty}^{\infty}\left(ik^2q_1 + 2ikq_2 - iV\right)dy \\ &\approx 1, \quad k \to \infty \end{aligned}$$

and

$$\begin{aligned} b(k) &\approx -\frac{1}{4k^3}\int_{-\infty}^{\infty} e^{iky}\left(kq_1ike^{iky} + ikq_2e^{iky} - ikq_2e^{iky} + iVe^{iky}\right)dy \\ &\approx -\frac{1}{4k^3}\int_{-\infty}^{\infty} e^{2iky}\left(ik^2q_1 + iV\right)dy, \quad k \to \infty. \end{aligned}$$

Thus it is reasonable to choose $b(k)$ as the object of interest. Setting $q_1 = 0$, we can formally write

$$V(y) \approx F\left(\frac{ik^3}{2\sqrt{2\pi}}b\left(\frac{k}{2}\right)\right)(y).$$

We have now solved the direct scattering problem for operator (1) on real line and studied its asymptotic behaviour. Next we turn our attention to the inverse problem that can be formulated as follows: find and construct the jumps and discontinuities of the potential V given the reflection coefficient b. Recall that $q_2 \in W_1^1(\mathbb{R})$ and therefore is continuous and vanishes at infinity, but V is only an integrable function, so it is natural to study the discontinuities of V. For simplicity we consider only the case $q_1 = 0$.

4 Inverse Born approximation of potential V

In previous section we obtained the reflection and the transmission coefficients for the operator (1). We focus on the first order perturbation by letting $q_1 = 0$. Then we can write operator (1) in a simpler form

$$L_4 u = \frac{d^4}{dx^4} u + 2iqu' + iq'u + Vu.$$

We do not repeat the calculations for this simpler operator since they are completely analogous to what we did in Sections 2 and 3. In this case we can use the simpler iterations as noted in Remark 2.5 that can be written as

$$u_j(z,k) = -\frac{1}{4k^3} \int_{-\infty}^{\infty} \left(ie^{ik|z-y|} - e^{-k|z-y|} \right) (2iqu'_{j-1} + iq'u_{j-1} + Vu_{j-1}) dy$$

for $k > 0$ and

$$u_j(z,k) = -\frac{1}{4k^3} \int_{-\infty}^{\infty} \left(ie^{ik|z-y|} + e^{k|z-y|} \right) (-2iqu'_{j-1} - iq'u_{j-1} + Vu_{j-1}) dy$$

for $k < 0$. The derivative u'_j can be calculated as in the proof of Lemma 2.4 to obtain

$$u'_j(z,k) = \frac{1}{4k^2} \int_{-\infty}^{\infty} \operatorname{sgn}(z-y) \left(e^{ik|z-y|} - e^{-k|z-y|} \right) \times (2iqu'_{j-1} + iq'u_{j-1} + Vu_{j-1}) dy$$

for $k > 0$ and

$$u'_j(z,k) = \frac{1}{4k^2} \int_{-\infty}^{\infty} \operatorname{sgn}(z-y) \left(e^{ik|z-y|} - e^{k|z-y|} \right) \times (-2iqu'_{j-1} - iq'u_{j-1} + Vu_{j-1}) dy$$

for $k < 0$. As noted in Remark 3.3, while studying the asymptotic behaviour of the solution, we saw that $|k|$ had to be chosen to be large enough to conclude certain behaviour for the tails of the series representations of u and u'. We take this into account while defining the reflection coefficient $b(k)$. By defining $k_0 := \sqrt{2(\|q\|_1 + \|q'\|_1 + \|V\|_1)}$ we can write the reflection coefficient $b(k/2)$ as

$$b\left(\frac{k}{2}\right) = \frac{-2}{k^3} \int_{-\infty}^{\infty} ie^{i\frac{k}{2}z} \left[2iq(z)u'(z,k/2) + iq'(z)u(z,k/2) + V(z)u(z,k/2) \right] dz$$

for $k > k_0$ and

$$b\left(\frac{k}{2}\right) = \frac{2}{k^3}\int_{-\infty}^{\infty} ie^{i\frac{k}{2}z}\left[2iq(z)u'(z,k/2) + iq'(z)u(z,k/2) - V(z)u(z,k/2)\right]dz$$

for $k < -k_0$. We set $b(k/2) = 0$ when $|k| \leq 2k_0$. To simplify the notation we define new function $\chi(k)$ as follows

$$\chi(k) = \begin{cases} 1, & \text{when } |k| > k_0 \\ 0, & \text{when } |k| \leq k_0. \end{cases}$$

Then coefficient $b(k/2)$ has the simple form

$$\begin{aligned} b\left(\frac{k}{2}\right) &= -\frac{2\chi(k/2)}{k^3}\int_{-\infty}^{\infty} ie^{i\frac{k}{2}z} \\ &\quad \times \left(\operatorname{sgn}(k)\left[2iq(z)u'(z,k/2) + iq'(z)u(z,k/2)\right] + V(z)u(z,k/2)\right)dz \end{aligned}$$

for all $k \in \mathbb{R}$.

Inspired by the asymptotic behaviour of b as $k \to \infty$ we have following definition.

Definition 4.1. The inverse Born approximation of potential $V(\xi)$ is

$$V_B(\xi) = F\left(\frac{ik^3}{2\sqrt{2\pi}}b\left(\frac{k}{2}\right)\right)(\xi). \tag{11}$$

Remark 4.2. Because $b(k) = \overline{b(-k)}$ for $k < 0$ we see that

$$\begin{aligned} V_B(y) &= \frac{i}{4\pi}\left(\int_{-\infty}^{-k_0} k^3 b(k/2)e^{-iky}dk + \int_{k_0}^{\infty} k^3 b(k/2)e^{-iky}dk\right) \\ &= \frac{i}{4\pi}\left(-\overline{\int_{k_0}^{\infty} k^3 b(k/2)e^{-iky}dk} + \int_{k_0}^{\infty} k^3 b(k/2)e^{-iky}dk\right), \end{aligned}$$

so V_B is real-valued.

We want to show that $V - V_B$ is, at least, a continuous function. This would imply that the discontinuities and jumps of the potential V could be recovered by calculating V_B. We will let $C > 0$ denote a generic constant that can have different values from step to step when it is not important to keep track of its precise value. Let us consider the reflection coefficient $b(k)$ now more carefully. Since

$$u(x,k) = \sum_{j=0}^{\infty} u_j(x,k),$$

where $u_0(x,k) = e^{ikx}$, we can write

$$\begin{aligned}
b\left(\frac{k}{2}\right) &= -\frac{2\chi(k/2)}{k^3}\int_{-\infty}^{\infty} ie^{i\frac{k}{2}z}\left(\operatorname{sgn}(k)\left[-kq(z)+iq'(z)\right]+V(z)\right)e^{i\frac{k}{2}z}dz \\
&- \frac{2\chi(k/2)}{k^3}\int_{-\infty}^{\infty} ie^{i\frac{k}{2}z} \\
&\quad\times(\operatorname{sgn}(k)\left[2iq(z)u_1'(z,k/2)+iq'(z)u_1(z,k/2)\right]+V(z)u_1(z,k/2))dz \\
&- \frac{2\chi(k/2)}{k^3}\sum_{j=2}^{\infty}\int_{-\infty}^{\infty} ie^{i\frac{k}{2}z} \\
&\quad\times(\operatorname{sgn}(k)\left[2iq(z)u_j'(z,k/2)+iq'(z)u_j(z,k/2)\right]+V(z)u_j(z,k/2))dz \\
&=: \chi(k/2)(b_0+b_1+b_{rest})\left(\frac{k}{2}\right).
\end{aligned}$$

Investigating the first integral and especially the term $iq'(z)$ shows that if we integrate by parts we obtain

$$-\frac{2\chi(k/2)}{k^3}\int_{-\infty}^{\infty} ie^{ikz}\operatorname{sgn}(k)iq'(z)dz = -\frac{2\chi(k/2)}{k^3}\int_{-\infty}^{\infty} ie^{ikz}\operatorname{sgn}(k)kq(z)dz$$

and $b_0(k/2)$ can be written in form

$$b_0\left(\frac{k}{2}\right) = -\frac{2}{k^3}\int_{-\infty}^{\infty} ie^{ikz}V(z)dz = -\frac{2\sqrt{2\pi}i}{k^3}F^{-1}(V)(k).$$

This means that the inverse Born approximation is

$$\begin{aligned}
V_B(\xi) &= F\left(\frac{\chi(k/2)ik^3}{2\sqrt{2\pi}}b_0\left(\frac{k}{2}\right)\right)(\xi) + F\left(\frac{\chi(k/2)ik^3}{2\sqrt{2\pi}}b_1\left(\frac{k}{2}\right)\right)(\xi) \\
&\quad + F\left(\frac{\chi(k/2)ik^3}{2\sqrt{2\pi}}b_{rest}\left(\frac{k}{2}\right)\right)(\xi) \\
&= V(\xi) + F\left(\frac{\chi(k/2)ik^3}{2\sqrt{2\pi}}b_1\left(\frac{k}{2}\right)\right)(\xi) \\
&\quad + F\left(\frac{(\chi(k/2)-1)ik^3}{2\sqrt{2\pi}}b_0\left(\frac{k}{2}\right)\right)(\xi) + F\left(\frac{\chi(k/2)ik^3}{2\sqrt{2\pi}}b_{rest}\left(\frac{k}{2}\right)\right)(\xi) \\
&=: V(\xi) + V_1(\xi) + \widetilde{V}(\xi) + V_{rest}(\xi). \qquad (12)
\end{aligned}$$

Here $\widetilde{V}(\xi)$ is a Fourier transform of a compactly supported distribution. By the Paley–Wiener theorem [5] $\widetilde{V} \in C^{\infty}(\mathbb{R})$, so it is enough to consider only V_1 and V_{rest}. We will study V_{rest} and V_1 separately to confirm their continuity.

Lemma 4.3. *If V_{rest} is as above then $V_{rest} \in W_2^s(\mathbb{R})$ for any $s < \frac{5}{2}$.*

Proof. We start by estimating b_{rest}. By applying Remark 3.3 we find that when $|k| > k_0$

$$\begin{aligned}\left|b_{rest}\left(\frac{k}{2}\right)\right| &\leq \frac{2}{|k|^3}\int_{-\infty}^{\infty}\left[2|q(y)|\sum_{j=2}^{\infty}|u_j'| + |q'(y)|\sum_{j=2}^{\infty}|u_j| + |V(y)|\sum_{j=2}^{\infty}|u_j|\right]dy \\ &\leq \frac{2}{|k|^3}\int_{-\infty}^{\infty}\left[|q(y)|\frac{16\widetilde{C_2}}{|k|^3} + |q'(y)|\frac{16\widetilde{C_2}}{|k|^4} + |V(y)|\frac{16\widetilde{C_2}}{|k|^4}\right]dy \\ &\leq \frac{C}{|k|^3}\left(\frac{\|q\|_1}{|k|^3} + \frac{\|q'\|_1}{|k|^4} + \frac{\|V\|_1}{|k|^4}\right),\end{aligned}$$

where u_j and u_j' are calculated at $(y, k/2)$ and $\widetilde{C_2} = 2\widetilde{C_0}^2 = 2(\|q\|_1 + \|q'\|_1 + \|V\|_1)^2$. Then we can estimate the inverse Fourier transform of V_{rest} by

$$|F^{-1}(V_{rest})(k)| \leq \frac{|k|^3\chi(k/2)}{2\sqrt{2\pi}}\left|b_{rest}\left(\frac{k}{2}\right)\right|.$$

We have an equivalent norm for function f in Sobolev space $W_2^s(\mathbb{R})$ that can be written as

$$\|f\|_{W_2^s}^2 = \int_{-\infty}^{\infty}(1+k^2)^s|\widehat{f}(k)|^2 dk,$$

where $\widehat{f}$ denotes the Fourier transform of f [5]. Since $F^{-1}(V_{rest})(y) = F(V_{rest})(-y)$ we can estimate the norm of V_{rest} by

$$\begin{aligned}\|V_{rest}\|_{W_2^s}^2 &= \int_{-\infty}^{\infty}(1+k^2)^s|\widehat{V_{rest}}(k)|^2 dk \\ &\leq \int_{|k|\geq k_0}(1+k^2)^s\frac{C}{k^6}dk \\ &\leq C\int_{k\geq k_0}\frac{1}{k^{6-2s}}dk.\end{aligned}$$

This integral converges if and only if $s < \frac{5}{2}$, so V_{rest} belongs to the Sobolev space $W_2^s(\mathbb{R})$. □

Now the only thing left is to check that V_1 is a continuous function. We will show that actually V_1 can be written as sum $V_1 = \widetilde{V_1} + V_{1,rest}$ where $\widetilde{V_1} \in C^\infty(\mathbb{R})$ and $V_{1,rest} \in W_2^s(\mathbb{R})$ with $s < \frac{3}{2}$.

Lemma 4.4. *If V_1 is as in equation* (12) *then $V_1 = \widetilde{V_1} + V_{1,rest}$, where $\widetilde{V_1} \in C^\infty(\mathbb{R})$ and $V_{1,rest} \in W_2^s(\mathbb{R})$ for any $s < \frac{3}{2}$.*

Proof. Since $u_0(x, k/2) = e^{i\frac{k}{2}y}$ then

$$u_1(z, k/2) = -\frac{2}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right)(-kq + iq + V)e^{i\frac{k}{2}y}dy$$

for $k > 0$ and

$$u_1(z, k/2) = -\frac{2}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|} + e^{\frac{k}{2}|z-y|}\right)(kq - iq + V)e^{i\frac{k}{2}y}dy$$

for $k < 0$. Similarly the derivative is

$$u_1'(z, k/2) = \frac{1}{k^2}\int_{-\infty}^{\infty}\operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right)(-kq + iq' + V)e^{i\frac{k}{2}y}dy$$

for $k > 0$ and

$$u_1'(z, k/2) = \frac{1}{k^2}\int_{-\infty}^{\infty}\operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|} - e^{\frac{k}{2}|z-y|}\right)(kq - iq' + V)e^{i\frac{k}{2}y}dy$$

for $k < 0$. Applying these formulas we can find estimates for $b_1(k/2)$. If $k > 0$ we have

$$\begin{aligned}
b_1\left(\frac{k}{2}\right) &= -\frac{2}{k^3}\int_{-\infty}^{\infty} ie^{i\frac{k}{2}z} \\
&\quad \times (2iq(z)u_1'(z, k/2) + iq'(z)u_1(z, k/2) + V(z)u_1(z, k/2))dz \\
&= -\frac{2i}{k^3}\int_{-\infty}^{\infty} e^{i\frac{k}{2}z} \\
&\times \left[\frac{2iq(z)}{k^2}\int_{-\infty}^{\infty}\operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right)(-kq + iq' + V)e^{i\frac{k}{2}y}dy\right. \\
&\quad - \frac{2iq'(z)}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right)(-kq + iq' + V)e^{i\frac{k}{2}y}dy \\
&\quad \left. -\frac{2V(z)}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right)(-kq + iq' + V)e^{i\frac{k}{2}y}dy\right] dz.
\end{aligned}$$

These integrals can be further separated to parts so that the first integral has only $\frac{1}{k}$ and the remaining integrals, even after multiplying by the terms with $\frac{1}{k^2}$ and $\frac{1}{k^3}$, will contain only $\frac{1}{k^2}$ or $\frac{1}{k^3}$ inside the integrals. We can write this fact as

$$
\begin{aligned}
b_1\left(\frac{k}{2}\right) &= -\frac{4}{k^3}\int_{-\infty}^{\infty} e^{i\frac{k}{2}z}\frac{q(z)}{k}\int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|}-e^{-\frac{k}{2}|z-y|}\right)q(y)e^{i\frac{k}{2}y}dydz \\
&\quad -\frac{2i}{k^3}\int_{-\infty}^{\infty} e^{i\frac{k}{2}z} \\
&\quad \times\left[\frac{2iq(z)}{k^2}\int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|}-e^{-\frac{k}{2}|z-y|}\right)(iq'+V)e^{i\frac{k}{2}y}dy\right. \\
&\quad -\frac{2iq'(z)}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|}-e^{-\frac{k}{2}|z-y|}\right)(-kq+iq'+V)e^{i\frac{k}{2}y}dy \\
&\quad \left.-\frac{2V(z)}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|}-e^{-\frac{k}{2}|z-y|}\right)(-kq+iq'+V)e^{i\frac{k}{2}y}dy\right]dz. \\
&=: (\widetilde{b_1}+b_{1,rest})\left(\frac{k}{2}\right).
\end{aligned}
$$

Analogously, if $k<0$ we have

$$
\begin{aligned}
b_1\left(\frac{k}{2}\right) &= -\frac{4}{k^3}\int_{-\infty}^{\infty} e^{i\frac{k}{2}z}\frac{q(z)}{k}\int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|}-e^{\frac{k}{2}|z-y|}\right)q(y)e^{i\frac{k}{2}y}dydz \\
&\quad -\frac{2i}{k^3}\int_{-\infty}^{\infty} e^{i\frac{k}{2}z} \\
&\quad \times\left[\frac{-2iq(z)}{k^2}\int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|}+e^{\frac{k}{2}|z-y|}\right)(-iq'+V)e^{i\frac{k}{2}y}dy\right. \\
&\quad +\frac{2iq'(z)}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|}-e^{\frac{k}{2}|z-y|}\right)(kq-iq'+V)e^{i\frac{k}{2}y}dy \\
&\quad \left.-\frac{2V(z)}{k^3}\int_{-\infty}^{\infty}\left(ie^{i\frac{k}{2}|z-y|}-e^{\frac{k}{2}|z-y|}\right)(kq-iq'+V)e^{i\frac{k}{2}y}dy\right]dz. \\
&=: (\widetilde{b_1}+b_{1,rest})\left(\frac{k}{2}\right).
\end{aligned}
$$

Similarly as in the proof of Lemma 4.3 we find the following estimate

$$
\left|b_{1,rest}\left(\frac{k}{2}\right)\right| \le \frac{C}{|k|^5},
$$

when $|k|>k_0$. This means that if we define

$$
V_{1,rest}(y) := F\left(\frac{ik^3\chi(k/2)}{2\sqrt{2\pi}}b_{1,rest}\left(\frac{k}{2}\right)\right)(y),
$$

then

$$\begin{aligned}
\|V_{1,rest}\|^2_{W_2^s} &= \int_{-\infty}^{\infty} (1+k^2)^s |\widehat{V_{1,rest}}(k)|^2 dk \\
&\le \int_{|k|\ge k_0} (1+k^2)^s \frac{C}{k^4} dk \\
&\le C \int_{k\ge k_0} \frac{1}{k^{4-2s}} dk.
\end{aligned}$$

This integral converges if and only if $s < \frac{3}{2}$ and thus $V_{1,rest} \in W_2^s(\mathbb{R})$ for $s < \frac{3}{2}$.

We still have to check that

$$\widetilde{V_1}(\xi) := F\left(\frac{ik^3\chi(k/2)}{2\sqrt{2\pi}} \widetilde{b_1}\left(\frac{k}{2}\right)\right)(\xi)$$

is a smooth function. To confirm this we calculate $\widetilde{V_1}$ precisely. We can write

$$\widetilde{V_1}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{|k|\ge k_0} \frac{ik^3 e^{-ik\xi}}{2\sqrt{2\pi}} \widetilde{b_1}(k/2) dk$$

and divide this integral into two parts as follows

$$\begin{aligned}
\widetilde{V_1}(\xi) &= \frac{i}{4\pi} \int_{k\le -k_0} k^3 e^{-ik\xi} \widetilde{b_1}(k/2) dk + \frac{i}{4\pi} \int_{k\ge k_0} k^3 e^{-ik\xi} \widetilde{b_1}(k/2) dk \\
&=: I_1 + I_2.
\end{aligned}$$

Now the first integral has an form

$$\begin{aligned}
I_1 &= -\frac{i}{\pi} \int_{-\infty}^{-k_0} e^{-ik\xi} \\
&\quad \times \int_{-\infty}^{\infty} e^{i\frac{k}{2}z} \frac{q(z)}{k} \int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|} - e^{\frac{k}{2}|z-y|}\right) q(y) e^{i\frac{k}{2}y} dydzdk \\
&= \frac{i}{\pi} \int_{k_0}^{\infty} e^{ik\xi} \\
&\quad \times \int_{-\infty}^{\infty} e^{-i\frac{k}{2}z} \frac{q(z)}{k} \int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{-i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right) q(y) e^{-i\frac{k}{2}y} dydzdk,
\end{aligned}$$

where we did the change of variable k to $-k$. The second integral is

$$\begin{aligned}
I_2 &= -\frac{i}{\pi} \int_{k_0}^{\infty} e^{-ik\xi} \\
&\quad \times \int_{-\infty}^{\infty} e^{i\frac{k}{2}z} \frac{q(z)}{k} \int_{-\infty}^{\infty} \operatorname{sgn}(z-y)\left(e^{i\frac{k}{2}|z-y|} - e^{-\frac{k}{2}|z-y|}\right) q(y) e^{i\frac{k}{2}y} dydzdk.
\end{aligned}$$

We consider these integrals to be in the sense of distributions, so that we can change the order of integration. Now the sum $I_1 + I_2$ equals

$$\begin{aligned}\widetilde{V}_1(\xi) = \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\operatorname{sgn}(z-y)q(z)q(y) \\ \times \int_{k_0}^{\infty}&\left[\frac{e^{-i\frac{k}{2}(|z-y|+z+y-2\xi)} - e^{-\frac{k}{2}|z-y|}e^{-i\frac{k}{2}(z+y-2\xi)}}{k}\right. \\ &\left. - \frac{e^{i\frac{k}{2}(|z-y|+z+y-2\xi)} - e^{-\frac{k}{2}|z-y|}e^{i\frac{k}{2}(z+y-2\xi)}}{k}\right] dkdydz \\ = \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\operatorname{sgn}(z-y)q(z)q(y)\int_{k_0}^{\infty}\left[\frac{\sin(\frac{k}{2}(|z-y|+z+y-2\xi))}{k}\right. \\ &\left. - \frac{e^{-\frac{k}{2}|z-y|}\sin(\frac{k}{2}(z+y-2\xi))}{k}\right] dkdydz.\end{aligned}$$

Here the innermost integral has a crucial part in determining the behaviour of the whole integral, so we consider it first separately. We can write

$$\begin{aligned}\int_{k_0}^{\infty}\frac{\sin(\frac{k}{2}(|z-y|+z+y-2\xi))}{k}dk = &\frac{\pi}{2}\operatorname{sgn}(|z-y|+z+y-2\xi) \\ &- \int_{0}^{k_0}\frac{\sin(\frac{k}{2}(|z-y|+z+y-2\xi))}{k}dk\end{aligned}$$

and

$$\begin{aligned}\int_{k_0}^{\infty}\frac{e^{-\frac{k}{2}|z-y|}\sin(\frac{k}{2}(z+y-2\xi))}{k}dk = &\arctan\left(\frac{z+y-2\xi}{|z-y|}\right) \\ &- \int_{0}^{k_0}\frac{e^{-\frac{k}{2}|z-y|}\sin(\frac{k}{2}(z+y-2\xi))}{k}dk,\end{aligned}$$

where we have applied the well-known [3] integrals

$$\int_0^{\infty}\frac{\sin(ax)}{x}dx = \frac{\pi}{2}\operatorname{sgn}(a), \quad a \in \mathbb{R}$$

and

$$\int_0^{\infty}\frac{\sin(ax)}{x}e^{-bx}dx = \arctan\left(\frac{a}{b}\right), \quad b > 0,\ a \in \mathbb{R}.$$

One method to calculate the latter integral is provided in Appendix B.

Now we have an open form for $\widetilde{V}_1$, that is,

$$\begin{aligned}\widetilde{V}_1(\xi) = \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\operatorname{sgn}(z-y)q(z)q(y)\operatorname{sgn}(|z-y|+z+y-2\xi)dydz \\ &- \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{sgn}(z-y)q(z)q(y)\arctan\left(\frac{z+y-2\xi}{|z-y|}\right)dydz \\ - \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\operatorname{sgn}(z-y)q(z)q(y)\int_0^{k_0}\frac{\sin(\frac{k}{2}(|z-y|+z+y-2\xi))}{k}dkdydz \\ + \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\operatorname{sgn}(z-y)q(z)q(y)\int_0^{k_0}\frac{e^{-\frac{k}{2}|z-y|}\sin(\frac{k}{2}(z+y-2\xi))}{k}dkdydz.\end{aligned}$$

It is clear that the last two integrals are smooth functions with respect to ξ. The first integral of $\widetilde{V}_1$ can be separated into four parts with respect to the region of integration to obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\operatorname{sgn}(z-y)q(z)q(y)\operatorname{sgn}(|z-y|+z+y-2\xi)dydz \\ &= -\iint\limits_{\substack{y<z,\\ z<\xi}} q(y)q(z)dydz + \iint\limits_{\substack{y<z,\\ z\geq\xi}} q(y)q(z)dydz \\ &\quad + \iint\limits_{\substack{z<y,\\ y<\xi}} q(y)q(z)dydz - \iint\limits_{\substack{z<y,\\ y\geq\xi}} q(y)q(z)dydz = 0,\end{aligned}$$

because the last two integrals are symmetric with respect to z and y. Similarly the second integral yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{sgn}(z-y)q(z)q(y)\arctan\left(\frac{z+y-2\xi}{|z-y|}\right)dydz$$
$$=\iint\limits_{\substack{y<z,\\ z<\xi}} q(y)q(z)\arctan\left(\frac{z+y-2\xi}{|z-y|}\right)dydz$$
$$+\iint\limits_{\substack{y<z,\\ z\geq\xi}} q(y)q(z)\arctan\left(\frac{z+y-2\xi}{|z-y|}\right)dydz$$
$$-\iint\limits_{\substack{z<y,\\ y<\xi}} q(y)q(z)\arctan\left(\frac{z+y-2\xi}{|z-y|}\right)dydz$$
$$-\iint\limits_{\substack{z<y,\\ y\geq\xi}} q(y)q(z)\arctan\left(\frac{z+y-2\xi}{|z-y|}\right)dydz=0.$$

This means that $\widetilde{V}_1$ is a smooth function. □

We have now proved that the inverse Born approximation V_B of potential V can be expressed as

$$V_B = V + \widetilde{V}_1 + V_{1,rest} + \widetilde{V} + V_{rest},$$

and further, if we include $\widetilde{V}_1$ in $\widetilde{V}$ and $V_{1,rest}$ in V_{rest}, then

$$V_B = V + \widetilde{V} + V_{rest}, \tag{13}$$

where $\widetilde{V} \in C^\infty(\mathbb{R})$ and $V_{rest} \in W_2^s(\mathbb{R})$ for $s < \frac{3}{2}$.

Next we use certain, somewhat delicate, embeddings of the Sobolev spaces. If $\frac{1}{2} < s < \frac{3}{2}$ then we can apply the generalized Sobolev embedding theorem to obtain $W_2^s(\mathbb{R}) \subset W_\infty^{s-\frac{1}{2}}(\mathbb{R})$ (see [2], Theorems 6.4.4 and 6.5.1). Furthermore, it is possible to prove that $W_\infty^{s-\frac{1}{2}}(\mathbb{R}) \subset C^{s-\frac{1}{2}}(\mathbb{R})$ (see [2], Theorem 6.2.5), so actually $V_{rest} \in C^\alpha(\mathbb{R})$ for $0 < \alpha := s - \frac{1}{2} < 1$.

Theorem 4.5. *If the potential q belongs to the Sobolev space $W_1^1(\mathbb{R})$ and the unknown potential V is in $L^1(\mathbb{R})$ then $V - V_B \in C^\alpha(\mathbb{R}) + C^\infty(\mathbb{R})$ for $0 < \alpha < 1$.*

Corollary 4.6. *The difference $V - V_B$ is a continuous function i.e. the jumps and singularities of unknown potential V can be recovered from V_B.*

5 Conclusion

We studied a differential operator of order four on the real line from the perspective of scattering problems by applying methods of [6]. We obtained a solution to the direct scattering problem and studied its asymptotic behaviour to define analogies of the reflection and transmission coefficients. Based on the reflection coefficient we defined the inverse Born approximation of unknown potential V and showed that it can be used to find the jumps and singularities of V. The method proved to be very powerful as we found out that the difference $V - V_B$ consists of two parts, where one is a smooth function and the second is in the Hölder space $C^\alpha(\mathbb{R})$ for $0 < \alpha < 1$.

Due to the continuity of $V - V_B$ we may conclude that it is possible to completely determine potential V for example when V is the characteristic function of an unknown interval. More complicated potentials may require special treatment that is beyond the scope of this thesis.

Lastly, in this thesis we chose the potential $q_1 = 0$ so that operator (1) had only a first order perturbation. This let us to calculate the inverse Born approximation directly and to estimate it. If $q_1 \neq 0$, then we might need to proceed differently since the terms in the inverse Born approximation do not behave so well.

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A Finding a candidate for fundamental solution

To obtain the fundamental solution G_k^+ (6) we start by taking the Fourier transform of the equation

$$u^{(4)} - k^4 u = \delta.$$

This yields

$$\xi^4 \widehat{u}(\xi) - k^4 \widehat{u}(\xi) = \frac{1}{\sqrt{2\pi}},$$

where $\widehat{u}$ denotes the Fourier transform of u. Now rearranging the terms and taking the inverse Fourier transform we get

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2\pi}} F^{-1}\left(\frac{1}{\xi^4 - k^4}\right)(x) \\ &= \frac{1}{\sqrt{2\pi}} F^{-1}\left(\frac{1}{(\xi^2 - k^2)(\xi^2 + k^2)}\right)(x) \\ &= \frac{1}{2k^2\sqrt{2\pi}}\left(F^{-1}\left(\frac{1}{\xi^2 - k^2}\right)(x) - F^{-1}\left(\frac{1}{\xi^2 + k^2}\right)(x)\right). \end{aligned}$$

Let us first assume that $k > 0$ is real and calculate the integral

$$F^{-1}\left(\frac{1}{\xi^2 + k^2}\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{\xi^2 + k^2} d\xi.$$

The integral can be calculated by applying Jordan's lemma. The integrand has poles at $\xi_1 = ik$ and $\xi_2 = -ik$. In the case $x > 0$ we choose the path of integration to be the positively oriented curve defined by $\Gamma_R^+ = \Gamma_R \cup]-R, R[$, where

$$\Gamma_R = \{z \in \mathbb{C} \mid z = Re^{i\theta}, \theta \in [0, \pi]\}$$

for $R > 0$. By Jordan's lemma we have

$$\lim_{R\to\infty} \int_{\Gamma_R^+} \frac{e^{ixz}}{z^2 + k^2} dz = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{\xi^2 + k^2} d\xi = 2\pi i \operatorname{Res}\left(\frac{e^{ix\xi}}{\xi^2 + k^2}, \xi_1\right) = \frac{\pi e^{-xk}}{k},$$

where $\operatorname{Res}(f, x)$ denotes the residue of f at the point x.

Similarly, if $x < 0$ we choose the path of integration as $\Gamma_R^- = \Gamma_R \cup]-R, R[$, where

$$\Gamma_R = \{z \in \mathbb{C} \mid z = Re^{i\theta}, \theta \in [\pi, 2\pi]\}$$

for $R > 0$. Then applying Jordan's lemma we find that

$$\lim_{R\to\infty}\int_{\Gamma_R^-}\frac{e^{ixz}}{z^2+k^2}dz=\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{\xi^2+k^2}d\xi=-2\pi i\text{Res}\left(\frac{e^{ix\xi}}{\xi^2+k^2},\xi_2\right)=\frac{\pi e^{xk}}{k}.$$

Combining these results gives

$$F^{-1}\left(\frac{1}{\xi^2+k^2}\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{\xi^2+k^2}d\xi=\sqrt{\frac{\pi}{2}}\frac{e^{-|x|k}}{k}.$$

The same calculations for $k < 0$ give

$$F^{-1}\left(\frac{1}{\xi^2+k^2}\right)(x)=-\sqrt{\frac{\pi}{2}}\frac{e^{|x|k}}{k},$$

so we have obtained that for $k \in \mathbb{R}$

$$F^{-1}\left(\frac{1}{\xi^2+k^2}\right)(x)=\sqrt{\frac{\pi}{2}}\frac{e^{-|x||k|}}{|k|}.$$

To calculate the second inverse Fourier-transform

$$F^{-1}\left(\frac{1}{\xi^2-k^2}\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{\xi^2-k^2}d\xi$$

we use different argument, because this integrand has poles on real line at $\xi_1 = k$ and $\xi_2 = -k$. We understand this integral as a Fourier transform in the sense of distributions and want to avoid using principal value integrals. Instead, we calculate this integral by using a regularization parameter $i\epsilon$, where $\epsilon > 0$, so that the integral becomes

$$F^{-1}\left(\frac{1}{\xi^2-k^2-i\epsilon}\right)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{\xi^2-k^2-i\epsilon}d\xi$$

and take the limit when $\epsilon \to +0$. Now the integral

$$\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{\xi^2-k^2-i\epsilon}d\xi$$

can be calculated similarly as before. The integrand has poles at $z_1 = \sqrt{k^2 - i\epsilon}$ and $z_2 = -\sqrt{k^2 - i\epsilon}$. Applying previous results we see that

$$\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{\xi^2-k^2-i\epsilon}d\xi\to\sqrt{\frac{\pi}{2}}\frac{ie^{i|x||k|}}{|k|}$$

as $\epsilon \to +0$. Combining these calculations yields

$$u(x)=\frac{1}{4|k|^3}\left(ie^{i|k||x|}-e^{-|k||x|}\right)=:G_k^+(|x|).$$

Remark A.1. If we calculated the last Fourier transform as a principal value integral, we would have obtained a different result. The use of the regularization parameter and the calculation of the Fourier transform in the sense distributions can be justified in the context of differential operators.

B Remark on a certain integral

In the proof of Lemma 4.4 we used a well-known integral (see [3])

$$\int_0^\infty \frac{\sin(ax)}{x} e^{-bx} dx = \arctan\left(\frac{a}{b}\right), \quad b > 0,\, a \in \mathbb{R}.$$

Confirming this result is not too difficult. Let us define a function $f(a)$ by formula

$$f(a) := \int_0^\infty \frac{\sin(ax)}{x} e^{-bx} dx,$$

where $b > 0$ is a parameter. Then differentiation and integrating by parts twice yields

$$\begin{aligned} f'(a) &= \int_0^\infty \cos(ax) e^{-bx} dx = \int_0^\infty \frac{b}{a} \sin(ax) e^{-bx} dx \\ &= \frac{b}{a^2} - \int_0^\infty \frac{b^2}{a^2} \cos(ax) e^{-bx} dx = \frac{b}{a^2} - \frac{b^2}{a^2} f'(a). \end{aligned}$$

This implies that

$$f'(a) = \frac{b}{a^2 + b^2}$$

and after integrating

$$f(a) = \arctan\left(\frac{a}{b}\right) + C.$$

By definition we have initial value $f(0) = 0$ and because $\arctan(0) = 0$ then also $C = 0$ as claimed. It may be interesting to note that as $b \to +0$ we have

$$\arctan\left(\frac{a}{b}\right) \to \frac{\pi}{2} \operatorname{sgn}(a).$$

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