

# Validity of the semiquantum approximation in the ultrastrong coupling regime of cavity QED

M.Sc. Thesis

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## Abstract

With the recent advancements made in the coupling of quantum systems, reaching the (ultra)strong coupling regime, where the coupling constant exceeds the dissipation rate and approaches the frequencies of the coupled systems, is getting closer. In this regime, the much used semiclassical model is no longer valid, and the solving of the full quantum master equation can be computationally too demanding. Thus, we must develop a new approach to be able to simulate the system.

One solution is to use the semiquantum approximation. In this thesis, we present a systematic way of employing the semiquantum approximation and compare it to the full master equation. Our aim is to determine the range for the coupling strength where the semiquantum model is applicable. We chose to do the comparison of the semiquantum approximation and the master equation within the Rabi model, which can be used to describe e.g. the interaction of light and matter. The Rabi model describes a system with a coupled two-level system and a harmonic oscillator. It was chosen because its behaviour is reasonably well known.

We compared the semiquantum approximation to the quantum master equation in two cases. First we studied the steady state results from the two models, and then moved on to the spectral properties. The results show that the steady state and the spectrum obtained from the semiquantum model agree with the ones obtained by using the master equation, until the coupling reaches a considerable fraction of the resonant frequency of the oscillator, while still exceeding the experimentally reasonable dissipation rate by an order of magnitude.

In the future, one could use the semiquantum approximation in the field of cavity optomechanics. There a mechanical oscillator is coupled to optical radiation confined in a cavity, e.g. a Fabry-Pérot cavity. Especially in optomechanics, the solving of the master equation can turn out to be a formidable task, and by using the semiquantum approximation one could reduce the computation time considerably.

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# Chapter 1

## Introduction

In quantum dynamics the central problem is to solve the time-evolution of the system. In the end, one is usually interested in expectation values of operators and how they change in time. Due to the nature of quantum mechanics there are several methods to obtain the expectation values. One approach would be to solve the time-evolution of the *operator* whose expectation value we want to discover. Another path would be to solve the time-evolution of the *states* in respect of which we take the expectation value. These would mean solving of the *quantum master equation*. This can be a formidable task if the system is complex, or if we have high excited states involved, and, thus, have to include a large number of states into the calculations. Third option would be to solve the time-evolution of the expectation values *directly*. This, however, poses a new problem. If we write equations of motion for the expectation values we encounter an infinite chain of equations of increasing order. Thus, we have to figure out a way to truncate the series. One way to do this truncation is to use the *semiquantum approximation*. The aim of this thesis is to present a clear and systematic way to employ the semiquantum approximation, and to investigate in which situations it is possible to use it instead of the computationally more demanding quantum master equation. Also, we want to discover possible problematics. The semiquantum model developed here assumes that the operators behave as random variables which obey the Gaussian distribution. To my knowledge this kind of systematic justification has not been performed earlier.

In the near future (ultra)strong coupling can be achieved between two or more quantum systems, creating the need to develop more accurate approximation methods in order we can make simulations. By strong coupling we mean coupling strengths that, at minimum, exceed the damping rates of the systems involved, preferably couplings that are in the order of the frequencies of the systems. In natural systems, e.g. interaction

between light and matter, reaching the strong coupling regime is difficult, owing to the small intrinsic coupling rate between the systems. We can overcome this problem by studying systems where we have control over the critical properties which determine the coupling strength. As a matter of fact, (ultra)strong coupling has already been observed in *circuit QED* [1, 2], in which superconducting qubits are coupled to microwave frequency resonators. The qubits and oscillators are manufactured to a circuit board, which allows the tuning of the parameters in the systems. Another field of research where strong coupling regime is pursued is that of *quantum optomechanics*. There electromagnetic cavities are coupled to mechanical oscillators through radiation-pressure coupling. Earlier the *single-photon coupling rate* has been in the order of  $10^{-3} - 10^{-5}$  compared to the dissipation rate [3–7]. However, strong coupling regime has been reached using Bose-Einstein condensates [8], where the role of a mechanical oscillator is played by a collective density excitation of the Bose-Einstein condensate. There has been theoretical suggestions to increase the single-photon coupling rate to be comparable to the damping rates [9] and even orders of magnitude larger [10].

This thesis is organized as follows: In the second chapter we go through the basic theory needed for understanding the discussion in the later chapters. First we discuss time-dependence in quantum mechanics. We go through the different pictures of time-evolution and present the tools needed to move between them. We move on to discuss open quantum systems. We begin by introducing the density operator and continue by deriving the quantum master equation. In [section 2.3](#) we introduce the semiquantum approximation, followed by its predecessor semiclassical approximation in [section 2.4](#). We finish the chapter by deriving a way to obtain the spectrum of a system from auto-correlation functions. The third chapter includes the derivation of equations of motion needed in the numerical analysis. In the fourth chapter we review the results of the simulations and draw conclusion based in them. Finally we discuss the obtained results and give propositions on paths that could be taken in the future.

# Chapter 2

## Theory

In this chapter we will present the theory and tools needed for understanding the rest of the thesis. We begin by briefly reviewing the different *pictures of quantum mechanics* and *unitary transformations*. We proceed with the derivation of the *quantum master equation*, which is used to derive the equations of motion, and present a transformation which can be used to eliminate external driving forces from the master equation. After this we introduce the central concept of the thesis, the *semiquantum approximation*, followed by the predecessor, the *semiclassical approximation*. We conclude by deriving a method to calculate the spectrum of a system by using autocorrelation functions.

### 2.1 Time-evolution in quantum mechanics

We begin by reviewing the different ways of handling time-evolution in quantum mechanics which are often referred to as pictures. There are two fundamentally different ways to think about time-dependence, the *Schrödinger picture* and the *Heisenberg picture*. Also a third picture exists, the so called interaction picture, which is a combination of the Schrödinger and Heisenberg pictures. Before talking more about the different pictures, we must discuss *unitary transformations*.

#### 2.1.1 Unitary transformations

Unitary operator is an operator which satisfies the relation

$$U^\dagger U = U U^\dagger = 1, \tag{2.1}$$

where 1 is the identity operator. From this it immediately follows that

$$U^\dagger = U^{-1}, \tag{2.2}$$

i.e. the inverse of the operator is its Hermitian conjugate. Unitary transformation is then a similarity transformation carried out with a unitary operator. We define the transformed operators and state vectors to be [11]

$$X \rightarrow U^\dagger X U, \quad (2.3)$$

$$|\psi\rangle \rightarrow U^\dagger |\psi\rangle, \quad (2.4)$$

where  $|\psi\rangle$  is an arbitrary state and  $X$  an arbitrary operator acting on the Hilbert space. Under a unitary transformation the inner product between states and the expectation values remain unchanged ( $|\alpha\rangle$  and  $|\beta\rangle$  are arbitrary state vectors)

$$\langle\alpha|\beta\rangle \rightarrow \langle\alpha|U U^\dagger|\beta\rangle = \langle\alpha|\beta\rangle, \quad (2.5)$$

$$\langle\alpha|X|\beta\rangle \rightarrow \langle\alpha|U U^\dagger X U U^\dagger|\beta\rangle = \langle\alpha|X|\beta\rangle. \quad (2.6)$$

In particular the probability is conserved.

## 2.1.2 Pictures of time-evolution

### Schrödinger picture

In the Schrödinger picture time-dependence is thought to be in the state vectors. The time-evolution is governed by the *Schrödinger equation*

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle, \quad (2.7)$$

where  $|\psi\rangle$  is an arbitrary state and  $H$  is the Hamiltonian of the system. The operators acting in the Hilbert space of the system are assumed to have only explicit time dependence ( $X$  is an arbitrary operator)

$$\frac{dX}{dt} = \frac{\partial X}{\partial t}. \quad (2.8)$$

Schrödinger's picture is the most common way of thinking about time-dependence in quantum mechanics.

### Heisenberg picture

An alternate way of thinking about the time-evolution is to consider the operators to be time-dependent and the state vectors to be independent on time. This approach is called Heisenberg picture. We define the Heisenberg picture operators to be [11]

$$X_H = U^\dagger X U, \quad (2.9)$$



where  $X$  is the corresponding operator in Schrödinger picture and  $U$  is the time-evolution operator. It is unitary and defined by the equation

$$i\hbar \frac{\partial U}{\partial t} = HU, \quad (2.10)$$

with the initial condition  $U(0) = 1$ . Equation (2.10) is the Schrödinger equation for the time-evolution operator. The state vectors are defined as [11]

$$|\psi\rangle_H = U^\dagger |\psi\rangle. \quad (2.11)$$

Differentiating the defining equations (2.9) and (2.11), and using Schrödinger equations for the state vectors and the time-evolution operator, one obtains the governing equations in the Heisenberg picture [11]

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle_H = 0, \quad (2.12)$$

$$i\hbar \frac{dX_H}{dt} = [X_H, H_H] + i\hbar \frac{\partial X_H}{\partial t}. \quad (2.13)$$

The differential equation for the Heisenberg picture operators is called the *Heisenberg equation of motion*.

### Interaction picture

The third picture of time-evolution in quantum mechanics is the *interaction picture*. In this picture we assume that the Hamiltonian of the system can be divided into two parts: an 'easy' part,  $H_0$ , and a 'hard' part, the interaction part  $H_I(t)$ ,

$$H = H_0 + H_I(t). \quad (2.14)$$

In this situation, we fix the transformation from the Schrödinger picture,  $U$ , with the easy part of the Hamiltonian only

$$i\hbar \frac{\partial U}{\partial t} = H_0 U \quad (2.15)$$

and define the operators and state vectors to be [11]

$$X_I = U^\dagger X U, \quad (2.16)$$

$$|\psi\rangle_I = U^\dagger |\psi\rangle. \quad (2.17)$$

Equations governing the time-evolution can, again, be obtained by differentiating the defining equations for the operators and states, and using the equations for  $U$  and  $|\psi\rangle$ .

We obtain the equations of motion in the interaction picture [11]

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle_I = H_I(t) |\psi\rangle_I, \quad (2.18)$$

$$i\hbar \frac{dX_I}{dt} = [X_I, H_0] + i\hbar \frac{\partial X_I}{\partial t}. \quad (2.19)$$

We see that the time-evolution of the states is determined by the interaction Hamiltonian alone, and the trivial time-dependence, generated by the time-independent Hamiltonian  $H_0$ , is transferred into the operators. In many respects the interaction picture is a mixture of the Schrödinger and Heisenberg pictures, as is clear from the treatment above. Thus, it is not a fundamentally different way of dealing with time-dependence in quantum mechanics, but can be very useful in some situations.

## 2.2 Open quantum systems

The equation governing the time-evolution of the system in Schrödinger's picture is the Schrödinger equation and in Heisenberg's picture the Heisenberg equation of motion. They are good for studying systems where energy is conserved, i.e. we do not have any loss mechanisms in the system. Where they fall short is when we want to deal with systems where there is dissipation involved. This is because they are based on Hamiltonian dynamics and, thus, the resulting equations are energy conserving. One could try to bolt on to the equations terms creating dissipation, but proceeding this way is not extremely systematic. Thus, to be able to study systems with loss mechanisms we need to come up with something else.

### Density operator

Before talking about the governing equations we need to introduce the *density operator*. We introduce it because it gives an easy and systematic way of taking dissipation into account. The density operator is an operator which carries in itself all of the physical information that can be obtained from the system [11]. It is defined as

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|, \quad (2.20)$$

where  $|\psi_i\rangle$  is a state in the Hilbert space of the system and  $p_i$  is the probability to find the system in the corresponding state  $|\psi_i\rangle$ . Note that states  $|\psi_i\rangle$  need not be orthogonal, but they are normalized, and the probabilities  $p_i$  satisfy  $\sum_i p_i = 1$ . The representation of the density operator in some basis  $|n\rangle$  is called the density matrix. One can derive

an equation governing the time-evolution of the density operator from the Schrödinger equation

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \quad (2.21)$$

where  $H$  is the Hamiltonian of the system. This equation is called the *von Neumann equation*. It holds in the case when there is no loss mechanisms present in the system.

The density operator is extremely useful when dealing with ensembles of states. The expectation value of an operator  $A$  can be calculated as [11]

$$\langle A \rangle = \text{Tr}(\rho A), \quad (2.22)$$

where  $\text{Tr}$  stands for trace, i.e. a sum over the diagonal elements of  $\rho A$ . This property is beneficial when we want to include dissipation which includes averaging over degrees of freedom we are not interested in.

### Damping

Dissipation occurs because energy leaks from the system to its environment. The number of degrees of freedom in the environment is extremely large, and because of this the energy, effectively, does not return to the system. An intuitive guess to include loss in equation (2.21) is to add a description of the environment to the equation. This poses a new problem, how do we determine the Hamiltonian for the environment? Because we are not interested in the exact dynamics of the environment, but just in the effect it has on the system, we assume that a crude model of the environment is sufficient. Simplest model imaginable is the harmonic oscillator. Thus, we assume that the environment can be described by the Hamiltonian

$$H_{\text{env}} = \hbar \sum_k \omega_k b_k^\dagger b_k, \quad (2.23)$$

where  $\omega_k$  is the frequency of environment mode  $k$  and  $b(b^\dagger)$  is the annihilation (creation) operator of the mode. Next we need to couple the system to the environment. To keep the total Hamiltonian sufficiently simple we assume that the coupling is linear in  $b_k$  and  $b_k^\dagger$  and also in the operators of the system. Also the interaction needs to be weak enough so that we can treat it perturbatively.

In this thesis we are dealing with a system consisting of a harmonic oscillator and a two-level-system. Thus, we need to have a description of the damping for both of the subsystems. Let us turn our attention first to the harmonic oscillator.

We assume that the environment couples bi-linearly to the position of the oscillator resulting in an interaction Hamiltonian of form

$$H_{\text{int}} = -\hbar x (\Gamma + \Gamma^\dagger) = -\hbar (a + a^\dagger) (\Gamma + \Gamma^\dagger), \quad (2.24)$$

where  $\Gamma = \sum_k g_k b_k$  and  $g_k$  gives the strength of the coupling between the studied oscillator and the mode  $k$  of the environment. Now the total Hamiltonian  $H$  in equation (2.21) is

$$H = H_{\text{sys}} + H_{\text{env}} + H_{\text{int}}. \quad (2.25)$$

Here  $H_{\text{env}}$  and  $H_{\text{int}}$  are as above and  $H_{\text{sys}}$  is the Hamiltonian for the system. In the case of a harmonic oscillator we have  $H_{\text{sys}} = \hbar\omega_0 a^\dagger a$ . We denote the density operator of the coupled system and environment with  $w$ . The von Neuman equation corresponding to it reads

$$i\hbar \frac{dw}{dt} = [H, w] = [H_{\text{sys}} + H_{\text{env}} + H_{\text{int}}, w]. \quad (2.26)$$

We move to the interaction picture with respect to the Hamiltonian  $H_0 = H_{\text{sys}} + H_{\text{env}}$ , and the above equation becomes

$$i\hbar \frac{dw}{dt} = [H_{\text{int}}(t), w]. \quad (2.27)$$

### 2.2.1 Master equation

Derivation presented here is based on [12, 13]. To solve the above equation we integrate it iteratively, in principle, infinitely many times and arrive at

$$w(t) = w(0) + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right) \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \quad (2.28)$$

$$\times [H_{\text{int}}(t_1), [H_{\text{int}}(t_2), \dots, [H_{\text{int}}(t_n), w(t)] \dots]].$$

With the inclusion of the Hamiltonian of the environment, the density operator  $w$  contains also all of the physical information from the environment. Thus, we average over the environmental degrees of freedom, that is take the trace over the environment of the above, to get the average effect the environment has on the system, and obtain an equation for the reduced density operator of the system alone  $\rho = \text{Tr}_{\text{env}}(w)$

$$\rho(t) = \rho(0) + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right) \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \quad (2.29)$$

$$\times \text{Tr}_{\text{env}} ([H_{\text{int}}(t_1), [H_{\text{int}}(t_2), \dots, [H_{\text{int}}(t_n), \rho(t)] \dots]]).$$

After this the derivation proceeds as follows:

- I Differentiate equation (2.29) to obtain a differential equation for  $\rho$ .
- II Interaction (2.24) is such that it satisfies  $\text{Tr}_{\text{env}}([H_{\text{int}}, \rho_{\text{env}}(0)]) = 0$ , so that the first order term in the interaction Hamiltonian is zero. Next we make the Born approximation, i.e. assume the interaction is weak enough, so that we can neglect terms that are higher than second order in  $H_{\text{env}}$ .
- III Use RWA on the equation for  $d\rho/dt$ . This means neglecting terms in the interaction proportional to terms  $ab_k$  and  $a^\dagger b_k^\dagger$ , the so called 'counter rotating' terms.
- IV Evaluate the resulting integrals with the assumptions that the environment oscillators were originally in thermal equilibrium, and that they have a continuous spectrum with density  $\varrho(\omega)$ .
- V Neglect the imaginary part of the integral as a small effect. Physically the imaginary part corresponds to a frequency shift to the oscillator due to the coupling with the environment [12]. It is analogous to the *Lamb shift* observed in atoms.

After performing these steps one is left with the *quantum master equation* for a damped harmonic oscillator

$$\frac{d\rho}{dt} = \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \rho + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \rho. \quad (2.30)$$

Here  $\kappa = 2\pi\varrho(\omega_0)|g(\omega_0)|^2$  determines the magnitude of the damping caused by the environment and  $N_c$  is the thermal number of excitations in the environment evaluated at the oscillator frequency  $\omega_0$ .  $\mathcal{D}[O] \rho = 2O\rho O^\dagger - \rho O^\dagger O - O^\dagger O \rho$  is the *Lindblad superoperator*. In the Schrödinger picture equation (2.30) reads

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_{\text{sys}}, \rho] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \rho + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \rho. \quad (2.31)$$

Now we turn our attention to the qubit. We follow the trail laid above while dealing with the harmonic oscillator. Inspired by (2.24), we assume an interaction

$$H_{\text{int}} = -\hbar(\sigma_+ + \sigma_-) (\Gamma + \Gamma^\dagger), \quad (2.32)$$

where  $\sigma_{+(-)}$  is the raising (lowering) operator for the two-level-system and  $\Gamma$  is same as with the harmonic oscillator. Naturally the coupling constants  $g_k$  are different than with the oscillator. In addition to this coupling one could think of an interaction of form

$$H_{\text{dep}} = \hbar\sigma_+\sigma_-\tilde{\Gamma}, \quad (2.33)$$

where  $\tilde{\Gamma}$  is some hermitian operator from the environment's Hilbert space. We take the interaction between the environment and the qubit to consist of both of the couplings mentioned above. After specifying the coupling we can continue the derivation of the master equation for the qubit, just like with the harmonic oscillator. The result is analogous to that of the oscillator, with an additional term generated by equation (2.33). In Schrödinger's picture the master equation reads

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_q, \rho] + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \rho + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \rho + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \rho, \quad (2.34)$$

where  $\gamma$  is analogous to  $\kappa$ , environment's thermal population  $N_q$  has to be evaluated now at the level separation of the qubit  $\hbar\Omega$  and the Hamiltonian of the two-level-system reads  $H_q = \hbar\Omega\sigma_+\sigma_-$ .

Let us look at the extra term in the master equation closer. Just like  $\gamma$  and  $\kappa$  also  $\gamma_\phi$  is determined from the properties of the environment by  $\gamma_\phi = 4 \int_0^\infty dt \langle \tilde{\Gamma}(t) \tilde{\Gamma}(0) \rangle$ . The additional term also does not have an effect on the diagonal elements of the density matrix, but it produces additional damping to the off-diagonal elements. This can be understood as the coupling induces fluctuations to the level separation of the qubit, making the different realisations to rotate at different speeds in the complex plane and, thus, to lose the phase coherence between the ground and excited state faster [13]. This effect is called *dephasing*.

In the derivation of the master equation, for both the harmonic oscillator and the two-level-system, we have also used a more subtle assumption. We have presumed that the derivative of the density operator,  $\partial\rho(t)/\partial t$ , depends only on  $\rho(t)$ , and not on any previous history of  $\rho$ . This assumption is called *Markovian*. Also non-Markovian master equations exist, e.g. the *Nakajima-Zwanzig equation* [14, 15].

## 2.2.2 Eliminating the effect of driving

In this thesis we study a system of a driven harmonic oscillator coupled to a two-level-system. To make the numerical calculations easier, we want to eliminate the effect of driving from the master equation. Driving causes the functions to be solved to oscillate, which, in turn, reduces the stability and effectiveness of the numerical calculations. In our case the driving can be removed by using the *displacement operator* [13]

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \quad (2.35)$$

where  $\alpha^{(*)}$  is the expectation value of operator  $a^{(\dagger)}$ . Physically this transformation means that we displace the cavity into the vacuum [16], i.e. after the transformation  $\langle a^{(\dagger)} \rangle = 0$ .

Transformation (2.35) is unitary, so it conserves all important properties. Using the *Baker-Campbell-Hausdorff formula* one can write (2.35) in the alternate form

$$D(\alpha) = e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger}, \quad (2.36)$$

which holds, since  $[a, a^\dagger] = 1$ . This form is more useful than the definition (2.35), because it gives a way to calculate the effect of the transformation. Using the above formula we calculate the effect for the operator  $a$ :

$$D^\dagger(\alpha) a D(\alpha) = e^{-|\alpha|^2/2} e^{-\alpha a^\dagger} e^{\alpha^* a} a e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger}. \quad (2.37)$$

Since  $a$  commutes with itself and  $\alpha$  is a scalar, we can move them to the other side of  $a$  and obtain

$$\begin{aligned} D^\dagger(\alpha) a D(\alpha) &= e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} \\ &= e^{-\alpha a^\dagger} a \sum_{k=0}^{\infty} \frac{(\alpha a^\dagger)^k}{k!}. \end{aligned} \quad (2.38)$$

With the help of commutator  $[a, a^\dagger] = 1$ , one can show that  $a (a^\dagger)^k = k (a^\dagger)^{k-1} + (a^\dagger)^k a$ . Using this on the above series expansion gives

$$\begin{aligned} D^\dagger(\alpha) a D(\alpha) &= e^{-\alpha a^\dagger} \left( \alpha \sum_{k=1}^{\infty} \frac{k (\alpha a^\dagger)^{k-1}}{(k-1)!} + \sum_{k=0}^{\infty} \frac{(\alpha a^\dagger)^k}{k!} a \right) \\ &= e^{-\alpha a^\dagger} \left( \alpha \sum_{l=0}^{\infty} \frac{(\alpha a^\dagger)^l}{l!} + \sum_{k=0}^{\infty} \frac{(\alpha a^\dagger)^k}{k!} a \right) \\ &= e^{-\alpha a^\dagger} e^{\alpha a^\dagger} (\alpha + a) \\ &= \alpha + a. \end{aligned} \quad (2.39)$$

Similarly one can show that

$$D^\dagger(\alpha) a^\dagger D(\alpha) = \alpha^* + a^\dagger. \quad (2.40)$$

Taking the expectation value of the transformed operator, e.g.  $a^\dagger$ , with respect to the transformed density operator,  $\tilde{\rho} = D^\dagger(\alpha) \rho D(\alpha)$ , gives

$$\begin{aligned} \left\langle D^\dagger(\alpha) a^\dagger D(\alpha) \right\rangle_{\tilde{\rho}} &= \text{Tr} \left( D^\dagger(\alpha) a^\dagger D(\alpha) \tilde{\rho} \right) = \text{Tr} \left( D^\dagger(\alpha) a^\dagger D(\alpha) D^\dagger(\alpha) \rho D(\alpha) \right) \\ &= \text{Tr} \left( D^\dagger(\alpha) a^\dagger \rho D(\alpha) \right) = \text{Tr} \left( a^\dagger \rho D(\alpha) D^\dagger(\alpha) \right) = \text{Tr} \left( a^\dagger \rho \right) \\ &= \left\langle a^\dagger \right\rangle_{\rho} = \alpha^*, \end{aligned} \quad (2.41)$$

where we have denoted expectation value with respect to  $\tilde{\rho}$  with  $\langle \cdot \rangle_{\tilde{\rho}}$  and similarly for  $\rho$ . On the other hand, the right hand side of (2.40) gives

$$\langle D^\dagger(\alpha)a^\dagger D(\alpha) \rangle_{\tilde{\rho}} = \langle \alpha^* + a^\dagger \rangle_{\tilde{\rho}} = \alpha^* + \langle a^\dagger \rangle_{\tilde{\rho}}. \quad (2.42)$$

Thus, the expectation value of the untransformed operator  $a^\dagger$  with respect to the transformed density operator vanishes

$$\langle a^\dagger \rangle_{\tilde{\rho}} = 0, \quad (2.43)$$

as we asserted earlier. Similarly one can show that  $\langle a \rangle_{\tilde{\rho}} = 0$ . Now this transformation can be used on the Hamiltonian and the master equation when deriving equations of motion for the system and isolate the driving to affect only  $\alpha^{(*)}$ .

The system under study in this thesis is comprised of a harmonic oscillator coupled to a qubit. The harmonic oscillator is driven with an external force, but because of the coupling between the two subsystems, the drive affects also the qubit. This means we could try to displace the qubit as well to get rid of the effect of driving completely. Our aim would be to find an unitary operator  $U$  with the properties

$$U\sigma_-U^\dagger = \sigma_- + s_-, \quad (2.44)$$

$$U\sigma_+U^\dagger = \sigma_+ + s_+, \quad (2.45)$$

where  $s_\pm$  is the expectation value of  $\sigma_\pm$ , respectively. Interestingly this kind of unitary transform *does not exist*. In fact there is no transform, unitary or not, of this kind. This can be easily seen. Firstly, the second equation is just the hermitian conjugate of the first one and, thus, does not give any new information. For that reason we need to study only (2.44). Let  $U$  be an arbitrary  $2 \times 2$  matrix with elements  $u_{ij}$ ,  $i, j = 1, 2$ . Writing equation (2.44) in matrix form gives

$$\begin{pmatrix} u_{11}u_{21}^* & u_{12}u_{21}^* \\ u_{11}u_{22}^* & u_{12}u_{22}^* \end{pmatrix} = \begin{pmatrix} s_- & 0 \\ 1 & s_- \end{pmatrix}. \quad (2.46)$$

For this to hold we would have to have  $u_{12} = 0$  or  $u_{21} = 0$ . In both cases this results in  $u_{11}u_{21}^* = 0 \neq s_-$  or  $u_{12}u_{22}^* = 0 \neq s_-$ , in general. Thus, transformation  $U$  does not exist. Nonetheless this kind of displacement has been used in earlier studies [17], which makes the results obtained in them questionable. In reference [17] they write operators  $a$ ,  $\sigma_-$  and  $\sigma_z$  like

$$a = \langle a \rangle + \delta a, \quad (2.47)$$

$$\sigma_- = \langle \sigma_- \rangle + \delta \sigma_-, \quad (2.48)$$

$$\sigma_z = \langle \sigma_z \rangle + \delta \sigma_z, \quad (2.49)$$



where  $\delta a$ ,  $\delta\sigma_-$  and  $\delta\sigma_z$  are the presumably small deviations from the expectation values of the corresponding operators. The expectation value of the displaced operators vanish,  $\langle\delta a\rangle = \langle\delta\sigma_- \rangle = \langle\delta\sigma_z\rangle = 0$ , clearly. For  $a$  this corresponds to the transformation (2.35), but, as shown above, for the two-level-system this kind of transformation cannot be found.

### 2.3 Semiquantum approximation

As stated, the main goal of this thesis is to determine the range of validity of the so called *semiquantum approximation* or *semiquantum model*. In this section we explain the idea behind the approximation and derive equations which are needed later in the thesis for the development of the equations of motion. To my knowledge a mathematical base as rigorous as we develop here, has not been used earlier in the justification of the semiquantum approximation. The first appearance of the semiquantum model was in a paper by J. Shirley [17]. There it was used to investigate the validity of the so called *semiclassical* or linear model for describing masers. The linear model will be discussed in greater depth later in the thesis, to obtain some analytical results for comparison. A few years after Shirley's paper P. Mandel wrote an article where he uses a semiquantum model developed by himself [18]. Later work done using the semiquantum approximation is based on these two publications.

Correlators and the order of a correlator are central concepts in the semiquantum approximation. Thus, we need to define them. Correlator of order  $n$  is defined as

$$\langle X_1 X_2 \cdots X_{n-1} X_n \rangle, \quad (2.50)$$

where  $X_i, i \in \mathbb{Z}_+$ , are 'fundamental' operators for the system at hand. What we mean here by 'fundamental' operators is, that every other operator can be expressed as a linear combination or a product of these operators. For example, for a harmonic oscillator one can choose canonical operators  $x$  and  $p$  or the ladder operators  $a$  and  $a^\dagger$ . For a qubit they would be Pauli's  $\sigma_x$  and  $\sigma_y$  matrices or the ladder operators  $\sigma_-$  and  $\sigma_+$ . In both cases it is vital to notice that the Hamiltonian for the system,  $\hbar\omega a^\dagger a$  for the harmonic oscillator and  $\hbar\omega\sigma_+\sigma_-$  for the qubit, are second order in these operators and, thus, result in second order correlators when taking expectation values. We call these expectation values correlators because they give information about the statistical correlation between the operators in question. Naturally the operators inside the expectation value need not be different, there can be several occurrences of the same operator.

When we want to solve the dynamics of a quantum system, we can always solve its master equation. This can be, however, numerically demanding so we would like to

develop an easier way of obtaining the information in which we are interested. In the Rabi model solving of the master equation is not an insurmountable task, but in the field of optomechanics it can be for the contemporary computers. In this thesis we are only interested in testing the functionality of the semiquantum approximation. Thus, we chose to use the Rabi model whose behaviour is better known. Most of the time we would like to obtain some expectation values for the system, e.g. the number of quanta in a harmonic oscillator. Thus, we can write equations of motion for the correlators we are interested in and solve those instead of the full master equation. There is a problem however, as it is possible that the equations of motion for correlators of order  $n$  depend on the values of higher order correlators. This means we would have to derive equations for those higher order correlators as well, but in turn they depend on even higher order correlators. This generates an infinite series of equations for correlators of increasing order. Clearly we have to truncate the series at some point and discard correlators above some order. How does one exactly do this truncation? In the semiquantum approximation one decides to take into account correlators of order 2. This doesn't mean that one assumes that higher order correlators vanish. Rather the idea is to express these correlators with first and second order ones. Mathematically this is equivalent to neglecting third order *cumulants* [19]. Cumulants give information about that part of the correlator which *can not* be expressed with lower order correlators [19]. The order of a cumulant is defined, similarly as the order of a correlator, as the number of 'fundamental' operators in the product. We will not dive deeper into cumulants in this thesis, just state formulae which are needed for the following. We will denote the cumulant of a random variable  $X_j$  with  $\langle X_j \rangle_c$ . One can show [20] that the cumulant  $\langle X_1^{\nu_1} X_2^{\nu_2} \dots X_N^{\nu_N} \rangle_c$  can be obtained from

$$\langle X_1^{\nu_1} X_2^{\nu_2} \dots X_N^{\nu_N} \rangle_c = \left[ \frac{\partial^{\Upsilon}}{\partial \xi_1^{\nu_1} \partial \xi_2^{\nu_2} \dots \partial \xi_N^{\nu_N}} \ln \left\langle e^{\xi_1 X_1} e^{\xi_2 X_2} \dots e^{\xi_N X_N} \right\rangle \right]_{\xi=0}, \quad (2.51)$$

where  $\Upsilon = \sum_j \nu_j$  and we used the short hand notation  $\xi = 0 \Leftrightarrow \xi_j = 0$  for all  $j$ . It is shown in reference [19], that third and fourth order cumulants can be written as

$$\begin{aligned} \langle X_j X_k X_l \rangle_c &= \langle X_j X_k X_l \rangle - [\langle X_j \rangle \langle X_k X_l \rangle + \langle X_k \rangle \langle X_j X_l \rangle + \langle X_l \rangle \langle X_j X_k \rangle] \\ &\quad + 2 \langle X_j \rangle \langle X_k \rangle \langle X_l \rangle, \end{aligned} \quad (2.52)$$

$$\begin{aligned} \langle X_j X_k X_l X_m \rangle_c &= \langle X_j X_k X_l X_m \rangle - [\langle X_j \rangle \langle X_j X_k X_l \rangle + \langle X_k \rangle \langle X_j X_l X_m \rangle \\ &\quad + \langle X_l \rangle \langle X_j X_k X_m \rangle + \langle X_m \rangle \langle X_j X_k X_l \rangle] \\ &\quad - [\langle X_j X_k \rangle \langle X_l X_m \rangle + \langle X_j X_l \rangle \langle X_k X_m \rangle + \langle X_j X_m \rangle \langle X_k X_l \rangle] \\ &\quad + 2[\langle X_j \rangle \langle X_k \rangle \langle X_l X_m \rangle + \langle X_j \rangle \langle X_l \rangle \langle X_k X_m \rangle + \langle X_j \rangle \langle X_m \rangle \langle X_k X_l \rangle] \end{aligned} \quad (2.53)$$

$$\begin{aligned}
 & + \langle X_l \rangle \langle X_m \rangle \langle X_j X_k \rangle + \langle X_k \rangle \langle X_m \rangle \langle X_j X_l \rangle + \langle X_k \rangle \langle X_l \rangle \langle X_j X_m \rangle \\
 & - 6 \langle X_j \rangle \langle X_k \rangle \langle X_l \rangle \langle X_m \rangle.
 \end{aligned}$$

Now, if one sets the right-hand-side of these equations equal to zero, one obtains representations for the third and fourth order correlators where only lower order correlators appear. We shall do this and obtain

$$\langle X_j X_k X_l \rangle = \langle X_j \rangle \langle X_k X_l \rangle + \langle X_k \rangle \langle X_j X_l \rangle + \langle X_j \rangle \langle X_j X_k \rangle - 2 \langle X_j \rangle \langle X_k \rangle \langle X_l \rangle, \quad (2.54)$$

$$\begin{aligned}
 \langle X_j X_k X_l X_m \rangle &= \langle X_j \rangle \langle X_k X_l X_m \rangle + \langle X_k \rangle \langle X_j X_l X_m \rangle & (2.55) \\
 & + \langle X_l \rangle \langle X_j X_k X_m \rangle + \langle X_m \rangle \langle X_j X_k X_l \rangle \\
 & + \langle X_j X_k \rangle \langle X_l X_m \rangle + \langle X_j X_l \rangle \langle X_k X_m \rangle + \langle X_j X_m \rangle \langle X_k X_l \rangle \\
 & - 2 \left[ \langle X_j \rangle \langle X_k \rangle \langle X_l X_m \rangle + \langle X_j \rangle \langle X_l \rangle \langle X_k X_m \rangle + \langle X_j \rangle \langle X_m \rangle \langle X_k X_l \rangle \right. \\
 & \left. + \langle X_l \rangle \langle X_m \rangle \langle X_j X_k \rangle + \langle X_k \rangle \langle X_m \rangle \langle X_j X_l \rangle + \langle X_k \rangle \langle X_l \rangle \langle X_j X_m \rangle \right] \\
 & + 6 \langle X_j \rangle \langle X_k \rangle \langle X_l \rangle \langle X_m \rangle.
 \end{aligned}$$

There are third order correlators appearing in equation (2.55). To get rid of these we use equation (2.54). Substituting the representation (2.54) in place of the third order correlators gives

$$\begin{aligned}
 \langle X_j X_k X_l X_m \rangle &= \langle X_j X_k \rangle \langle X_l X_m \rangle + \langle X_j X_l \rangle \langle X_k X_m \rangle + \langle X_j X_m \rangle \langle X_k X_l \rangle & (2.56) \\
 & - 2 \langle X_j \rangle \langle X_k \rangle \langle X_l \rangle \langle X_m \rangle.
 \end{aligned}$$

Equations (2.54) and (2.56) are used in [chapter 3](#) in the derivation of the equations of motion for the system under study.

In the semiclassical model used in reference [17] the orders of the correlators included into the equations of motion differ from what we include. The system studied there is a maser coupled with  $N$  identical atoms. The maser is modelled with a single harmonic oscillator with frequency  $\omega$  and the atoms with two-level-systems with energy level spacing of  $\hbar\omega$ . In the article they consider, like we, the number operator for the harmonic oscillator  $n = a^\dagger a$  as a second order operator, but handle Pauli's  $\sigma_z$  matrix as a first order operator even though  $\sigma_z = [\sigma_+, \sigma_-]$ , and, according to our treatment, is second order in the operators  $\sigma_\pm$ . There is also another problem in the formalism used in reference [17], which was discussed in [subsection 2.2.2](#).

In the papers by Mandel, [18, 21], he studies a laser coupled to  $N$  atoms. The semiclassical model used there is characterized as follows: When one encounters correlators,

one factorizes the correlator into two parts, one with the qubit operators and the other with the operators for the laser, *if the qubit part of the correlator is diagonal*. According to this, e.g.  $\langle \sigma_+ \sigma_- a \rangle$  would decompose to  $\langle \sigma_+ \sigma_- \rangle \langle a \rangle$  and  $\langle \sigma_+ \sigma_- a^\dagger a \rangle$  to  $\langle \sigma_+ \sigma_- \rangle \langle a^\dagger a \rangle$ , but  $\langle \sigma_\pm a^\dagger a \rangle$  would not be decomposed at all. Based on the discussion in this section, this way of factorizing the correlators is not consistent, at least not with respect to the order of cumulants to neglect. It is not explained in papers [18, 21] how one arrives to this version of the semiquantum approximation. It is also unclear if the model is supposed to be used only when studying systems where a laser interacts with atoms, or if it is a general guideline for other types of systems as well.

Even though we use in this thesis a firm mathematical base, keeping track of the orders of the cumulants included, there is still some ambiguity in the use of semiquantum approximation. Results seem to depend on where you use the anticommutation rule for the ladder operators of the qubit. For example, let us look at the correlator  $\langle \sigma_z n \rangle$ . This is a fourth order correlator so we need to use equation (2.56) to break it up. We have

$$\begin{aligned}
\langle \sigma_z n \rangle &= \langle [\sigma_+, \sigma_-] a^\dagger a \rangle = \langle \sigma_+ \sigma_- a^\dagger a \rangle - \langle \sigma_- \sigma_+ a^\dagger a \rangle \\
&= \langle \sigma_+ \sigma_- \rangle \langle a^\dagger a \rangle + \langle \sigma_+ a^\dagger \rangle \langle \sigma_- a \rangle + \langle \sigma_+ a \rangle \langle \sigma_- a^\dagger \rangle - 2 |\langle \sigma_- \rangle|^2 |\langle a \rangle|^2 \\
&\quad - \langle \sigma_- \sigma_+ \rangle \langle a^\dagger a \rangle - \langle \sigma_- a^\dagger \rangle \langle \sigma_+ a \rangle - \langle \sigma_- a \rangle \langle \sigma_+ a^\dagger \rangle + 2 |\langle \sigma_- \rangle|^2 |\langle a \rangle|^2 \\
&= \langle \sigma_+ \sigma_- \rangle \langle a^\dagger a \rangle - \langle \sigma_- \sigma_+ \rangle \langle a^\dagger a \rangle \\
&= (2 \langle \sigma_+ \sigma_- \rangle - 1) \langle a^\dagger a \rangle, \tag{2.57}
\end{aligned}$$

where we used the anticommutator  $\{\sigma_+, \sigma_-\} = 1$  to obtain the last equality. Let us now use *first* the anticommutator for the ladder operators and then equation (2.56) to divide the correlator. We obtain

$$\begin{aligned}
\langle [\sigma_+, \sigma_-] a^\dagger a \rangle &= 2 \langle \sigma_+ \sigma_- a^\dagger a \rangle - \langle a^\dagger a \rangle \\
&= 2 \left( \langle \sigma_+ \sigma_- \rangle \langle a^\dagger a \rangle + \langle \sigma_+ a^\dagger \rangle \langle \sigma_- a \rangle + \langle \sigma_+ a \rangle \langle \sigma_- a^\dagger \rangle \right) \\
&\quad - 4 |\langle \sigma_- \rangle|^2 |\langle a \rangle|^2 - \langle a^\dagger a \rangle \\
&= (2 \langle \sigma_+ \sigma_- \rangle - 1) \langle a^\dagger a \rangle + 2 \left( |\langle \sigma_- a \rangle|^2 + |\langle \sigma_- a^\dagger \rangle|^2 - 2 |\langle \sigma_- \rangle|^2 |\langle a \rangle|^2 \right). \tag{2.58}
\end{aligned}$$

This time we get the same as before, plus some qubit-cavity correlators and a term proportional to the magnitudes of  $\langle a \rangle$  and  $\langle \sigma_- \rangle$ . I haven't been able to justify which method is the right one, but we calculated the true value for the fourth order correlator

and for the two approximations above from the master equation, and the extra terms in (2.58) seem to correct the result to the right direction. In this thesis we use the latter method to divide the correlators, i.e. first use the anticommutation relation for  $\sigma_{\pm}$  and then do the semiclassical approximation.

## 2.4 Semiclassical approximation

In section 2.3 we discussed the semiclassical approximation. It dealt with a way to truncate infinite series of equations of motion for the correlators. A cruder way to handle this truncation is to use the *semiclassical theory*. It treats operators as statistically independent variables, i.e. we neglect all quantum coherence between operators. This is equivalent to neglecting the second order cumulants. In reference [19] it is shown that the second order cumulant can be written as

$$\langle AB \rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle. \quad (2.59)$$

Now setting the right-hand-side equal to zero gives

$$\langle AB \rangle = \langle A \rangle \langle B \rangle. \quad (2.60)$$

So, to put it simpler, we write everything with first order correlators, which are just plain expectation values of operators. Again these operators must be 'fundamental' for the system.

The semiclassical model has been used extensively in the study of lasers and masers [22–27]. We shall make use of the semiclassical model in section 3.3 to derive analytical solutions to which we compare our numerical results.

## 2.5 Calculation of the spectrum

When one wants to study the internal energy structure of a system, a good way is to study its emission or absorption spectrum. The spectrum gives information about the energy levels and transitions between them. The standard way to study the spectrum is to use Fermi's golden rule, which gives the transition rate between some initial state,  $|i\rangle$ , and a final state,  $|f\rangle$ . In this section the aim is to derive an alternative way to obtain the spectrum through two-time correlation functions, or autocorrelation functions. The rest of the discussion in this section is based on the references [28, 29].

### 2.5.1 From Fermi's golden rule to autocorrelation functions

Assume that we are perturbing the system with a weak harmonic probe which is coupled to the system through the Hermitian operator  $V$ . Then the Hamiltonian of the perturbation can be written as

$$H_P = 2A_P V \cos \omega_P t. \quad (2.61)$$

Using first order time-dependent perturbation theory, one can derive [30] the following expression for the transition rate from the initial state  $|i\rangle$  to the final state  $|f\rangle$  at frequency  $\omega_P$  ( $|i\rangle$  and  $|f\rangle$  are eigenstates of the undriven Hamiltonian)

$$W_{i \rightarrow f} = \frac{2\pi A_P^2}{\hbar^2} \left( |\langle f|V|i\rangle|^2 \delta(\omega_{fi} + \omega_P) + |\langle f|V^\dagger|i\rangle|^2 \delta(\omega_{fi} - \omega_P) \right), \quad (2.62)$$

where  $\omega_{fi} = (E_f - E_i)/\hbar$  and  $\omega_P > 0$  is the angular frequency of the probe. This is *Fermi's golden rule*. Let us concentrate on the absorptive transitions, i.e. on transitions for which  $\omega_{fi} > 0$ . Thus, we neglect the first term in (2.62), which corresponds then to emission. To get the total absorptive transition rate from state  $|i\rangle$ , we sum over all final states

$$W_i^{\text{abs}} = \sum_f W_{i \rightarrow f}. \quad (2.63)$$

We can extend the sum, in the above, over all states  $|f\rangle$ , because negative  $\omega_{fi}$  produce vanishing  $W_{i \rightarrow f}$ . The total transition rate from any initial state,  $|i\rangle$ , to any final state,  $|f\rangle$ , is obtained by summing (2.63) over all the initial states, weighted by the occupation probability of the corresponding state. Thus,

$$W_T = \sum_i p_i W_i^{\text{abs}} = \sum_{i,f} p_i W_{i \rightarrow f} = \frac{2\pi A_P^2}{\hbar^2} \sum_{i,f} p_i \left| \langle f|V^\dagger|i\rangle \right|^2 \delta(\omega_{fi} - \omega_P). \quad (2.64)$$

Using the property  $\delta(x) = \delta(-x)$  of the Dirac delta function and writing the delta function in the above expression as an integral  $\left( \delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} d\tau \right)$ , we get

$$\begin{aligned} W_T &= \frac{A_P}{\hbar^2} \sum_{i,f} \int_{-\infty}^{\infty} p_i \left| \langle f|V^\dagger|i\rangle \right|^2 e^{-i(\omega_{fi} - \omega_P)\tau} d\tau \\ &= \frac{A_P}{\hbar^2} \sum_{i,f} \int_{-\infty}^{\infty} e^{i\omega_P\tau} p_i \langle f|V^\dagger|i\rangle \langle f|V^\dagger|i\rangle^* e^{-i\omega_{fi}\tau} d\tau \\ &= \frac{A_P}{\hbar^2} \sum_{i,f} \int_{-\infty}^{\infty} e^{i\omega_P\tau} p_i \langle f|V^\dagger|i\rangle \langle i|V|f\rangle e^{-i\omega_{fi}\tau} e^{i\omega_i\tau} d\tau \end{aligned}$$

$$= \frac{A_P}{\hbar^2} \int_{-\infty}^{\infty} e^{i\omega_P \tau} \sum_f \langle f | V^\dagger \sum_i p_i |i\rangle \langle i| e^{i\omega_i \tau} V e^{-i\omega_f \tau} |f\rangle d\tau. \quad (2.65)$$

We use the relation  $e^{-i\omega_g \tau} |g\rangle = e^{-iH\tau/\hbar} |g\rangle$ , where  $H$  is the undriven Hamiltonian and  $g = i, f$ , and recall, that in the Heisenberg picture  $V_H(\tau) = e^{iH\tau/\hbar} V e^{-iH\tau/\hbar}$ , and obtain

$$W_T = \frac{A_P}{\hbar^2} \int_{-\infty}^{\infty} e^{i\omega_P \tau} \sum_f \langle f | V^\dagger(0) \left( \sum_i p_i |i\rangle \langle i| \right) V(\tau) |f\rangle d\tau. \quad (2.66)$$

Because states  $|i\rangle$  and  $|f\rangle$  form a complete orthonormal basis for the Hilbert space, we have that  $\sum_i p_i |i\rangle \langle i| = \rho$ ,  $\rho$  is the density operator for the undriven system, and  $\sum_f \langle f | \cdot |f\rangle = \text{Tr}(\cdot)$ . Thus,

$$\begin{aligned} W_T &= \frac{A_P}{\hbar^2} \int_{-\infty}^{\infty} e^{i\omega_P \tau} \text{Tr} \left( V^\dagger(0) \rho V(\tau) \right) d\tau \\ &= \frac{A_P}{\hbar^2} \int_{-\infty}^{\infty} e^{i\omega_P \tau} \text{Tr} \left( V(\tau) V^\dagger(0) \rho \right) d\tau \\ &= \frac{A_P}{\hbar^2} \int_{-\infty}^{\infty} e^{i\omega_P \tau} \left\langle V(\tau) V^\dagger(0) \right\rangle d\tau. \end{aligned} \quad (2.67)$$

We have obtained the result, that the total transition rate from state  $|i\rangle$  to state  $|f\rangle$  can be extracted from the Fourier transform of the autocorrelation function  $\langle V(\tau) V^\dagger(0) \rangle$ . This is exactly what we set out for. Often one denotes

$$S(\omega) \equiv \frac{A_P}{\hbar^2} \int_{-\infty}^{\infty} e^{i\omega_P \tau} \left\langle V(\tau) V^\dagger(0) \right\rangle d\tau. \quad (2.68)$$

This result was used to obtain the spectra in [chapter 4](#).

## Chapter 3

# Derivation of equations of motion

In this chapter we derive the differential equations governing the time-evolution of the system under study. The system consist of a harmonic oscillator coupled to a two-level-system. We also take into account the possibility to drive the oscillator with an external force. The equations of motion are obtained through the quantum master equation and the semiquantum approximation presented in [chapter 2](#). System of a harmonic oscillator and a qubit was chosen because of its simplicity, so that the essential features of the approximation would be clearly emphasized.

### 3.1 The Hamiltonian

Often [\[31–36\]](#), when dealing with a system of a coupled two-level-system and an oscillator, the used model is that developed by E. Jaynes and F. Cummings [\[22\]](#), or the generalization of that for  $N$  identical qubits, the Tavis-Cummings model [\[37\]](#). Jaynes-Cummings model is valid when the frequencies of the two sub-systems are similar and the coupling between them is sufficiently weak [\[12\]](#). In this thesis, however, we want to study regimes where these assumptions do not hold, particularly, we want to have the coupling between the systems to be strong. Because of this, we use the *Rabi model*, presented by I. I. Rabi in the 1930s [\[38, 39\]](#). Rabi and Jaynes-Cummings models are not all that different, one can obtain the latter from the former by making the rotating wave approximation. In recent years use of the Rabi model has gained in popularity [\[1, 2, 40–43\]](#), when the coupling between the subsystems has been getting stronger. Advantage of the Jaynes-Cummings model is that it is much easier to solve. Until recently [\[44\]](#), an analytical solution for the Rabi model was not even found. Even though there now exist a solution for the Rabi Hamiltonian, it is given by the power series of a transcendental



function, and is, thus, not that convenient to use.

The Rabi Hamiltonian reads

$$H_R = \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-), \quad (3.1)$$

where  $\omega_c$  is the natural frequency of the harmonic oscillator,  $\hbar\omega_0$  is the level separation of the qubit and  $g$  gives the strength of the coupling,  $a^{(\dagger)}$  is the lowering (raising) operator for the oscillator and  $\sigma_{-(+)}$  are the corresponding operators for the qubit, respectively. (One obtains the Jaynes-Cummings Hamiltonian by ignoring the 'counter rotating' terms  $a^\dagger \sigma_+$  and  $a \sigma_-$ .) Here we have neglected the zero point energy of the harmonic oscillator. In addition to this, we take into account the possibility to drive the oscillator. We choose to couple the drive to the momentum of the oscillator, and assume that the driving field can be treated as a classical variable. With these assumptions the Hamiltonian for the drive reads

$$H_d = i\hbar f(t) (a^\dagger - a). \quad (3.2)$$

Here  $f(t)$  is some real function of time changing the amplitude of the drive. Thus, the system is described by total Hamiltonian

$$H = H_R + H_d. \quad (3.3)$$

## 3.2 Equations of motion

In [chapter 2](#) we discussed how to include damping into quantum systems. The end result was the quantum master equation. In the derivation of the master equations for the qubit and the oscillator we considered them as separate systems, interacting with just the bath of oscillators. Now, however, the two systems are coupled together. How should one proceed in this case? If one assumes that the coupling between the two subsystems is weak, one can discard it in the derivation of the master equation, and proceed as in the previous chapter. When the coupling gets stronger this is not a valid treatment. The result of neglecting the coupling can be seen, e.g. in the fact that the environment does not drive the system to its true ground state. However, in this thesis we are interested in demonstrating the semiquantum approximation and, thus, chose to use this approximation. Neglecting the coupling results in the standard Born-Markovian master equation in Lindblad form [\[12\]](#)

$$\begin{aligned} \dot{\rho} = & -\frac{i}{\hbar} [H, \rho] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \rho + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \rho \\ & + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \rho + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \rho + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \rho. \end{aligned} \quad (3.4)$$

The notation is the same as in [chapter 2](#). From the master equation one can obtain the governing equations for all correlators, e.g. for any operator  $A$  in the Schrödinger picture holds

$$\frac{d\langle A \rangle}{dt} = \frac{d}{dt} \text{Tr}(A\rho) = \text{Tr}\left(A \frac{d\rho}{dt}\right). \quad (3.5)$$

We begin by deriving an equation for the expectation value of the annihilation operator for the harmonic oscillator. We denote  $\alpha = \langle a \rangle$ .

$$\begin{aligned} \frac{d\alpha}{dt} &= \text{Tr}\left(a \frac{d\rho}{dt}\right) \\ &= -\frac{i}{\hbar} \text{Tr}(a[H, \rho]) + \frac{\kappa}{2} (N_c + 1) \text{Tr}(a\mathcal{D}[a]\rho) + \frac{\kappa}{2} N_c \text{Tr}(a\mathcal{D}[a^\dagger]\rho) \\ &\quad + \frac{\gamma}{2} (N_q + 1) \text{Tr}(a\mathcal{D}[\sigma_-]\rho) + \frac{\gamma}{2} N_q \text{Tr}(a\mathcal{D}[\sigma_+]\rho) + \gamma_\phi \text{Tr}(a\mathcal{D}[\sigma_+\sigma_-]\rho) \end{aligned} \quad (3.6)$$

Let us take a closer look on the term  $\text{Tr}(a\mathcal{D}[\sigma_-]\rho)$ . Because  $a$  commutes with  $\sigma_\pm$  we obtain

$$\begin{aligned} \text{Tr}(a\mathcal{D}[\sigma_-]\rho) &= 2\text{Tr}(a\sigma_- \rho \sigma_+) - \text{Tr}(a\rho \sigma_+ \sigma_-) - \text{Tr}(a\sigma_+ \sigma_-) \\ &= \text{Tr}(a\rho \sigma_+ \sigma_-) + \text{Tr}(a\sigma_+ \sigma_- \rho) - \text{Tr}(a\rho \sigma_+ \sigma_-) - \text{Tr}(a\sigma_+ \sigma_-) \\ &= 0, \end{aligned} \quad (3.7)$$

where we have used the cyclic property of trace. In a similar manner, one can show that terms proportional to  $\text{Tr}(a\mathcal{D}[\sigma_+]\rho)$  and  $\text{Tr}(a\mathcal{D}[\sigma_+\sigma_-]\rho)$  vanish as well. In general, if the operator that is traced commutes with the argument of the Lindblad superoperator and its hermite conjugate, the trace vanishes. The term with the Hamiltonian can be written as

$$\begin{aligned} \text{Tr}(a[H, \rho]) &= \text{Tr}(aH\rho) - \text{Tr}(a\rho H) = \text{Tr}(aH\rho) - \text{Tr}(Ha\rho) \\ &= \text{Tr}([a, H]\rho) = \langle [a, H] \rangle. \end{aligned} \quad (3.8)$$

Plugging in Hamiltonian [\(3.3\)](#) gives

$$\begin{aligned} -\frac{i}{\hbar} \text{Tr}(a[H, \rho]) &= \langle [a, H_R] \rangle + \langle [a, H_d] \rangle \\ &= -i\omega_c \langle [a, a^\dagger a] \rangle + -i\omega_0 \langle [a, \sigma_+\sigma_-] \rangle \\ &\quad - ig \langle [a, (a^\dagger + a)(\sigma_+ + \sigma_-)] \rangle + f(t) \langle [a, (a^\dagger - a)] \rangle \\ &= -i\omega_c \langle a \rangle - ig \langle \sigma_+ + \sigma_- \rangle + f(t) \\ &= -i\omega_c \alpha - 2ig\Re(s_-) + f(t), \end{aligned} \quad (3.9)$$

where we have used the commutators  $[a, a^\dagger a] = a$  and  $[a, a^\dagger] = 1$  and the fact that  $a$  commutes with  $\sigma_\pm$  and, naturally, with itself. We also denoted  $\langle \sigma_- \rangle = s_-$  and used  $\langle \sigma_+ \rangle = \langle \sigma_- \rangle^*$ .  $\Re$  denotes taking the real part. The first of the damping terms reduces to

$$\begin{aligned} \text{Tr}(a\mathcal{D}[a]\rho) &= 2\text{Tr}(aa\rho a^\dagger) - \text{Tr}(a\rho a^\dagger a) - \text{Tr}(aa^\dagger a\rho) \\ &= 2\text{Tr}(a^\dagger aa\rho) - \text{Tr}(a^\dagger aa\rho) - \text{Tr}(aa^\dagger a\rho) \\ &= \langle a^\dagger aa \rangle - \langle (a^\dagger a + 1)a \rangle \\ &= -\langle a \rangle = -\alpha \end{aligned} \quad (3.10)$$

and the second one to

$$\begin{aligned} \text{Tr}(a\mathcal{D}[a^\dagger]\rho) &= 2\text{Tr}(aa^\dagger\rho a) - \text{Tr}(a\rho aa^\dagger) - \text{Tr}(aaa^\dagger\rho) \\ &= 2\text{Tr}(aaa^\dagger\rho) - \text{Tr}(aa^\dagger a\rho) - \text{Tr}(aaa^\dagger\rho) \\ &= \langle aaa^\dagger \rangle - \langle a(aa^\dagger - 1) \rangle \\ &= \langle a \rangle = \alpha, \end{aligned} \quad (3.11)$$

where we have used  $[a, a^\dagger] = 1$  in the above. Collecting the results calculated above and substituting them into (3.6) gives

$$\begin{aligned} \frac{d\alpha}{dt} &= -i\omega_c\alpha - 2ig\Re(s_-) + f(t) - \frac{\kappa}{2}(N_c + 1)\alpha + \frac{\kappa}{2}N_c\alpha \\ &= -\left(\frac{\kappa}{2} + i\omega_c\right)\alpha - 2ig\Re(s_-) + f(t). \end{aligned} \quad (3.12)$$

### 3.2.1 Eliminating the drive

As we discussed in the previous chapter, we would like to eliminate the driving from the equations of motion. The solution was to use the displacement operator,  $D(\alpha)$ , equation (2.35). Next we carry out the transformation for the Hamiltonian and the master equation. Because we have  $\alpha$  appearing in the operator  $D(\alpha)$ , it is time-dependent according to equation (3.12). For this reason, the transformation cannot be carried out just by calculating  $D^\dagger(\alpha)HD(\alpha)$ , but we must take into account the time-dependence generated by  $\alpha$ . Easiest way to determine the transformation of the Hamiltonian is to look at the Schrödinger equation for the transformed states  $|a'\rangle = D^\dagger(\alpha)|a\rangle$ . We have,

$$\begin{aligned} i\hbar\frac{d|a'\rangle}{dt} &= i\hbar\left(D^\dagger(\alpha)\frac{d|a\rangle}{dt} + \frac{dD^\dagger(\alpha)}{dt}|a\rangle\right) = D^\dagger(\alpha)H|a\rangle + i\hbar\frac{dD^\dagger(\alpha)}{dt}|a\rangle \\ &= \left(D^\dagger(\alpha)HD(\alpha) + i\hbar\frac{dD^\dagger(\alpha)}{dt}D(\alpha)\right)|a'\rangle, \end{aligned} \quad (3.13)$$

where we have used the Schrödinger equation for the original states,  $i\hbar d|a\rangle/dt = H|a\rangle$ , and  $|a\rangle = D(\alpha)|a'\rangle$ . Thus, the transformed Hamiltonian,  $\tilde{H}$ , is the expression in the brackets

$$\tilde{H} = D^\dagger(\alpha)HD(\alpha) + i\hbar \frac{dD^\dagger(\alpha)}{dt}D(\alpha). \quad (3.14)$$

In addition to the normal unitary transformation part, we have a derivative term appearing in the transformed Hamiltonian. The derivative of  $D^\dagger(\alpha)$  becomes

$$\begin{aligned} \frac{dD^\dagger(\alpha)}{dt} &= \frac{d}{dt}e^{|\alpha|^2/2}e^{\alpha^*a}e^{-\alpha a^\dagger} \\ &= \frac{d}{d\alpha}e^{|\alpha|^2/2}e^{\alpha^*a}e^{-\alpha a^\dagger} \frac{d\alpha}{dt} + \frac{d}{d\alpha^*}e^{|\alpha|^2/2}e^{\alpha^*a}e^{-\alpha a^\dagger} \frac{d\alpha^*}{dt} \\ &= \left[ \left( \frac{\alpha}{2} + a \right) \frac{d\alpha^*}{dt} + \left( \frac{\alpha^*}{2} - a^\dagger \right) \frac{d\alpha}{dt} \right] e^{|\alpha|^2/2}e^{\alpha^*a}e^{-\alpha a^\dagger} \\ &= \left[ \left( \frac{\alpha}{2} + a \right) \frac{d\alpha^*}{dt} + \left( \frac{\alpha^*}{2} - a^\dagger \right) \frac{d\alpha}{dt} \right] D^\dagger(\alpha). \end{aligned} \quad (3.15)$$

Thus, the latter term is

$$\begin{aligned} i\hbar \frac{dD^\dagger(\alpha)}{dt}D(\alpha) &= i\hbar \left[ \left( \frac{\alpha}{2} + a \right) \frac{d\alpha^*}{dt} + \left( \frac{\alpha^*}{2} - a^\dagger \right) \frac{d\alpha}{dt} \right] D^\dagger(\alpha)D(\alpha) \\ &= i\hbar \left[ \left( \frac{\alpha}{2} + a \right) \frac{d\alpha^*}{dt} + \left( \frac{\alpha^*}{2} - a^\dagger \right) \frac{d\alpha}{dt} \right] \\ &= i\hbar \left( a \frac{d\alpha^*}{dt} - a^\dagger \frac{d\alpha}{dt} \right) + \frac{i\hbar}{2} \frac{d|\alpha|^2}{dt}. \end{aligned} \quad (3.16)$$

The last term in the above is a scalar and can, thus, be left out from the Hamiltonian, since it has no effect on the equations of motion. Using the properties  $D^\dagger(\alpha)aD(\alpha) = \alpha + a$ ,  $D^\dagger(\alpha)a^\dagger D(\alpha) = \alpha^* + a^\dagger$ , derived in [subsection 2.2.2](#), the first term in (3.14) becomes (transformation (2.35) does not affect  $\sigma_\pm$ , since they commute with  $a$  and  $a^\dagger$ )

$$\begin{aligned} D^\dagger(\alpha)HD(\alpha) &= D^\dagger(\alpha)H_R D(\alpha) + D^\dagger(\alpha)H_d D(\alpha) \\ &= \hbar\omega_c (\alpha^* + a^\dagger) (\alpha + a) + \hbar\omega_0 \sigma_+ \sigma_- + i\hbar f(t) \left[ (\alpha^* + a^\dagger) - (\alpha + a) \right] \\ &\quad + \hbar g \left[ (\alpha^* + a^\dagger) + (\alpha + a) \right] (\sigma_+ + \sigma_-) \\ &= \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-) + \hbar g (\alpha^* + \alpha) (\sigma_+ + \sigma_-) \\ &\quad + \hbar (\omega_c \alpha + i f(t)) a^\dagger + \hbar (\omega_c \alpha^* - i f(t)) a + \hbar \left[ \omega_c |\alpha|^2 + 2f(t) \Im(\alpha) \right]. \end{aligned} \quad (3.17)$$

Also here, the term in the square brackets is a scalar, so we can leave it out from the Hamiltonian. Combining these, gives the transformed Hamiltonian

$$\begin{aligned}
 \tilde{H} &= \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-) + \hbar g (\alpha^* + \alpha) (\sigma_+ + \sigma_-) \\
 &\quad + \hbar (\omega_c \alpha + i f(t)) a^\dagger + \hbar (\omega_c \alpha^* - i f(t)) a + i \hbar \left( a \frac{d\alpha^*}{dt} - a^\dagger \frac{d\alpha}{dt} \right) \\
 &= \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-) + \hbar g (\alpha^* + \alpha) (\sigma_+ + \sigma_-) \\
 &\quad + \hbar \left( \omega_c \alpha + i f(t) - i \frac{d\alpha}{dt} \right) a^\dagger + \hbar \left( \omega_c \alpha^* - i f(t) + i \frac{d\alpha^*}{dt} \right) a \\
 &\equiv \tilde{H}_0 + \hbar \left( \omega_c \alpha + i f(t) - i \frac{d\alpha}{dt} \right) a^\dagger + \hbar \left( \omega_c \alpha^* - i f(t) + i \frac{d\alpha^*}{dt} \right) a, \tag{3.18}
 \end{aligned}$$

where we have denoted

$\tilde{H}_0 = \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-) + \hbar g (\alpha^* + \alpha) (\sigma_+ + \sigma_-)$ . Next, we must carry out the transformation for the master equation (3.4). The outcome is ( $\tilde{\rho}$  is the transformed density operator)

$$\begin{aligned}
 \dot{\tilde{\rho}} &= -\frac{i}{\hbar} [H_{\text{eff}}, \tilde{\rho}] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \tilde{\rho} + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \tilde{\rho} + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \tilde{\rho} \\
 &\quad + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \tilde{\rho} + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \tilde{\rho}, \tag{3.19}
 \end{aligned}$$

with effective Hamiltonian

$$\begin{aligned}
 H_{\text{eff}} &= \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-) \\
 &\quad + 2\hbar g \left[ \Re(\alpha) (\sigma_+ + \sigma_-) - \Re(s_-) (a^\dagger + a) \right]. \tag{3.20}
 \end{aligned}$$

We have used equation (3.12) in the above. Thus, we have achieved what we were after, the driving force,  $f(t)$ , has vanished from the master equation. Its effect is not completely gone, though, but can be seen through  $\alpha$  and  $s_-$ . The transformation of the master equation is carried out in [Appendix A](#).

Now we can continue the derivation of the equations of motion. For the rest of the correlators we use the transformed master equation. We do not present all of the derivations here, but just one of each kind of correlator: oscillator-oscillator, qubit-qubit and qubit-oscillator. We drop the tilde from the transformed density operator, since there is little possibility for misunderstanding.

Let us proceed with  $\langle a^\dagger a \rangle$ : like with  $\alpha$ ,  $\langle a^\dagger a \rangle$  commutes with  $\sigma_\pm$ , so the damping terms with  $\mathcal{D}[\sigma_\pm]$  vanish. Likewise in the Hamiltonian part, the commutator with the qubit Hamiltonian vanishes. With these simplifications we get

$$\begin{aligned}
\frac{d\langle a^\dagger a \rangle}{dt} &= -i\omega_c \langle [a^\dagger a, a^\dagger a] \rangle - ig \langle [a^\dagger a, (a^\dagger + a)(\sigma_+ + \sigma_-)] \rangle \\
&\quad + 2ig\Re(s_-) \langle [a^\dagger a, (a^\dagger + a)] \rangle + \frac{\kappa}{2}(N_c + 1) \text{Tr}(a^\dagger a \mathcal{D}[a] \rho) \\
&\quad + \frac{\kappa}{2} N_c \text{Tr}(a^\dagger a \mathcal{D}[a^\dagger] \rho). \tag{3.21}
\end{aligned}$$

Operator  $a^\dagger a$  commutes with itself, so the first term is zero. For the second and third terms we use commutators  $[a^\dagger a, a] = -a$  and  $[a^\dagger a, a^\dagger] = a^\dagger$ . The first of the damping terms becomes

$$\begin{aligned}
\text{Tr}(a^\dagger a \mathcal{D}[a] \rho) &= 2\text{Tr}(a^\dagger a a \rho a^\dagger) - \text{Tr}(a^\dagger a a^\dagger a \rho) - \text{Tr}(a^\dagger a \rho a^\dagger a) \\
&= 2\text{Tr}(a^\dagger a^\dagger a a \rho) - \text{Tr}(a^\dagger a a^\dagger a \rho) - \text{Tr}(a^\dagger a a^\dagger a \rho) \\
&= 2\text{Tr}(a^\dagger (a a^\dagger - 1) a \rho) - 2\text{Tr}(a^\dagger a a^\dagger a \rho) \\
&= -2\text{Tr}(a^\dagger a) = -2\langle a^\dagger a \rangle \tag{3.22}
\end{aligned}$$

and the latter one

$$\begin{aligned}
\text{Tr}(a^\dagger a \mathcal{D}[a^\dagger] \rho) &= 2\text{Tr}(a^\dagger a a^\dagger \rho a) - \text{Tr}(a^\dagger a a a^\dagger \rho) - \text{Tr}(a^\dagger a \rho a a^\dagger) \\
&= 2\text{Tr}(a a^\dagger a a^\dagger \rho) - \text{Tr}(a^\dagger a a a^\dagger \rho) - \text{Tr}(a a^\dagger a^\dagger a \rho) \\
&= 2\text{Tr}([a^\dagger a + 1][a^\dagger a + 1] \rho) - \text{Tr}(a^\dagger a [a^\dagger a + 1] \rho) \\
&\quad - \text{Tr}([a^\dagger a + 1] a^\dagger a \rho) \\
&= 2\text{Tr}([2a^\dagger a + 1]) - 2\text{Tr}(a^\dagger a) \\
&= 2\text{Tr}([a^\dagger a + 1]) = 2\langle a^\dagger a \rangle + 2. \tag{3.23}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{d\langle a^\dagger a \rangle}{dt} &= -ig \langle (a^\dagger - a)(\sigma_+ + \sigma_-) \rangle + 2ig\Re(s_-) \langle a^\dagger - a \rangle \\
&\quad - \kappa(N_c + 1) \langle a^\dagger a \rangle + \kappa N_c (\langle a^\dagger a \rangle + 1) \\
&= -ig \langle a^\dagger \sigma_+ + a^\dagger \sigma_- - a \sigma_+ - a \sigma_- \rangle - \kappa (\langle a^\dagger a \rangle - N_c) \\
&= 2g [\Im(\langle a^\dagger \sigma_- \rangle) - \Im(\langle a \sigma_- \rangle)] - \kappa (\langle a^\dagger a \rangle - N_c). \tag{3.24}
\end{aligned}$$

Where we have used  $\langle a \sigma_+ \rangle = \langle a^\dagger \sigma_- \rangle^*$ ,  $\langle a^\dagger \sigma_+ \rangle = \langle a \sigma_- \rangle^*$  and  $\langle a^{(\dagger)} \rangle = 0$ , because of the transformation (2.35). It is noticeable that the above equation does not contain correlators higher than order 2, so we do not have to employ the semiclassical approximation.

This is also the case for other correlators with operators just from the harmonic oscillator space.

Next we tackle  $\langle \sigma_+ \sigma_- \rangle$ . In this case the damping terms with oscillator operators vanish, and in the Hamiltonian part the oscillator Hamiltonian has no effect. With these notions, we obtain

$$\begin{aligned} \frac{d\langle \sigma_+ \sigma_- \rangle}{dt} &= -i\omega_0 \langle [\sigma_+ \sigma_-, \sigma_+ \sigma_-] \rangle - ig \left\langle \left[ \sigma_+ \sigma_-, \left( a^\dagger + a \right) (\sigma_+ + \sigma_-) \right] \right\rangle \\ &\quad - 2ig\Re(\alpha) \langle [\sigma_+ \sigma_-, (\sigma_+ + \sigma_-)] \rangle + \frac{\gamma}{2} (N_q + 1) \text{Tr}(\sigma_+ \sigma_- \mathcal{D}[\sigma_-] \rho) \\ &\quad + \frac{\gamma}{2} N_q \text{Tr}(\sigma_+ \sigma_- \mathcal{D}[\sigma_+] \rho) + \gamma\phi \text{Tr}(\sigma_+ \sigma_- \mathcal{D}[\sigma_+ \sigma_-] \rho). \end{aligned} \quad (3.25)$$

The first term vanishes since  $\sigma_+ \sigma_-$  commutes with itself. On the second and third terms we use the commutator  $[\sigma_+ \sigma_-, \sigma_\pm] = \pm \sigma_\pm$ . The first dissipator gives

$$\begin{aligned} \text{Tr}(\sigma_+ \sigma_- \mathcal{D}[\sigma_-] \rho) &= 2\text{Tr}(\sigma_+ \sigma_- \sigma_- \rho \sigma_+) - \text{Tr}(\sigma_+ \sigma_- \rho \sigma_+ \sigma_-) - \text{Tr}(\sigma_+ \sigma_- \sigma_+ \sigma_- \rho) \\ &= -2\text{Tr}(\sigma_+ \sigma_- \sigma_+ \sigma_- \rho) \\ &= -2\text{Tr}([1 - \sigma_- \sigma_+] \sigma_+ \sigma_- \rho) \\ &= -2\text{Tr}(\sigma_+ \sigma_- \rho) = -2\langle \sigma_+ \sigma_- \rangle. \end{aligned} \quad (3.26)$$

The second one becomes

$$\begin{aligned} \text{Tr}(\sigma_+ \sigma_- \mathcal{D}[\sigma_+] \rho) &= 2\text{Tr}(\sigma_+ \sigma_- \sigma_+ \rho \sigma_-) - \text{Tr}(\sigma_+ \sigma_- \rho \sigma_- \sigma_+) - \text{Tr}(\sigma_+ \sigma_- \sigma_- \sigma_+ \rho) \\ &= 2\text{Tr}(\sigma_- \sigma_+ \sigma_- \sigma_+ \rho) - \text{Tr}(\sigma_- \sigma_+ \sigma_+ \sigma_- \rho) \\ &= 2\text{Tr}([1 - \sigma_+ \sigma_-] \sigma_- \sigma_+ \rho) \\ &= 2\text{Tr}([1 - \sigma_+ \sigma_-] \rho) = 2 - 2\langle \sigma_+ \sigma_- \rangle. \end{aligned} \quad (3.27)$$

Above, we used the anticommutator  $\{\sigma_+, \sigma_-\} = 1$  and that  $\sigma_\pm^2 = 0$ . The last term in (3.25), causing dephasing, vanishes because of commutation. Thus, the equation for  $\langle \sigma_+ \sigma_- \rangle$  becomes

$$\begin{aligned} \frac{d\langle \sigma_+ \sigma_- \rangle}{dt} &= -ig \left\langle \left( a^\dagger + a \right) (\sigma_+ - \sigma_-) \right\rangle - 2ig\Re(\alpha) \langle \sigma_+ - \sigma_- \rangle \\ &\quad - \gamma (N_q + 1) \langle \sigma_+ \sigma_- \rangle + \gamma N_q (\langle \sigma_+ \sigma_- \rangle + 1) \\ &= -2g \left[ \Im \left( \left\langle a^\dagger \sigma_- \right\rangle \right) + \Im(\langle a \sigma_- \rangle) + 2\Re(\alpha) \Im(\langle \sigma_- \rangle) \right] \end{aligned} \quad (3.28)$$

$$- \gamma [(2N_q + 1) \langle \sigma_+ \sigma_- \rangle - N_q]. \quad (3.29)$$

Lastly, we calculate the equation for  $\langle a \sigma_- \rangle$ . In this case, none of the dissipators nor terms in the Hamiltonian vanish because of commutation. However, one can easily see

that the effect of the damping terms, proportional to  $a$  and  $a^\dagger$ , on  $\langle a\sigma_- \rangle$  is the same as on  $\langle a \rangle$ . Analogously, dissipators for the qubit affect  $\langle a\sigma_- \rangle$  similarly as  $\langle \sigma_- \rangle$ . Thus, for simplicity, we calculate the latter:

$$\begin{aligned}\mathrm{Tr}(\sigma_- \mathcal{D}[\sigma_-] \rho) &= 2\mathrm{Tr}(\sigma_- \sigma_- \rho \sigma_+) - \mathrm{Tr}(\sigma_- \rho \sigma_+ \sigma_-) - \mathrm{Tr}(\sigma_- \sigma_+ \sigma_- \rho) \\ &= -\mathrm{Tr}(\sigma_+ \sigma_- \sigma_- \rho) - \mathrm{Tr}([1 - \sigma_+ \sigma_-] \sigma_- \rho) \\ &= -\mathrm{Tr}(\sigma_- \rho) = -\langle \sigma_- \rangle,\end{aligned}\tag{3.30}$$

$$\begin{aligned}\mathrm{Tr}(\sigma_- \mathcal{D}[\sigma_+] \rho) &= 2\mathrm{Tr}(\sigma_- \sigma_+ \rho \sigma_-) - \mathrm{Tr}(\sigma_- \rho \sigma_- \sigma_+) - \mathrm{Tr}(\sigma_- \sigma_- \sigma_+ \rho) \\ &= 2\mathrm{Tr}(\sigma_- \sigma_- \sigma_+ \rho) - \mathrm{Tr}(\sigma_- \sigma_+ \sigma_- \rho) \\ &= -\mathrm{Tr}([1 - \sigma_+ \sigma_-] \sigma_- \rho) \\ &= -\mathrm{Tr}(\sigma_- \rho) = -\langle \sigma_- \rangle\end{aligned}\tag{3.31}$$

and

$$\begin{aligned}\mathrm{Tr}(\sigma_- \mathcal{D}[\sigma_+ \sigma_-] \rho) &= 2\mathrm{Tr}(\sigma_- \sigma_+ \sigma_- \rho \sigma_+ \sigma_-) - \mathrm{Tr}(\sigma_- \rho \sigma_+ \sigma_- \sigma_+ \sigma_-) \\ &\quad - \mathrm{Tr}(\sigma_- \sigma_+ \sigma_- \sigma_+ \sigma_- \rho) \\ &= 2\mathrm{Tr}(\sigma_+ \sigma_- \sigma_- \sigma_+ \sigma_- \rho) - \mathrm{Tr}(\sigma_+ \sigma_- \sigma_+ \sigma_- \sigma_- \rho) \\ &\quad - \mathrm{Tr}(\sigma_- \sigma_+ \sigma_- [1 - \sigma_- \sigma_+] \rho) \\ &= -\mathrm{Tr}(\sigma_- [1 - \sigma_- \sigma_+] \rho) \\ &= -\mathrm{Tr}(\sigma_- \rho) = -\langle \sigma_- \rangle.\end{aligned}\tag{3.32}$$

Above, we used the anticommutator for the ladder operators and  $\sigma_\pm^2 = 0$ . The time-evolution from the Hamiltonian is

$$\begin{aligned}-\frac{i}{\hbar} \langle [a\sigma_-, H] \rangle &= -i\omega_c \langle [a\sigma_-, a^\dagger a] \rangle + -i\omega_0 \langle [a\sigma_-, \sigma_+ \sigma_-] \rangle \\ &\quad - ig \left\langle \left[ a\sigma_-, \left( a^\dagger + a \right) (\sigma_+ + \sigma_-) \right] \right\rangle - 2ig\Re(\alpha) \langle [a\sigma_-, (\sigma_+ + \sigma_-)] \rangle \\ &\quad + 2ig\Re(s_-) \left\langle \left[ a\sigma_-, \left( a^\dagger + a \right) \right] \right\rangle.\end{aligned}\tag{3.33}$$

The first and last terms can be calculated like  $\sigma_-$  would not be there, because it commutes with the operators of the oscillator. Likewise,  $a$  has no effect on the second term. Thus, they can be calculated using the proper commutation and anticommutation relations. The points of interest are the middle terms. Here we encounter third and fourth order correlators for the first time. Let us simplify the third term first. We immediately see that  $[a\sigma_-, (a^\dagger + a) \sigma_-] = 0$ , because  $a\sigma_-$  commutes with itself and  $\sigma_-^2 = 0$ . Thus, we obtain



$$\begin{aligned}
\left\langle \left[ a\sigma_-, (a^\dagger + a)(\sigma_+ + \sigma_-) \right] \right\rangle &= \left\langle \left[ a\sigma_-, (a^\dagger + a)\sigma_+ \right] \right\rangle \\
&= \left\langle \left[ a\sigma_-, a^\dagger\sigma_+ \right] \right\rangle + \langle [a\sigma_-, a\sigma_+] \rangle \\
&= \left\langle a\sigma_- a^\dagger\sigma_+ - a^\dagger\sigma_+ a\sigma_- \right\rangle + \langle a^2 [\sigma_-, \sigma_+] \rangle \\
&= \left\langle (a^\dagger a + 1)\sigma_- \sigma_+ - a^\dagger a \sigma_+ \sigma_- \right\rangle + \langle a^2 (1 - 2\sigma_+ \sigma_-) \rangle \\
&= \langle 1 - \sigma_+ \sigma_- \rangle + \left\langle a^\dagger a (1 - 2\sigma_+ \sigma_-) \right\rangle + \langle a^2 \rangle \\
&\quad - 2 \langle a^2 \sigma_+ \sigma_- \rangle \\
&= 1 - \langle \sigma_+ \sigma_- \rangle + \left\langle a^\dagger a \right\rangle + \langle a^2 \rangle \\
&\quad - 2 \left\langle a^\dagger a \sigma_+ \sigma_- \right\rangle - 2 \langle a^2 \sigma_+ \sigma_- \rangle. \tag{3.34}
\end{aligned}$$

We have two fourth order correlators in the result,  $\langle a^\dagger a \sigma_+ \sigma_- \rangle$  and  $\langle a^2 \sigma_+ \sigma_- \rangle$ , thus, we must invoke the semiquantum approximation. Using equation (2.56) on the first correlator gives

$$\begin{aligned}
\left\langle a^\dagger a \sigma_+ \sigma_- \right\rangle &= \left\langle a^\dagger a \right\rangle \langle \sigma_+ \sigma_- \rangle + \left\langle a^\dagger \sigma_+ \right\rangle \langle a\sigma_- \rangle + \left\langle a^\dagger \sigma_- \right\rangle \langle a\sigma_+ \rangle - 2 \left\langle a^\dagger \right\rangle \langle a \rangle \langle \sigma_+ \rangle \langle \sigma_- \rangle \\
&= \left\langle a^\dagger a \right\rangle \langle \sigma_+ \sigma_- \rangle + |\langle a\sigma_- \rangle|^2 + \left| \left\langle a^\dagger \sigma_- \right\rangle \right|^2 \tag{3.35}
\end{aligned}$$

and for the latter one we obtain

$$\begin{aligned}
\langle a^2 \sigma_+ \sigma_- \rangle &= \langle a^2 \rangle \langle \sigma_+ \sigma_- \rangle + \langle a\sigma_+ \rangle \langle a\sigma_- \rangle + \langle a\sigma_- \rangle \langle a\sigma_+ \rangle - 2 \langle a \rangle \langle a \rangle \langle \sigma_+ \rangle \langle \sigma_- \rangle \\
&= \langle a^2 \rangle \langle \sigma_+ \sigma_- \rangle + 2 \left\langle a^\dagger \sigma_- \right\rangle^* \langle a\sigma_- \rangle. \tag{3.36}
\end{aligned}$$

We used  $\langle a \rangle = \langle a^\dagger \rangle = 0$ ,  $\langle a\sigma_+ \rangle = \langle a^\dagger \sigma_- \rangle^*$  and  $\langle a^\dagger \sigma_+ \rangle = \langle a\sigma_- \rangle^*$  in the above. In the fourth term of equation (3.33)  $a\sigma_-$  commutes with  $\sigma_-$ , so that does not contribute. For what is left we get

$$\langle [a\sigma_-, \sigma_+] \rangle = \langle a(1 - 2\sigma_+ \sigma_-) \rangle = -2 \langle a\sigma_+ \sigma_- \rangle. \tag{3.37}$$

Here we have a third order correlator present. We use formula (2.54) to break it up,

$$\begin{aligned}
\langle a\sigma_+ \sigma_- \rangle &= \langle a\sigma_+ \rangle \langle \sigma_- \rangle + \langle a\sigma_- \rangle \langle \sigma_+ \rangle + \langle \sigma_+ \sigma_- \rangle \langle a \rangle - 2 \langle \sigma_+ \rangle \langle \sigma_- \rangle \langle a \rangle \\
&= \left\langle a^\dagger \sigma_- \right\rangle^* s_- + \langle a\sigma_- \rangle s_-^*. \tag{3.38}
\end{aligned}$$

Combining the above, one obtains for the Hamiltonian time-evolution

$$\begin{aligned}
-\frac{i}{\hbar} \langle [a\sigma_-, H] \rangle &= -i(\omega_c + \omega_0) \langle a\sigma_- \rangle - ig \left( 1 - \langle \sigma_+ \sigma_- \rangle + \langle a^\dagger a \rangle + \langle a^2 \rangle \right) \\
&\quad + 2ig \left( \langle a^\dagger a \rangle \langle \sigma_+ \sigma_- \rangle + |\langle a\sigma_- \rangle|^2 + \left| \langle a^\dagger \sigma_- \rangle \right|^2 \right) \\
&\quad + 2ig \left( \langle a^2 \rangle \langle \sigma_+ \sigma_- \rangle + 2 \langle a^\dagger \sigma_- \rangle^* \langle a\sigma_- \rangle \right) \\
&\quad + 4ig\Re(\alpha) \left( \langle a^\dagger \sigma_- \rangle^* s_- + \langle a\sigma_- \rangle s_-^* \right) + 2ig\Re(s_-) s_-. \tag{3.39}
\end{aligned}$$

Combining this with what we have for the dissipators, we obtain

$$\begin{aligned}
\frac{d\langle a\sigma_- \rangle}{dt} &= - \left\{ \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] - i(\omega_c + \omega_0) \right\} \langle a^\dagger \sigma_- \rangle \\
&\quad - ig \left[ 1 - \langle \sigma_+ \sigma_- \rangle + \langle a^\dagger a \rangle + \langle a^2 \rangle - 2 \langle \sigma_+ \sigma_- \rangle \left( \langle a^\dagger a \rangle + \langle a^2 \rangle \right) \right] \\
&\quad + 2ig\Re(s_-) s_- + 2ig \left( |\langle a\sigma_- \rangle|^2 + \left| \langle a^\dagger \sigma_- \rangle \right|^2 + 2 \langle a^\dagger \sigma_- \rangle^* \langle a\sigma_- \rangle \right) \\
&\quad + 4ig\Re(\alpha) \left( \langle a^\dagger \sigma_- \rangle^* s_- + \langle a\sigma_- \rangle s_-^* \right). \tag{3.40}
\end{aligned}$$

To obtain a closed set of equations we still have to derive equations for  $s_- = \langle \sigma_- \rangle$ ,  $\langle a^2 \rangle$  and  $\langle a^\dagger \sigma_- \rangle$ . We will not present the derivation here, like mentioned earlier. The derivation follows the same lines as for the correlators covered here. As said earlier, we do not need to employ semiquantum approximation for  $\langle a^2 \rangle$ , but for  $s_-$  and  $\langle a^\dagger \sigma_- \rangle$  we do. With  $s_-$  we run into a third order correlator and with  $\langle a^\dagger \sigma_- \rangle$  to third and fourth order correlators, similarly as with  $\langle a\sigma_- \rangle$ . The resulting group of differential equations is

$$\frac{d\alpha}{dt} = - \left( \frac{\kappa}{2} + i\omega_c \right) \alpha - 2ig\Re(s_-) + f(t), \tag{3.41}$$

$$\begin{aligned}
\frac{ds_-}{dt} &= - \left\{ \left[ \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] + i\omega_0 \right\} s_- \\
&\quad + 2ig \left\{ \Re(\alpha) (2 \langle \sigma_+ \sigma_- \rangle - 1) + 2\Re \left( \left[ \langle a\sigma_- \rangle + \langle a^\dagger \sigma_- \rangle \right] s_-^* \right) \right\}, \tag{3.42}
\end{aligned}$$

$$\begin{aligned}
\frac{d\langle \sigma_+ \sigma_- \rangle}{dt} &= -2g \left[ \Im \left( \langle a^\dagger \sigma_- \rangle + \langle a\sigma_- \rangle \right) + 2\Re(\alpha) \Im(s_-) \right] \\
&\quad - \gamma \left[ (2N_q + 1) \langle \sigma_+ \sigma_- \rangle - N_q \right], \tag{3.43}
\end{aligned}$$

$$\frac{d\langle a^\dagger a \rangle}{dt} = 2g\Im \left( \langle a^\dagger \sigma_- \rangle - \langle a\sigma_- \rangle \right) - \kappa \left( \langle a^\dagger a \rangle - N_c \right), \tag{3.44}$$

$$\frac{d\langle a^2 \rangle}{dt} = -(\kappa + 2i\omega_c) \langle a^2 \rangle - 2ig \left( \langle a^\dagger \sigma_- \rangle^* + \langle a\sigma_- \rangle \right), \tag{3.45}$$

$$\begin{aligned}
\frac{d\langle a\sigma_- \rangle}{dt} = & - \left\{ \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] - i(\omega_c + \omega_0) \right\} \langle a\sigma_- \rangle \\
& - ig \left[ 1 - \langle \sigma_+ \sigma_- \rangle + \langle a^\dagger a \rangle + \langle a^2 \rangle - 2 \langle \sigma_+ \sigma_- \rangle \left( \langle a^\dagger a \rangle + \langle a^2 \rangle \right) \right] \\
& + 2ig\Re(s_-)s_- + 2ig \left( |\langle a\sigma_- \rangle|^2 + |\langle a^\dagger \sigma_- \rangle|^2 + 2 \langle a^\dagger \sigma_- \rangle^* \langle a\sigma_- \rangle \right) \\
& + 4ig\Re(\alpha) \left( \langle a^\dagger \sigma_- \rangle^* s_- + \langle a\sigma_- \rangle s_-^* \right),
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
\frac{d\langle a^\dagger \sigma_- \rangle}{dt} = & - \left\{ \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] + i(\omega_0 - \omega_c) \right\} \langle a^\dagger \sigma_- \rangle \\
& - ig \left[ \left\langle \left( a^\dagger \right)^2 \right\rangle + \langle a^\dagger a \rangle - \langle \sigma_+ \sigma_- \rangle - 2 \langle \sigma_+ \sigma_- \rangle \left( \langle a^\dagger a \rangle + \left\langle \left( a^\dagger \right)^2 \right\rangle \right) \right] \\
& - 2ig\Re(s_-)s_- + 2ig \left( |\langle a\sigma_- \rangle|^2 + |\langle a^\dagger \sigma_- \rangle|^2 + 2 \langle a\sigma_- \rangle^* \langle a^\dagger \sigma_- \rangle \right) \\
& + 4ig\Re(\alpha) \left( s_-^* \langle a^\dagger \sigma_- \rangle + s_- \langle a\sigma_- \rangle^* \right).
\end{aligned} \tag{3.47}$$

This is a complex set of coupled differential equations. There is no hope to obtain an analytical solution, thus, we need to solve it numerically. If we would have used the alternate way of breaking up the third and higher order correlators, discussed in [section 2.3](#), the group of equations would be slightly simpler. The last two terms in the equations for  $\langle a\sigma_- \rangle$  and  $\langle a^\dagger \sigma_- \rangle$  and the term proportional to the real part of  $[\langle a\sigma_- \rangle + \langle a^\dagger \sigma_- \rangle] s_-^*$ , in the equation for  $s_-$ , would drop off if the alternative way would be used. Still, the set of equations would be too complicated to solve analytically. These are the equations which we use in [chapter 4](#) to obtain the steady state properties for the system.

### 3.3 Jaynes-Cummings model with semiclassical approximation

We want to derive equations of motion for the system in the semiclassical model employing the rotating wave approximation. This allows us to obtain analytical solutions for the governing equations. We start again with the Hamiltonian [\(3.3\)](#)

$$H = H_R + H_d. \tag{3.48}$$

In RWA the Rabi Hamiltonian becomes the Jaynes-Cummings Hamiltonian [\[45\]](#)

$$H_{JC} = \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g \left( a^\dagger \sigma_- + a \sigma_+ \right). \tag{3.49}$$

The driving force  $f(t)$  can be expressed as a Fourier series like  $f(t) = \sum_n A_n e^{in\omega_d t}$ . Assuming that  $|\omega_d - \omega_c| \ll \omega_d + \omega_c$  we can write the Hamiltonian for the drive, in the

interaction picture, as  $H_d^I = A_1 (ae^{i(\omega_d - \omega_c)t} + a^\dagger e^{-i(\omega_d - \omega_c)t}) +$  rapidly oscillating part. We assume, that, in the time scale we are interested in, the rapidly oscillating part averages to zero, so that we can neglect it. With this, the drive Hamiltonian in Schrödinger picture reads

$$H_d = \frac{A}{2} (ae^{i\omega_d t} + a^\dagger e^{-i\omega_d t}), \quad (3.50)$$

where we denoted  $A_1 \equiv A/2$ , and  $A$  is the amplitude of the driving force. The damping terms in the master equation are not affected by the RWA. Because we are using the semiclassical approximation (2.60), the equations of motion for  $\langle a \rangle \equiv \alpha$  and  $\langle \sigma_- \rangle \equiv s_-$  constitute a closed set of equations. In this situation we are not displacing the oscillator to the vacuum by transformation (2.35) (because it has no effect on the equations). The equations of motion, before the semiclassical approximation, become

$$\frac{d\alpha}{dt} = -\left(\frac{\kappa}{2} + i\omega_c\right)\alpha - igs_- + \frac{A}{2}e^{-i\omega_d t}, \quad (3.51)$$

$$\frac{ds_-}{dt} = -\left\{\left[\frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right] + i\omega_0\right\}s_- - ig(\alpha - 2\langle a\sigma_+\sigma_- \rangle). \quad (3.52)$$

Now, we use (2.60) and approximate  $\langle a\sigma_+\sigma_- \rangle \approx \langle a \rangle \langle \sigma_+ \rangle \langle \sigma_- \rangle = \alpha |s_-|^2$ . We obtain

$$\frac{ds_-}{dt} = -\left\{\left[\frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right] + i\omega_0\right\}s_- - ig\alpha(1 - 2\alpha |s_-|^2). \quad (3.53)$$

This equation looks already quite simple, but  $|s_-|^2$  makes it hard to solve analytically. Thus, we make another approximation. We approximate  $|s_-|^2 \approx 0$  and finally arrive at

$$\frac{d\alpha}{dt} = -\left(\frac{\kappa}{2} + i\omega_c\right)\alpha - igs_- + \frac{A}{2}e^{-i\omega_d t}, \quad (3.54)$$

$$\frac{ds_-}{dt} = -\left\{\left[\frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right] + i\omega_0\right\}s_- - ig\alpha. \quad (3.55)$$

Notice, that in the linear model the equations of motion are symmetric with respect to  $\alpha$  and  $s_-$  (when neglecting the driving force).

We want to solve the steady state of the above pair of differential equations. To this end, we make an ansatz

$$\alpha(t) = \tilde{\alpha}e^{-i\omega_d t}, \quad (3.56)$$

$$s_-(t) = \tilde{s}_-e^{-i\omega_d t}, \quad (3.57)$$

where  $\tilde{\alpha}$  and  $\tilde{s}_-$  are time-independent. This is equivalent to a coordinate transform to a coordinate system rotating at the angular frequency of the external force [13]. Substituting these into equations (3.54), (3.55) gives

$$\left[i(\omega_d - \omega_c) - \frac{\kappa}{2}\right]\tilde{\alpha} - ig\tilde{s}_- + \frac{A}{2} = 0, \quad (3.58)$$

$$\left\{ i(\omega_d - \omega_0) - \left[ \frac{\gamma}{2}(2N_q + 1) + \gamma_\phi \right] \right\} \tilde{s}_- - ig\tilde{\alpha} = 0. \quad (3.59)$$

We obtained time-independent equations for  $\tilde{\alpha}$  and  $\tilde{s}_-$  as we wanted. The solution to equations (3.58), (3.59) gives the steady state of the system in the semiclassical model. The above pair of equations is a standard group of linear equations and can be readily solved. The solution is

$$\tilde{\alpha} = \frac{A[\gamma(2N_q + 1) + 2\gamma_\phi - 2i(\omega_d - \omega_0)]}{4g^2 - [2(\omega_d - \omega_c) + i\kappa][2(\omega_d - \omega_0) + i(\gamma(2N_q + 1) + 2\gamma_\phi)]}, \quad (3.60)$$

$$\tilde{s}_- = -\frac{2iAg}{4g^2 - [2(\omega_d - \omega_c) + i\kappa][2(\omega_d - \omega_0) + i(\gamma(2N_q + 1) + 2\gamma_\phi)]}. \quad (3.61)$$

For  $\alpha(t)$  and  $s_-(t)$  we have rotation in the complex plane with amplitudes  $\tilde{\alpha}$  and  $\tilde{s}_-$ , respectively.

From  $\alpha(t)$  and  $s_-(t)$  we can obtain all the quantities we want for the system. We are especially interested in the steady state occupation of the oscillator,  $\langle a^\dagger a \rangle$ , and the population in the excited state of the qubit,  $\langle \sigma_+ \sigma_- \rangle$ . In the linear model these are just  $|\alpha|^2$  and  $|s_-|^2$ , respectively. We obtain

$$\langle a^\dagger a \rangle = |\alpha|^2 = \frac{A^2 [4\delta^2 + \Gamma^2]}{16g^4 + 8g^2(\kappa\Gamma - 4\Delta\delta) + (4\Delta^2 + \kappa^2)(4\delta^2 + \Gamma^2)}, \quad (3.62)$$

$$\langle \sigma_+ \sigma_- \rangle = |s_-|^2 = \frac{4A^2 g^2}{16g^4 + 8g^2(\kappa\Gamma - 4\Delta\delta) + (4\Delta^2 + \kappa^2)(4\delta^2 + \Gamma^2)}, \quad (3.63)$$

where we have defined  $\Gamma \equiv \gamma(2N_q + 1) + 2\gamma_\phi$ ,  $\Delta \equiv \omega_d - \omega_c$  and  $\delta \equiv \omega_d - \omega_0$ . We see that both  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$  vanish if the amplitude of the driving force,  $A$ , is zero, i.e. the semiclassical approximation does not take into account the ability of the environment to excite the system to finite  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$ . Expectedly,  $\langle \sigma_+ \sigma_- \rangle$  vanishes when there is no coupling between the two subsystems. These analytical results are compared to the solutions obtained from the semiquantum calculation in [chapter 4](#).

### 3.4 Time-correlators

One of the goals starting the project, was to obtain the absorption spectrum of the coupled oscillator-qubit system. In [chapter 2](#) we showed that it could be obtained as the Fourier transformation of autocorrelation function  $\langle a^\dagger(t)a(0) \rangle$ ,

$$S(\omega) = \int_{-\infty}^{\infty} \langle a^\dagger(t)a(0) \rangle e^{-i\omega t} dt. \quad (3.64)$$

The next question is then: how can we determine  $\langle a^\dagger(t)a(0) \rangle$ ? The answer is the *quantum regression theorem*. Let us write the two-time correlation function open, using Heisenberg's picture,

$$\begin{aligned} \langle a^\dagger(t)a(0) \rangle &= \text{Tr} \left( a^\dagger(t)a(0)\rho(0) \right) = \text{Tr} \left( U^\dagger(t)a^\dagger(0)U(t)a(0)\rho(0) \right) \\ &= \text{Tr} \left( a^\dagger(0)U(t)a(0)\rho(0)U^\dagger(t) \right) = \text{Tr} \left( a^\dagger(0)\bar{\rho}(t) \right), \end{aligned} \quad (3.65)$$

where we denoted  $\bar{\rho}(t) = U(t)a(0)\rho(0)U^\dagger(t)$  and operator  $U(t)$  is the unitary transformation from the Heisenberg picture to the Schrödinger picture. Thus, we can use initial value  $a(0)\rho(0)$  for the density operator when we solve the master equation (we did not make any assumptions about  $\rho$  in the derivation of the master equation), and then obtain the autocorrelation function just by tracing over  $a^\dagger(0)\bar{\rho}(t)$ . Clearly,  $\bar{\rho}(t)$  is not any more an acceptable density operator. It is clearly not hermitian and its trace is not necessarily equal to unity. The quantum regression theorem holds, of course, for arbitrary operators.

In terms of using the semiquantum approximation, the quantum regression theorem means that we use the solution we obtain from differential equation group (3.41) - (3.47) as the *initial condition* for the time-correlator calculation. We can derive similar equations of motion for the time-correlators as we did for the equal-time correlation functions. In this situation it is important to notice that the closed group of equations is bigger than when dealing with simple correlation functions. This is because relations such as  $\langle a^\dagger(t)a(0) \rangle = \langle a(t)a(0) \rangle^*$  or  $\langle a^\dagger\sigma_-(t)a(0) \rangle = \langle a\sigma_+(t)a(0) \rangle^*$  do not hold any more, but we have to derive equations of motion also for them.

Let us derive the equation of motion for  $\langle a(t)a(0) \rangle$  as an example. According to the quantum regression theorem, we obtain for the time derivative of the autocorrelation function

$$\frac{d\langle a(t)a(0) \rangle}{dt} = \frac{d}{dt} \text{Tr} (a(0)\bar{\rho}(t)) = \text{Tr} \left( a(0) \frac{d\bar{\rho}(t)}{dt} \right). \quad (3.66)$$

Using master equation (3.19) for the derivative of  $\bar{\rho}(t)$ , we obtain, similarly as with  $\langle a \rangle$ ,

$$\begin{aligned} \frac{d\langle a(t)a(0) \rangle}{dt} &= -\frac{i}{\hbar} \text{Tr} (a(0) [H_{\text{eff}}, \bar{\rho}(t)]) + \frac{\kappa}{2} (N_c + 1) \text{Tr} (a(0)\mathcal{D} [a(0)] \bar{\rho}(t)) \\ &\quad + \frac{\kappa}{2} N_c \text{Tr} \left( a(0)\mathcal{D} [a^\dagger(0)] \bar{\rho}(t) \right). \end{aligned} \quad (3.67)$$

We wrote explicitly at what time we have to evaluate the operators  $a$  and  $a^\dagger$  in the Lindblad superoperators to avoid any ambiguity. Let us concentrate on the first term of the above equation. Simplifying gives

$$\begin{aligned}
 \text{Tr}(a(0)[H_{\text{eff}}, \bar{\rho}(t)]) &= \text{Tr}(a(0)H_{\text{eff}}\bar{\rho}(t)) - \text{Tr}(a(0)\bar{\rho}(t)H_{\text{eff}}) \\
 &= \text{Tr}([a(0), H_{\text{eff}}]\bar{\rho}(t)) \\
 &= \langle [a(0), H_{\text{eff}}] \rangle_{\bar{\rho}(t)}. \tag{3.68}
 \end{aligned}$$

We have put the subscript  $\bar{\rho}(t)$  to the 'expectation value' to remind ourselves that it is taken with respect to  $\bar{\rho}(t)$ , not  $\rho(t)$ . Naturally  $\langle [a(0), H_{\text{eff}}] \rangle_{\bar{\rho}(t)}$  is not really an expectation value, because  $\bar{\rho}(t)$  is not a valid density operator. Nonetheless, we can handle it just like it would be. For the commutator, we obtain

$$[a(0), H_{\text{eff}}] = \hbar\omega_c a(0) + \hbar g(\sigma_+(0) + \sigma_-(0)) - 2\hbar g \Re(s_-) \tag{3.69}$$

and, thus, have for the trace

$$\begin{aligned}
 -\frac{i}{\hbar} \text{Tr}(a(0)[H_{\text{eff}}, \bar{\rho}(t)]) &= -i\omega_c \langle a(0) \rangle_{\bar{\rho}(t)} - ig \left( \langle \sigma_+(0) \rangle_{\bar{\rho}(t)} + \langle \sigma_-(0) \rangle_{\bar{\rho}(t)} \right) \\
 &\quad - 2ig \langle \Re(s_-) \rangle_{\bar{\rho}(t)}. \tag{3.70}
 \end{aligned}$$

Next we write the  $\bar{\rho}(t)$  'expectation values' open, and express everything with respect to the normal density operator. For an arbitrary operator,  $Q$ , we obtain

$$\langle Q \rangle_{\bar{\rho}(t)} = \text{Tr}(QU^\dagger(t)a(0)\rho(0)U(t)) = \text{Tr}(U(t)QU^\dagger(t)a(0)\rho(0)) = \langle Q(t)a(0) \rangle. \tag{3.71}$$

Thus,

$$\begin{aligned}
 -\frac{i}{\hbar} \text{Tr}(a(0)[H_{\text{eff}}, \bar{\rho}(t)]) &= -i\omega_c \langle a(t)a(0) \rangle - ig \left( \langle \sigma_+(t)a(0) \rangle + \langle \sigma_-(t)a(0) \rangle \right) \\
 &\quad - 2ig \Re(s_-) \langle a(0) \rangle \\
 &= -i\omega_c \langle a(t)a(0) \rangle - ig \left( \langle \sigma_+(t)a(0) \rangle + \langle \sigma_-(t)a(0) \rangle \right), \tag{3.72}
 \end{aligned}$$

since  $\langle a(0) \rangle = 0$  because of the transformation (2.35). Next, we handle the damping terms. Because they are of the same form as with  $\langle a \rangle$  before, just with  $\bar{\rho}(t)$  instead of  $\rho$ , we can use results obtained for it by replacing  $\langle \cdot \rangle \rightarrow \langle \cdot \rangle_{\bar{\rho}(t)}$ . Thus,

$$\text{Tr}(a(0)\mathcal{D}[a(0)]\bar{\rho}(t)) = -\langle a(0) \rangle_{\bar{\rho}(t)} = -\langle a(t)a(0) \rangle, \tag{3.73}$$

$$\text{Tr}(a(0)\mathcal{D}[a^\dagger(0)]\bar{\rho}(t)) = \langle a(0) \rangle_{\bar{\rho}(t)} = \langle a(t)a(0) \rangle, \tag{3.74}$$

where we used the result, derived above, for the arbitrary operator  $Q$ . Combining the above gives the equation of motion

$$\frac{d\langle a(t)a(0) \rangle}{dt} = \left( \frac{\kappa}{2} - i\omega_c \right) \langle a(t)a(0) \rangle - ig \left( \langle \sigma_+(t)a(0) \rangle + \langle \sigma_-(t)a(0) \rangle \right). \tag{3.75}$$

When solving this equation, we use the steady state solution of  $\langle a^2 \rangle$  as the initial condition.

The complete set of equations for the time-correlators is a differential equation group of 12 equations. Above, the derivation of  $d\langle a(t)a(0) \rangle / dt$ , didn't involve making the semi-quantum approximation. However, we have to use it for other autocorrelation functions. The order of the time-correlation function is determined by the *order of the operator evaluated at time  $t$* . Thus, e.g. the above calculated term  $\langle a(t)a(0) \rangle$  is of order one and  $\langle a\sigma_-(t)a(0) \rangle$  is of order two. In the semi-quantum model, initial values for the second order autocorrelation functions are given by equation (2.54), i.e. they are approximate.

We have to continue solving the governing equations for the simple correlators, equations (3.41) - (3.47), in parallel with the time-correlators. This is, because the equations for the autocorrelation functions depend on the values of  $\alpha$  and  $s_-$  (See Appendix B for the complete set of equations.). We cannot just use the steady state value obtained from the equation group (3.41) - (3.47), because the external drive and the counter rotating terms cause oscillations, so that the steady state values are not completely constant.



## Chapter 4

# Numerical simulations

In this section we discuss the numerical methods used in the calculations and the obtained results. Results obtained are compared to full master equation solutions provided by M.Sc. Iivari Pietikäinen. In the pictures plotted, solutions from the full master equation are labelled as 'ME' and results obtained with the semiquantum model as 'SQ'. The aim is to determine a range for the parameters, especially for the coupling strength  $g$ , in which we can use the semiquantum approximation, and at the same time the results from it differ significantly from the semiclassical model.

All numerical values for the quantities are given in the units of  $\omega_c$  in this chapter.

### 4.1 Methods

All calculations in this thesis were performed with MATLAB and Mathematica. The equations of motion for the simple correlators and time-correlators were solved using the Dormand-Prince method, which is a Runge-Kutta type solver for ordinary differential equations [46]. MATLAB provides a built-in function for it, `ode45`. The absorption spectrum was calculated with the fast Fourier transform algorithm provided by MATLAB. Numerical diagonalization of the Rabi Hamiltonian was also done with MATLAB, using the built in tools for matrix diagonalization. We used Mathematica to obtain the analytical solution for the semiclassical steady state of the system, equations (3.58) and (3.59), and expressions for the steady state number of quanta in the oscillator and the population in the excited state of the two-level-system.

## 4.2 Results

This section is organized as follows. We begin by comparing the semiquantum approximation, for both the JC and Rabi models, to the more traditional semiclassical approximation. In particular, we look at how the steady states compare against each other. Then we proceed by looking how the full master equation compares to the semiquantum model. First we look at the steady state of the system and then move on to the spectral properties. The time-correlator calculation is reviewed briefly before looking at the spectrum.

While running the simulations we observed that the correlator calculation is not stable with all values of the coupling parameter  $g$ . The calculation fails when  $g \gtrsim 0.5$  while the dissipators are around 0.01. If one increases the damping, one can use slightly larger values of  $g$ . Because of this, we used only values  $g < 0.3$  in the calculations. The instability seems to be produced by the counter rotating terms,  $\langle a^2 \rangle$  and  $\langle a\sigma_- \rangle$ , in the equations of motion, because in the Jaynes-Cummings model this instability is not present.

### 4.2.1 Steady states of the Rabi and Jaynes-Cummings models

We begin by examining the difference between the steady states for  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$  obtained from the Rabi model, equation group (3.41) - (3.47), the JC model and the analytical solutions from section 2.4, obtained from JC model using the semiclassical approximation. Equations of motion for the Jaynes-Cummings model with semiquantum approximation can be found in Appendix C. In figures 4.1 and 4.2 we have plotted the steady state values of  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$ , respectively, as functions of the coupling strength  $g$ . There is a significant difference in the results between JC and Rabi in both of the correlators. The number of quanta in the cavity from the semiclassical calculation is identical to the JC semiquantum one. Whereas with  $\langle \sigma_+ \sigma_- \rangle$ , there is a notable difference also between the semiclassical and semiquantum JC results.

As we will discuss in subsection 4.2.2, the use of the standard master equation creates spurious excitations in the oscillator and the two-level-system. We would like to get rid of these fake excitations. To this end, we assume that the number of these excitations is independent on the driving amplitude, if it is sufficiently small. Thus, we can calculate a correction for  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$  obtained from the Rabi model, by taking the corrections to be the differences between the numerical diagonalization of the Rabi Hamiltonian and the steady state values obtained with the semiquantum model in figures 4.13 and 4.14. We have plotted the JC values, from SQ and SC models, and the corrected Rabi values

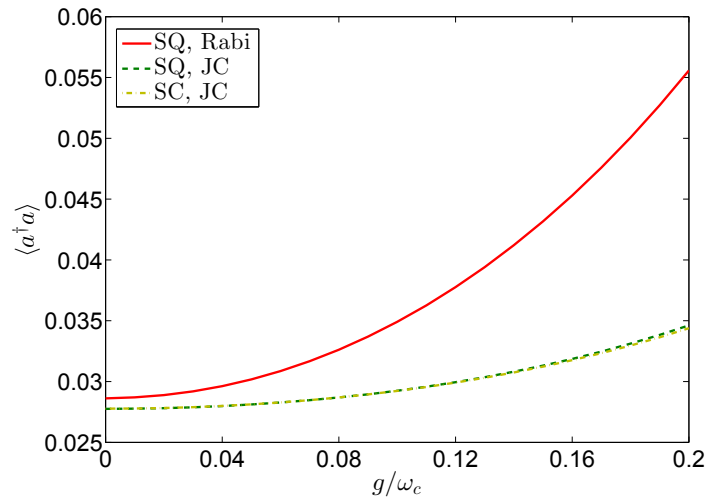


Figure 4.1: Steady state value of  $\langle a^\dagger a \rangle$  plotted as a function of  $g$ , calculated from the Rabi and Jaynes-Cummings models. Used parameters:  $\omega_0 = 2$ ,  $\omega_d = 0.7$ ,  $A = 0.1$ ,  $\kappa = \gamma = \gamma_\phi = 0.01$  and  $T = 0$ .

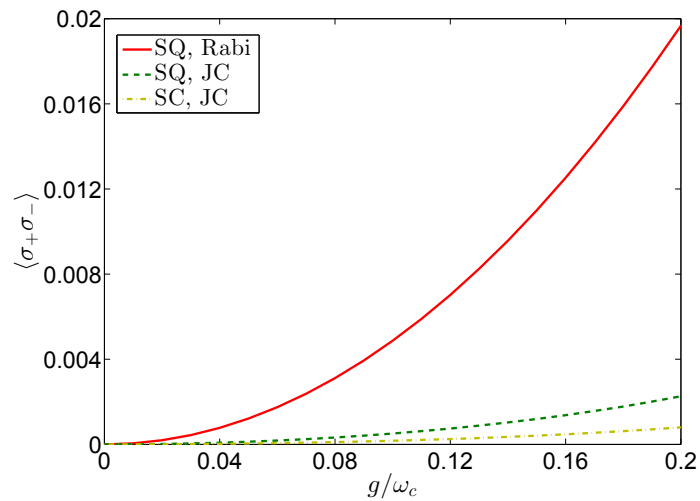


Figure 4.2: Steady state value of  $\langle \sigma_+ \sigma_- \rangle$  plotted as a function of  $g$ , calculated from the Rabi and Jaynes-Cummings models. Used parameters are the same as in [Figure 4.1](#)

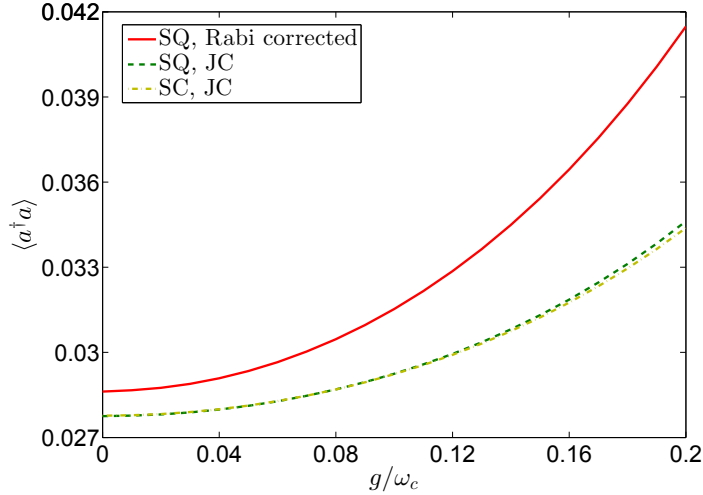


Figure 4.3: Same as Figure 4.1, with the corrected  $\langle a^\dagger a \rangle$  from the Rabi model.

in figures 4.3 and 4.4. Although the correction brings the Rabi results closer to the JC ones, there is still notable difference between the steady state values, both for  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$ . The difference in  $\langle a^\dagger a \rangle$  from the Rabi and JC models at  $g = 0$ , can be explained with the counter rotating terms included in the driving Hamiltonian in the Rabi model. If we use RWA for the drive, the difference at zero coupling vanishes, as it should.

### 4.2.2 Semiquantum approximation compared to the master equation

Let us now see how the steady states from the Rabi semiquantum calculation compares against the full master equation solution. In figures 4.5 and 4.6 we have plotted the steady state values of  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$ , respectively. In the presence of driving there are oscillations in the steady state solutions for the correlators. To plot the figures, we average over multiple periods of the oscillation, and take that average as the steady state value.

One sees that the values obtained from the semiquantum and master equation calculations agree well when  $g \lesssim 0.16$ . Similar results are obtained also at finite temperatures. Although  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$  from semiquantum and master equation calculations coincide rather well, there are reasonably big differences in some qubit-cavity correlators, as can be seen from figures 4.7 and 4.8. In those, we have plotted the real parts of  $\langle a^\dagger \sigma_- \rangle$  and  $\langle a \sigma_- \rangle$ , respectively. The results from the master equation and the semiquantum

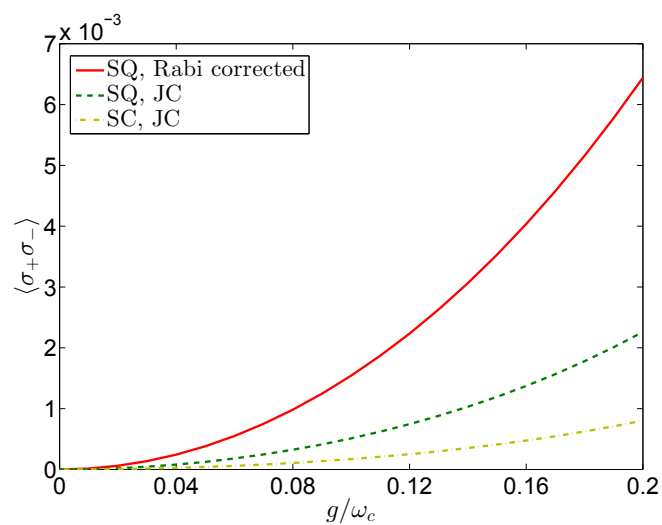


Figure 4.4: Same as Figure 4.2, with the corrected  $\langle \sigma_+ \sigma_- \rangle$  from the Rabi model.

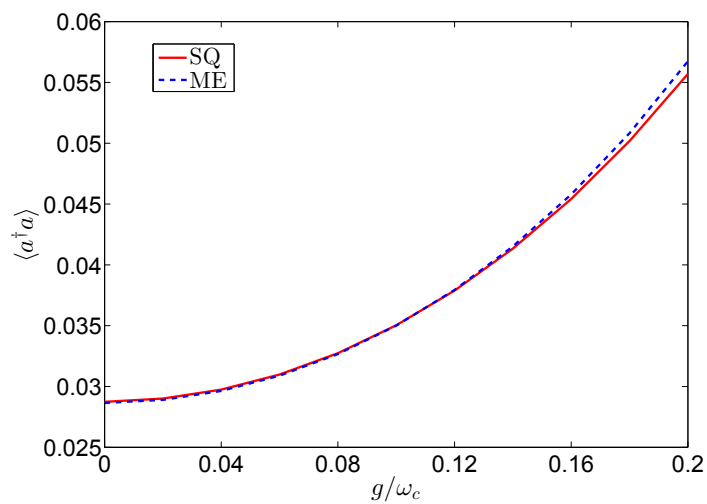


Figure 4.5: SS value of  $\langle a^\dagger a \rangle$  as a function of  $g$ .  $\kappa = \gamma = \gamma_\phi = 0.01$ ,  $A = 0.1$ ,  $\omega_0 = 2$  and temperature is at absolute zero.

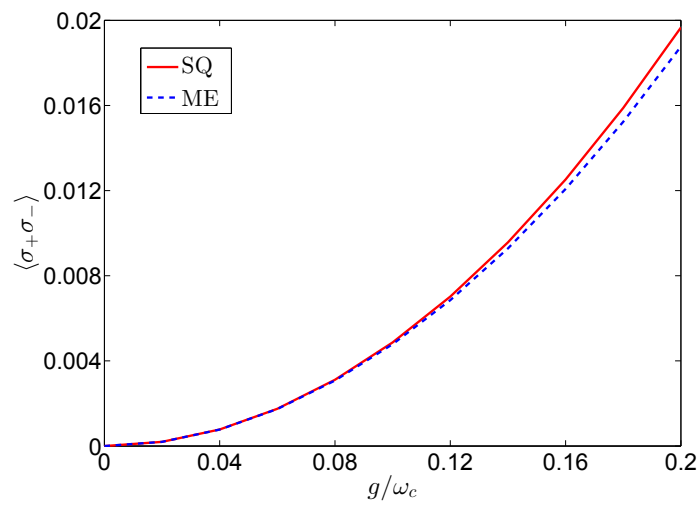


Figure 4.6: SS value of  $\langle \sigma_+ \sigma_- \rangle$  as a function of  $g$ . Parameters are the same as in Figure 4.5.

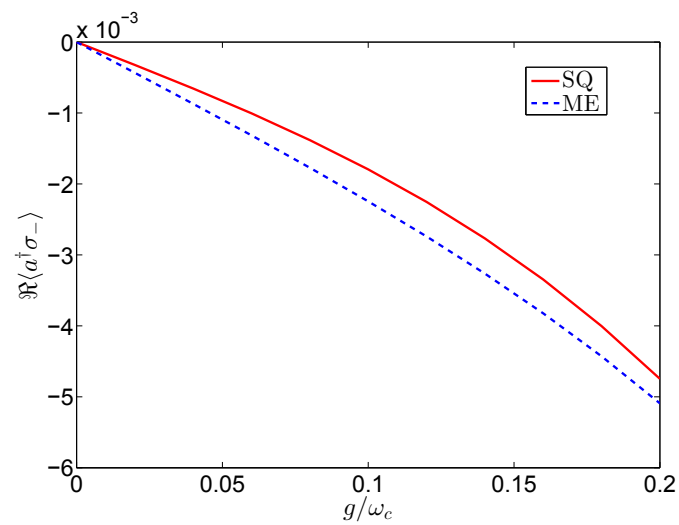


Figure 4.7: SS value of  $\Re(\langle a^\dagger \sigma_- \rangle)$  as a function of  $g$ . Parameters are the same as in Figure 4.5.

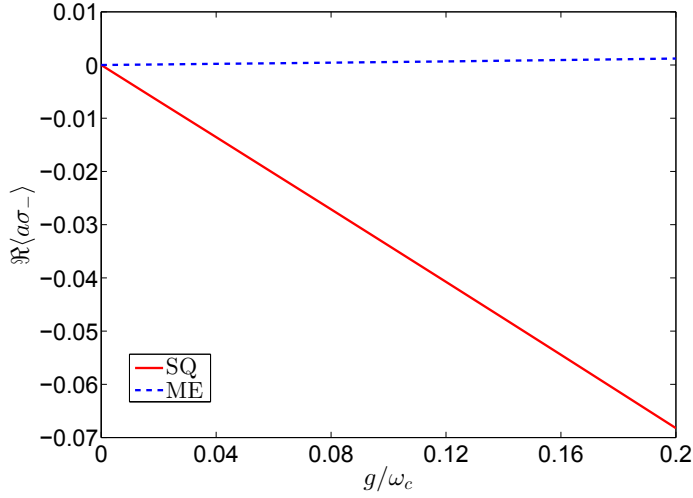


Figure 4.8: SS value of  $\Re\langle a\sigma_- \rangle$  as a function of  $g$ . Parameters are the same as in Figure 4.5.

calculations show roughly the same behaviour for  $\Re\langle a^\dagger\sigma_- \rangle$ , but for  $\Re\langle a\sigma_- \rangle$  it is completely different. The same holds for the imaginary parts of the correlation functions. There were big differences also in the other 'counter rotating' term,  $\langle a^2 \rangle$ , so it seems that the semiquantum calculation is not good in predicting values for those. Weak dependence of  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$  on the qubit-oscillator correlators has been found also in [37]. Based on figures 4.5 and 4.6, it seems that the semiquantum approximation over estimates the coupling between the two-level-system and the oscillator, at least when the coupling strength  $g$  gets large enough.

### Inadequacy of the standard quantum optical master equation

When studying the steady state properties of the system, we noticed that the steady state of the system, when the system is undriven and temperature is at absolute zero, depends on the values of the dissipation coefficients  $\kappa$ ,  $\gamma$  and  $\gamma_\phi$ . In this situation, however, the system should relax to its ground state and the coefficients dictate how fast this happens. There should be no dependence in the steady state values on the dissipation coefficients. Thus, this is not a real physical phenomenon, but a deficiency in the model used. The source of this error is the fact that the ground state of the Rabi Hamiltonian is *not* a simple tensor product of the qubit and oscillator states, but

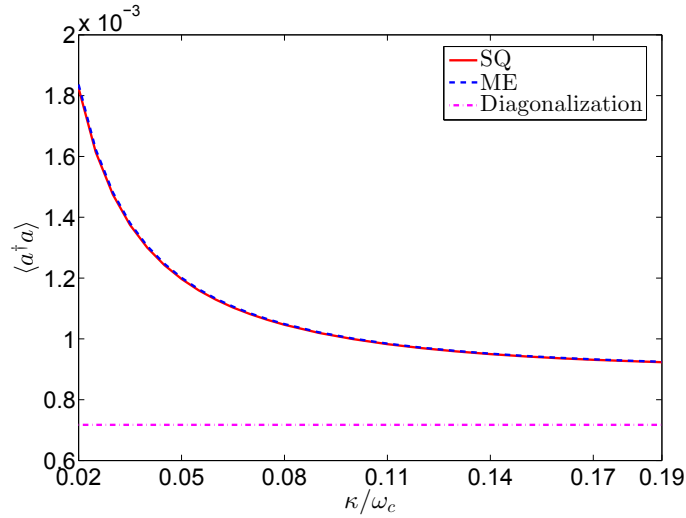


Figure 4.9: SS value of  $\langle a^\dagger a \rangle$  as a function of  $\kappa$ . Parameters are:  $\omega_0 = 2$ ,  $\omega_d = 0.7$ ,  $A = 0$ ,  $\gamma = \gamma_\phi = 0.01$ ,  $g = 0.08$  and  $T = 0$ .

a superposition of them including qubit and oscillator excited states. At absolute zero the master equation used drives the system to the state  $|g, 0\rangle$ , where the qubit is in its ground state and there is no quanta in the cavity. This is because in the derivation of master equation (3.4), we assumed we can neglect the coupling between the two-level-system and the cavity, leading to dissipative terms which are a sum of the dissipative terms for the two individual subsystems. Master equations for the separate systems drive the systems, naturally, to their ground states, the Fock-state  $|0\rangle$  for the oscillator and  $|g\rangle$  for the qubit. Thus, the combined system is driven to the state  $|g, 0\rangle$ . Due to this discrepancy, different values of the dissipators relax the system at different rates, and result in different equilibrium values for the correlators. This effect is illustrated in Figures 4.9 and 4.10, where we plot the correlators  $\langle a^\dagger a \rangle$  and  $\langle \sigma_+ \sigma_- \rangle$  as a function of  $\kappa$ . The horizontal magenta line is the true value for the correlator obtained by diagonalizing the Rabi Hamiltonian numerically. Dependence of  $\langle a^\dagger a \rangle$  on  $\kappa$  is as expected, it decreases with growing dissipation. Peculiarly, occupation probability of the excited state of the qubit grows with increasing  $\kappa$ . One would assume that also  $\langle \sigma_+ \sigma_- \rangle$  would decrease since larger ratio  $\kappa/g$  means weaker effective coupling. Analogous results hold for the correlators as functions of  $\gamma$ ,  $\langle \sigma_+ \sigma_- \rangle$  decreases as  $\gamma$  grows and  $\langle a^\dagger a \rangle$  gets larger. The dependence on  $\gamma_\phi$  is different, as both of the correlators increase as  $\gamma_\phi$  increases, as seen in



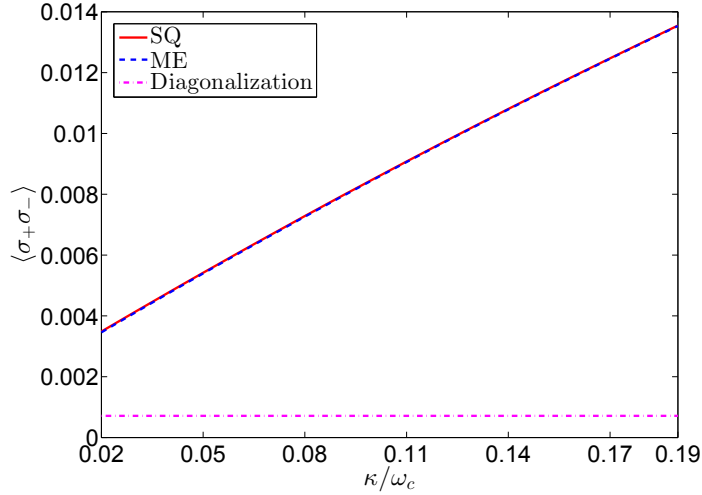


Figure 4.10: SS value of  $\langle \sigma_+ \sigma_- \rangle$  as a function of  $\kappa$ . Parameters are the same as in Figure 4.9.

figures 4.11 and 4.12. The results between the master equation and semiclassical model agree extremely well for  $\kappa$  and  $\gamma$ , but start to deviate early for  $\gamma_\phi$ , around  $\gamma_\phi \gtrsim 0.03$ . Also figures 4.11 and 4.12 indicate that in the semiclassical model the coupling is stronger than in the full master equation, since the semiclassical approximation predicts higher occupation of the qubit excited state than the master equation.

In figures 4.13 and 4.14, we plot the number of quanta in the cavity and the probability of the excited state of the qubit, respectively, obtained from the semiclassical model and from the numerical diagonalization of the Rabi Hamiltonian. For the comparison, we set the amplitude of the driving force and temperature to zero. For both correlators there is a significant difference between the real value and the one obtained from the semiclassical approximation. The difference grows as the coupling is increased, and is notable even when the coupling is relatively weak.

Resolution to the above explained problem is given in reference [47], by Beaudoin et al., where the authors derive a new master equation. The master equation used in our calculations treats the cavity and the qubit as separate entities, i.e. the coupling between the two systems is not taken into account when deriving the master equation. This approach works when dealing with small values of the coupling and using RWA, but breaks down as the coupling strength increases and RWA is no longer valid. The approach used in reference [47] takes the coupling into account by diagonalizing the Rabi

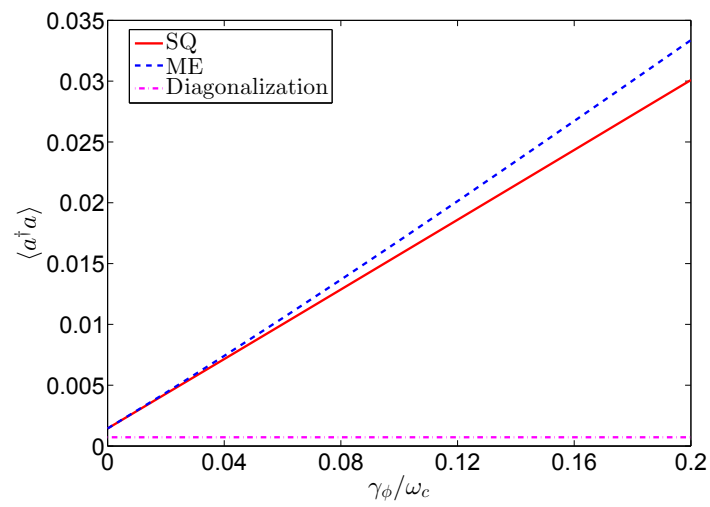


Figure 4.11: SS value of  $\langle a^\dagger a \rangle$  as a function of  $\gamma_\phi$ . Parameters are:  $\omega_0 = 2$ ,  $\omega_d = 0.7$ ,  $A = 0$ ,  $\gamma = \kappa = 0.01$ ,  $g = 0.08$  and  $T = 0$ .

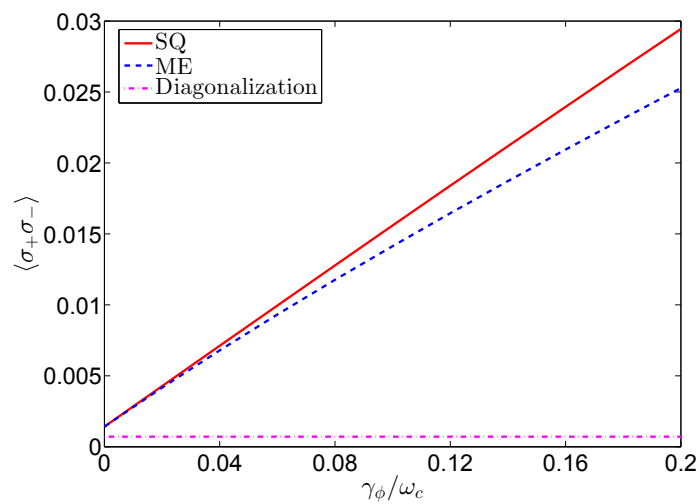


Figure 4.12: SS value of  $\langle \sigma_+ \sigma_- \rangle$  as a function of  $\gamma_\phi$ . Parameters are the same as in [Figure 4.11](#).

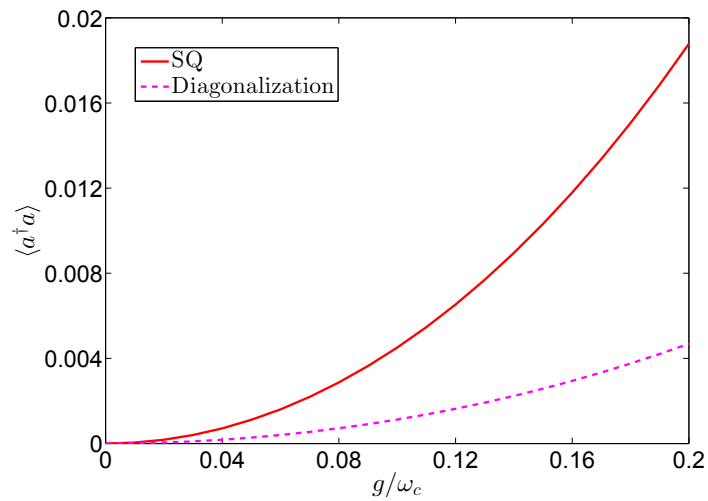


Figure 4.13: SS value for  $\langle a^\dagger a \rangle$ , obtained from the semiquantum calculation, compared against the real value, from the numerical diagonalization of the Rabi Hamiltonian. Parameters:  $A = 0$ ,  $\omega_0 = 2$ ,  $\gamma = \gamma_\phi = \kappa = 0.01$  and  $T = 0$

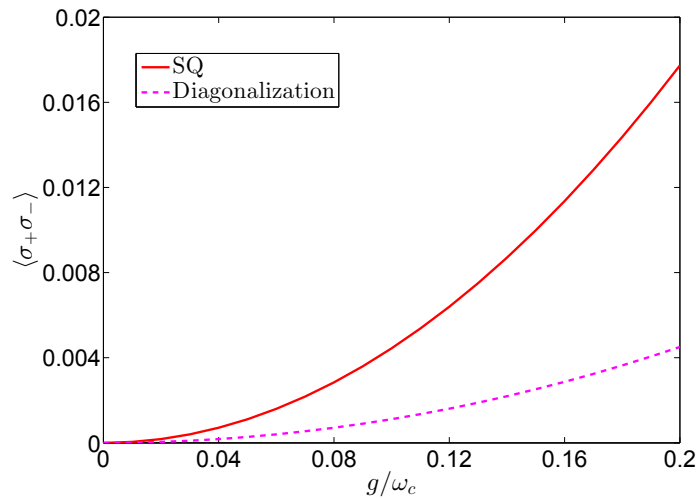


Figure 4.14: SS value for  $\langle \sigma_+ \sigma_- \rangle$ , obtained from the semiquantum calculation, compared against the real value, from the numerical diagonalization of the Rabi Hamiltonian. Parameters are the same as in [Figure 4.13](#).

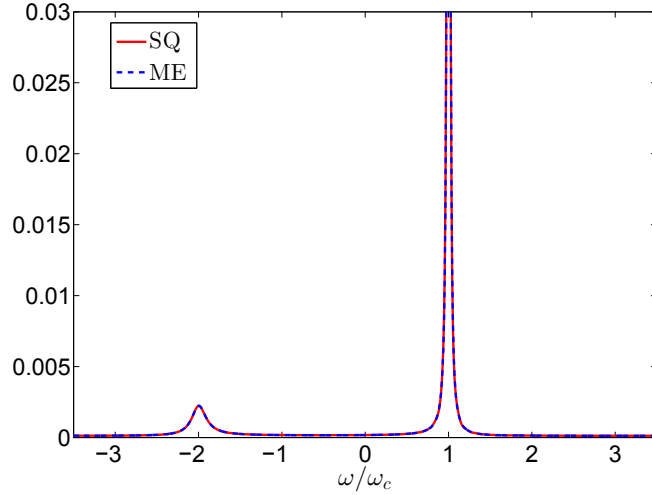


Figure 4.15: Absorption spectrum:  $\omega_0 = 2$ ,  $g = 0.01$ ,  $A = 0.01$ ,  $\omega_d = 0.7$ ,  $\kappa = \gamma = 0.01$ ,  $\gamma_\phi = 0.1$  and  $T = 0$ .

Hamiltonian to second order in the coupling parameter, with a unitary transformation, and using the obtained eigenstates as a basis in the derivation of the master equation. This way the coupling is included in the derivation. The new master equation drives the system to the ground state of the approximately diagonalized Hamiltonian, not the vacuum state  $|g, 0\rangle$ , thus avoiding the dissipator dependent steady state [47].

### 4.2.3 Time-correlators

In the time-correlator calculations, solutions from the full master equation show faster attenuation than the ones obtained with the semiquantum approximation when the coupling gets stronger,  $g \gtrsim 0.05$ . Additionally, also the effect of driving can be seen more prominently in the solutions of the master equation. The amplitude of the driven oscillations in the steady state of the time-correlator, from the master equation, is roughly 5 times larger than that obtained from the semiquantum model.

### 4.2.4 Spectrum

In figures 4.15 - 4.17 we plot the absorption spectrum with different coupling strengths and drive amplitudes. Figure 4.15 shows the spectrum with small coupling,  $g = 0.01$ . The spectra from the semiquantum and master equation calculations overlap very well.

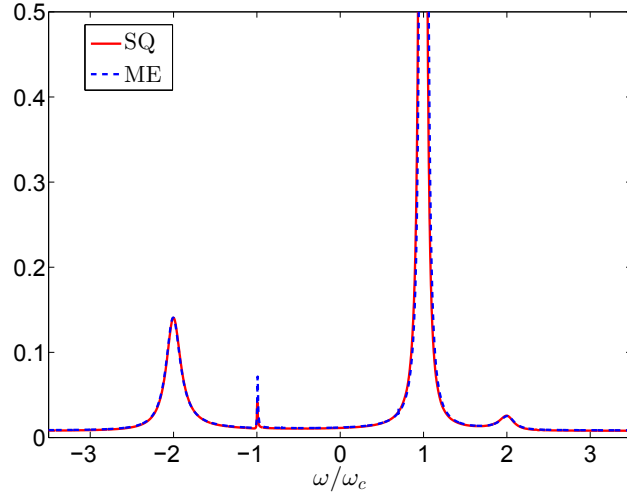


Figure 4.16: Absorption spectrum:  $\omega_0 = 2$ ,  $g = 0.08$ ,  $A = 0.01$ ,  $\omega_d = 0.7$ ,  $\kappa = \gamma = 0.01$ ,  $\gamma_\phi = 0.1$  and  $T = 0$ .

There are two peaks in the spectrum, at  $\omega \approx 0.999$  and at  $\omega \approx -1.997$ . Positive frequencies correspond to absorption and negative to emission. If one diagonalizes the Rabi Hamiltonian numerically, one notices that these frequencies are very close to the transitions from the 1st and 2nd excited states to the ground state. Small deviations are probably caused by the drive. The reason we see emission peaks in the absorption spectrum is the coupling. There is a finite occupation in the cavity, which can excite the qubit and, thus, energy leaves the oscillator at the coupling modified resonant frequency of the qubit (calculated below).

When the coupling increases, the peaks shift more from the uncoupled eigenfrequencies  $\omega_c = 1$  and  $\omega_0 = 2$ . They still coincide well with the energy levels of the Rabi Hamiltonian. Also new peaks emerge around  $\omega = -1$  and  $\omega = 2$ . But the peak around  $\omega = \pm 2$  now corresponds to a transition between the 3rd excited state and the ground state. The peak in figure 4.15 is not very sharp, so its exact position is ambiguous, and the 2nd and 3rd energy levels are nearly degenerate, so it might be also in a position corresponding to transition from 3rd excited state to ground state.

Using the methods given in reference [47], one can derive the following forms for the ground state and 1st and 3rd excited states of the Rabi Hamiltonian, to second order in

$g$ , for  $\omega_c < \omega_0$  (we use the notation of reference [47]):

$$|\widetilde{g, 0}\rangle = \left(1 - \frac{\Lambda^2}{2}\right) |g, 0\rangle - \Lambda |e, 1\rangle + \sqrt{2}\xi |g, 2\rangle, \quad (4.1)$$

$$|\widetilde{1, +}\rangle = \left(1 - \frac{g^2}{2} - \Lambda\right) |g, 1\rangle - g |e, 0\rangle - \sqrt{2}\Lambda |e, 2\rangle + \sqrt{6}\xi |g, 3\rangle, \quad (4.2)$$

$$|\widetilde{1, -}\rangle = \left(1 - \frac{g^2}{2}\right) |e, 0\rangle + g |g, 1\rangle - \sqrt{2}(g\Lambda + \xi) |e, 2\rangle. \quad (4.3)$$

Here  $|\widetilde{g, 0}\rangle$  is the ground state,  $|\widetilde{1, +}\rangle$  the first and  $|\widetilde{1, -}\rangle$  third excited state and the parameters are  $\Lambda = g/(\omega_c + \omega_0)$  and  $\xi = g\Lambda/2\omega_c$ . One can see that the state  $|\widetilde{1, +}\rangle$  is mostly like  $|g, 1\rangle$  in character and  $|\widetilde{1, -}\rangle$  is close to the state  $|e, 0\rangle$ . Using these, one can derive expressions for the transition frequencies between the states. They are:

$$\omega_{1\text{st} \rightarrow \text{gs}} = \frac{E_{1\text{st}} - E_{\text{gs}}}{\hbar} = \omega_c - \frac{2g^2\omega_0}{\omega_0^2 - \omega_c^2}, \quad (4.4)$$

$$\omega_{3\text{rd} \rightarrow \text{gs}} = \frac{E_{3\text{rd}} - E_{\text{gs}}}{\hbar} = \omega_0 + \frac{2g^2\omega_0}{\omega_0^2 - \omega_c^2}. \quad (4.5)$$

Both transitions show quadratic shift from the bare frequencies,  $\omega_c$  and  $\omega_0$ , of magnitude  $2g^2\omega_0/(\omega_0^2 - \omega_c^2)$ . This analytical result is in good agreement with the numerical diagonalization of the Rabi Hamiltonian and with the locations of the resonances in the calculated spectra, discussed above. The next correction to the transition frequencies is proportional to  $g^4$  and is, thus, extremely small. The magnitude of the fourth order correction in  $g$  is no longer the same for both of the transitions.

If one looks at the exact positions of the peaks in figures 4.16 and 4.17, one notices that the semiquantum calculation predicts an asymmetrical shift in the peaks. The peaks at negative frequencies shift less than those in positive ones. This effect is not seen in the spectra obtained from the master equation calculations and, thus, is due to the approximations made. When coupling gets stronger, the peak around  $\omega = 1$  splits into 2 peaks in the spectra from the master equation calculation, as seen in figure 4.17, while the semiquantum model does not capture this phenomenon. The position of the right peak near  $\omega = 1$  corresponds to a transition from the fourth to the first excited state. As mentioned when discussing the time-correlators, the effect of driving is seen more prominently in ME calculations. In figure 4.17, the 'peaks' at  $\pm 0.7 = \pm\omega_d$  are caused by the driving.

Based on the results presented here, the semiquantum calculation works relatively well when  $g \lesssim 0.1$ . This is notably smaller than what we had for the steady state results, those were in good agreement until  $g \gtrsim 0.14$ . Based on this, we deduce that the

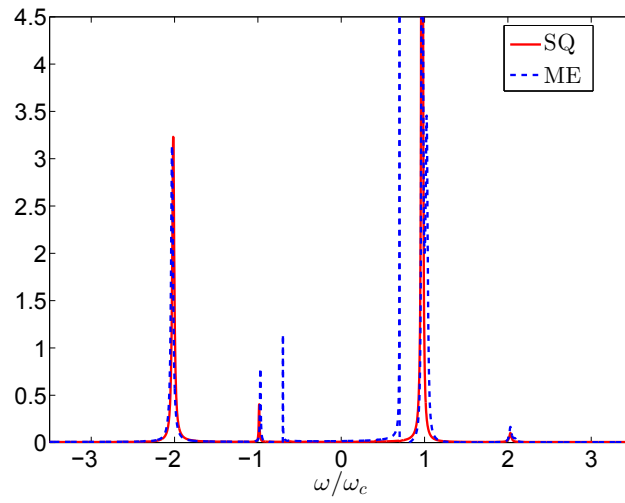


Figure 4.17: Absorption spectrum:  $\omega_0 = 2$ ,  $g = 0.15$ ,  $A = 0.10$ ,  $\omega_d = 0.7$ ,  $\kappa = \gamma = \gamma_\phi = 0.01$  and  $T = 0$ .

cavity-qubit correlations, where we found notable differences between the two models, are important for the spectral properties of the system. This has been found also in reference [37].

# Chapter 5

## Conclusions

In this thesis we studied the semi-quantum approximation in the Rabi model. The system is comprised of a driven harmonic oscillator coupled to a two-level-system. Our aim was to determine a range of parameters, on which the semi-quantum approximation could be used instead of the full quantum master equation, while producing different results compared to the semiclassical model.

### **Semi-quantum and semiclassical approximations**

As it should be, the semi-quantum approximation produces identical results with the semiclassical model, when the coupling,  $g$ , between cavity and qubit is small enough. This is a good sanity check, since we know that semiclassical approximation is valid when the coupling is weak enough. There is new physics to be found using the semi-quantum approximation and the Rabi model. The results differ significantly from the semiclassical ones when  $g/\omega_c \gtrsim 0.03$ , while still maintaining good agreement with the more accurate full master equation solution. Using semi-quantum approximation with the Jaynes-Cummings model does not seem to produce different results compared to the semiclassical model, while being in agreement with the Rabi model results.

### **Full master equation and the semi-quantum model**

When it comes to the steady states obtained from the master equation and the semi-quantum model, one can use the latter safely when  $g/\omega_c \leq 0.10$ . There is no practical difference in the results. Good agreement is obtained up to  $g/\omega_c \lesssim 0.14$ , even higher, if one is interested only in the number of quanta in the oscillator. Based on the behaviour of the correlators as functions of the coupling and the dissipators, the semi-quantum model overestimates the coupling between the cavity and the qubit. If one wants to study the



qubit-cavity correlators or  $\langle a^2 \rangle$ , the semiquantum approximation should not be used, especially the 'counter rotating' terms are given poorly.

The spectra obtained from the two calculations do not agree as well as the steady state results. Presumably, the error in the semiquantum model accumulates, as one first calculates the simple correlators and then the autocorrelation functions, using the steady state values as the initial values. Moreover, we have to give approximate initial values for some of the time-correlators. The spectra agree considerably well when  $g/\omega_c \lesssim 0.10$ , with a moderate drive. If one uses the semiquantum approximation to calculate the spectrum, one should remember that the asymmetric shift in the spectral lines is not a real phenomenon.

### Future improvements

In the future, one thing to study would be to derive equations in the semiquantum approximation from the master equation presented in reference [47], to get rid of the spurious excitations in the system and the dissipator dependent steady state. One could also go a step further, and try to diagonalize the Rabi Hamiltonian to even higher order in  $g$  than in reference [47], they do it only up to  $g^2$ . Then compare it again with the full master equation, to see if the good agreement we have seen here transfers to that situation. One should also investigate the discrepancy arising from when one uses the anticommutation relation for  $\sigma_{\pm}$ . One could also think about taking higher order correlators to the equations of motion, say up to third order. The amount of equations would rise, but it could be still lower than what one would have with the master equation.

The ideal place to use semiquantum model could be with systems including mechanical oscillators, i.e in optomechanics. Usually the frequency of the mechanical oscillator is so small, that there is notable amount of excitations at the environment, so that one has to include states with considerably high number of quanta into the master equation. For example, if we would have a system comprising of a coupled qubit, cavity and mechanical oscillator, and we would need to include, say, 10 states for the cavity and 50 states for the mechanical oscillator, then the density matrix would have  $1000^2$  elements. Thus, we would need to solve a differential equation group of 500499 coupled equations. With a normal desktop computer this would take at least several days, probably weeks.

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## Appendix A

# Transformation of the master equation

In this appendix we carry out the transformation (2.35) for the density operator. In the transformed picture the density operator reads

$$\tilde{\rho} = \sum_n p_n \left| \widetilde{\psi}_n \right\rangle \left\langle \widetilde{\psi}_n \right|, \quad (\text{A.1})$$

where  $\left| \widetilde{\psi}_n \right\rangle = D^\dagger(\alpha) |\psi_n\rangle$  are the transformed states. Writing the transformed density operator using the original states gives

$$\begin{aligned} \tilde{\rho} &= \sum_n p_n \left| \widetilde{\psi}_n \right\rangle \left\langle \widetilde{\psi}_n \right| = \sum_n p_n D^\dagger(\alpha) |\psi_n\rangle \langle \psi_n| D(\alpha) \\ &= D^\dagger(\alpha) \sum_n p_n |\psi_n\rangle \langle \psi_n| D(\alpha) = D^\dagger(\alpha) \rho D(\alpha). \end{aligned} \quad (\text{A.2})$$

Now we are ready to derive the master equation for the transformed density operator. Differentiating (A.2) with respect to time gives

$$\frac{d\tilde{\rho}}{dt} = \frac{d}{dt} \left( D^\dagger(\alpha) \rho D(\alpha) \right) = \frac{dD^\dagger(\alpha)}{dt} \rho D(\alpha) + D^\dagger(\alpha) \frac{d\rho}{dt} D(\alpha) + D^\dagger(\alpha) \rho \frac{dD(\alpha)}{dt}. \quad (\text{A.3})$$

The operator  $D(\alpha)$  is unitary, so  $D(\alpha)D^\dagger(\alpha) = 1$ . Inserting these expressions for the identity operator in convenient places, we obtain

$$\begin{aligned} \frac{d\tilde{\rho}}{dt} &= \frac{dD^\dagger(\alpha)}{dt} D(\alpha) D^\dagger(\alpha) \rho D(\alpha) + D^\dagger(\alpha) \frac{d\rho}{dt} D(\alpha) + D^\dagger(\alpha) \rho D(\alpha) D^\dagger(\alpha) \frac{dD(\alpha)}{dt} \\ &= \frac{dD^\dagger(\alpha)}{dt} D(\alpha) \tilde{\rho} + D^\dagger(\alpha) \frac{d\rho}{dt} D(\alpha) + \tilde{\rho} D^\dagger(\alpha) \frac{dD(\alpha)}{dt}. \end{aligned} \quad (\text{A.4})$$

Differentiating the identity operator with respect to time gives

$$\frac{d1}{dt} = \frac{d}{dt} \left( D^\dagger(\alpha) D(\alpha) \right) = \frac{dD^\dagger(\alpha)}{dt} D(\alpha) + D^\dagger(\alpha) \frac{dD(\alpha)}{dt} = 0, \quad (\text{A.5})$$

thus,

$$D^\dagger(\alpha) \frac{dD(\alpha)}{dt} = - \frac{dD^\dagger(\alpha)}{dt} D(\alpha). \quad (\text{A.6})$$

Using this result and substituting the master equation for  $\rho$  into the above gives

$$\begin{aligned} \frac{d\tilde{\rho}}{dt} &= -\frac{i}{\hbar} D^\dagger(\alpha) [H, \rho] D(\alpha) + \frac{\kappa}{2} (N_c + 1) D^\dagger(\alpha) \mathcal{D}[a] \rho D(\alpha) + \frac{\kappa}{2} N_c D^\dagger(\alpha) \mathcal{D}[a^\dagger] \rho D(\alpha) \\ &\quad + \frac{\gamma}{2} (N_q + 1) D^\dagger(\alpha) \mathcal{D}[\sigma_-] \rho D(\alpha) + \frac{\gamma}{2} N_q D^\dagger(\alpha) \mathcal{D}[\sigma_+] \rho D(\alpha) \\ &\quad + \gamma_\phi D^\dagger(\alpha) \mathcal{D}[\sigma_+ \sigma_-] \rho D^\dagger(\alpha) + \frac{dD^\dagger(\alpha)}{dt} D(\alpha) \tilde{\rho} - \tilde{\rho} \frac{dD^\dagger(\alpha)}{dt} D(\alpha). \end{aligned}$$

The transformation has no effect on the dissipators acting in the qubit space, because  $a^{(\dagger)}$  commute with  $\sigma_\pm$ . We will go through the rest term by term. First, the Hamiltonian part:

$$\begin{aligned} D^\dagger(\alpha) [H, \rho] D(\alpha) &= D^\dagger(\alpha) H \rho D(\alpha) - D^\dagger(\alpha) \rho H D(\alpha) \\ &= D^\dagger(\alpha) H D(\alpha) D^\dagger(\alpha) \rho D(\alpha) - D^\dagger(\alpha) \rho D(\alpha) D^\dagger(\alpha) H D(\alpha) \\ &= D^\dagger(\alpha) H D(\alpha) \tilde{\rho} - \tilde{\rho} D^\dagger(\alpha) H D(\alpha). \end{aligned} \quad (\text{A.7})$$

The dissipators for the cavity give:

$$\begin{aligned} D^\dagger(\alpha) \mathcal{D}[a] \rho D(\alpha) &= 2D^\dagger(\alpha) a \rho a^\dagger D(\alpha) - D^\dagger(\alpha) a^\dagger a \rho D(\alpha) - D^\dagger(\alpha) \rho a^\dagger a D(\alpha) \\ &= 2(\alpha + a) \tilde{\rho} (\alpha^* + a^\dagger) - (\alpha^* + a^\dagger) (\alpha + a) \tilde{\rho} - \tilde{\rho} (\alpha^* + a^\dagger) (\alpha + a) \\ &= 2a\tilde{\rho}a^\dagger - a^\dagger a \tilde{\rho} - \tilde{\rho} a^\dagger a + 2\alpha\tilde{\rho}a^\dagger + 2\alpha^* a \tilde{\rho} - \alpha^* a \tilde{\rho} - \alpha a^\dagger \tilde{\rho} \\ &\quad - \alpha^* \tilde{\rho} a - \alpha \tilde{\rho} a^\dagger + 2|\alpha|^2 \tilde{\rho} - |\alpha|^2 \tilde{\rho} - |\alpha|^2 \tilde{\rho} \\ &= \mathcal{D}[a] \tilde{\rho} - \alpha [a^\dagger, \tilde{\rho}] + \alpha^* [a, \tilde{\rho}]. \end{aligned} \quad (\text{A.8})$$

Similarly, we have for the other one

$$D^\dagger(\alpha) \mathcal{D}[a^\dagger] \rho D(\alpha) = \mathcal{D}[a^\dagger] \tilde{\rho} + \alpha [a^\dagger, \tilde{\rho}] - \alpha^* [a, \tilde{\rho}]. \quad (\text{A.9})$$

Combining the above results, we obtain for the transformed master equation

$$\begin{aligned} \frac{d\tilde{\rho}}{dt} &= -\frac{i}{\hbar} \left( D^\dagger(\alpha) H D(\alpha) \tilde{\rho} - \tilde{\rho} D^\dagger(\alpha) H D(\alpha) \right) + \frac{dD^\dagger(\alpha)}{dt} D(\alpha) \tilde{\rho} - \tilde{\rho} \frac{dD^\dagger(\alpha)}{dt} D(\alpha) \\ &\quad + \frac{\kappa}{2} (N_c + 1) \left( \mathcal{D}[a] \tilde{\rho} - \alpha [a^\dagger, \tilde{\rho}] + \alpha^* [a, \tilde{\rho}] \right) \\ &\quad + \frac{\kappa}{2} N_c \left( \mathcal{D}[a^\dagger] \tilde{\rho} + \alpha [a^\dagger, \tilde{\rho}] - \alpha^* [a, \tilde{\rho}] \right) + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \rho \\ &\quad + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \rho + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \rho \end{aligned}$$

$$\begin{aligned}
 &= -\frac{i}{\hbar} \left[ \left( D^\dagger(\alpha) H D(\alpha) + i\hbar \frac{dD^\dagger(\alpha)}{dt} D(\alpha) \right) \tilde{\rho} - \tilde{\rho} \left( D^\dagger(\alpha) H D(\alpha) + i\hbar \frac{dD^\dagger(\alpha)}{dt} D(\alpha) \right) \right] \\
 &\quad + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \tilde{\rho} + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \tilde{\rho} - \frac{\kappa}{2} \alpha [a^\dagger, \tilde{\rho}] + \frac{\kappa}{2} \alpha^* [a, \tilde{\rho}] \\
 &\quad + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \rho + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \rho + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \rho \\
 &= -\frac{i}{\hbar} [\tilde{H}, \tilde{\rho}] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \tilde{\rho} + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \tilde{\rho} - \frac{\kappa}{2} \alpha [a^\dagger, \tilde{\rho}] + \frac{\kappa}{2} \alpha^* [a, \tilde{\rho}] \\
 &\quad + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \rho + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \rho + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \rho.
 \end{aligned}$$

Substituting  $\tilde{H}$  into the above from [section 3.2](#), equation (3.18), gives

$$\begin{aligned}
 \frac{d\tilde{\rho}}{dt} &= -\frac{i}{\hbar} [\tilde{H}_0, \tilde{\rho}] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \tilde{\rho} + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \tilde{\rho} + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \tilde{\rho} \\
 &\quad + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \tilde{\rho} + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \tilde{\rho} + [a^\dagger, \tilde{\rho}] \left[ -\left(\frac{\kappa}{2} + i\omega_c\right) \alpha + f(t) - \frac{d\alpha}{dt} \right] \\
 &\quad - [a, \tilde{\rho}] \left[ -\left(\frac{\kappa}{2} - i\omega_c\right) \alpha^* + f(t) - \frac{d\alpha^*}{dt} \right]. \tag{A.10}
 \end{aligned}$$

We can simplify this, by solving for the two last expression in the brackets from the equations of motion for  $\alpha$  and  $\alpha^*$  (equation for  $\alpha^*$  is the complex conjugate of the equation for  $\alpha$ ). We obtain, from equation (3.12),

$$2ig\Re(s_-) = -\left(\frac{\kappa}{2} + i\omega_c\right) \alpha + f(t) - \frac{d\alpha}{dt}, \tag{A.11}$$

and, thus, have for  $\alpha^*$

$$-2ig\Re(s_-) = -\left(\frac{\kappa}{2} - i\omega_c\right) \alpha^* + f(t) - \frac{d\alpha^*}{dt}. \tag{A.12}$$

Substitute these into the transformed master equation and combine with the Hamiltonian part, to obtain

$$\begin{aligned}
 \dot{\tilde{\rho}} &= -\frac{i}{\hbar} \left[ \tilde{H}_0 - 2\hbar g \Re(s_-) (a^\dagger + a), \tilde{\rho} \right] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \tilde{\rho} + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \tilde{\rho} \\
 &\quad + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \tilde{\rho} + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \tilde{\rho} + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \tilde{\rho} \\
 &\equiv -\frac{i}{\hbar} [H_{eff}, \tilde{\rho}] + \frac{\kappa}{2} (N_c + 1) \mathcal{D}[a] \tilde{\rho} + \frac{\kappa}{2} N_c \mathcal{D}[a^\dagger] \tilde{\rho} + \frac{\gamma}{2} (N_q + 1) \mathcal{D}[\sigma_-] \tilde{\rho} \\
 &\quad + \frac{\gamma}{2} N_q \mathcal{D}[\sigma_+] \tilde{\rho} + \gamma_\phi \mathcal{D}[\sigma_+ \sigma_-] \tilde{\rho}, \tag{A.13}
 \end{aligned}$$

where we defined the effective Hamiltonian

$$\begin{aligned}
 H_{eff} &= \hbar\omega_c a^\dagger a + \hbar\omega_0 \sigma_+ \sigma_- + \hbar g (a^\dagger + a) (\sigma_+ + \sigma_-) \\
 &\quad + 2\hbar g \left[ \Re(\alpha) (\sigma_+ + \sigma_-) - \Re(s_-) (a^\dagger + a) \right]. \tag{A.14}
 \end{aligned}$$



## Appendix B

# Equations of motions for the time-correlators

We list the equations of motion for the autocorrelation functions, discussed in [section 3.4](#), here.

$$\frac{d\langle a^\dagger(t)a(0)\rangle}{dt} = \left(i\omega_c - \frac{\kappa}{2}\right) \langle a^\dagger(t)a(0)\rangle + ig(\langle \sigma_+(t)a(0)\rangle + \langle \sigma_-(t)a(0)\rangle), \quad (\text{B.1})$$

$$\frac{d\langle a(t)a(0)\rangle}{dt} = -\left(\frac{\kappa}{2} + i\omega_c\right) \langle a(t)a(0)\rangle - ig(\langle \sigma_+(t)a(0)\rangle + \langle \sigma_-(t)a(0)\rangle), \quad (\text{B.2})$$

$$\frac{d\langle \sigma_-(t)a(0)\rangle}{dt} = -\left\{\left[\frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right] + i\omega_0\right\} \langle \sigma_-(t)a(0)\rangle \quad (\text{B.3})$$

$$\begin{aligned} & + 2ig \langle \sigma_+ \sigma_-(t)a(0)\rangle \left(\langle a^\dagger(t)a(0)\rangle + \langle a(t)a(0)\rangle + 2\Re(\alpha)\right) \\ & + 2ig \langle \sigma_+(t)a(0)\rangle \left(\langle a\sigma_-(t)a(0)\rangle + \langle a^\dagger\sigma_-(t)a(0)\rangle\right) \\ & + 2ig \langle \sigma_-(t)a(0)\rangle \left(\langle a^\dagger\sigma_+(t)a(0)\rangle + \langle a\sigma_+(t)a(0)\rangle\right) \\ & - ig \langle a^\dagger(t)a(0)\rangle (1 + 4 \langle \sigma_+(t)a(0)\rangle \langle \sigma_-(t)a(0)\rangle) \\ & - ig \langle a(t)a(0)\rangle (1 + 4 \langle \sigma_+(t)a(0)\rangle \langle \sigma_-(t)a(0)\rangle), \end{aligned}$$

$$\begin{aligned} \frac{d\langle \sigma_+(t)a(0)\rangle}{dt} & = \left\{i\omega_0 - \left[\frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right]\right\} \langle \sigma_+(t)a(0)\rangle \quad (\text{B.4}) \\ & - 2ig \langle \sigma_+ \sigma_-(t)a(0)\rangle \left(\langle a^\dagger(t)a(0)\rangle + \langle a(t)a(0)\rangle + 2\Re(\alpha)\right) \\ & - 2ig \langle \sigma_+(t)a(0)\rangle \left(\langle a\sigma_-(t)a(0)\rangle + \langle a^\dagger\sigma_-(t)a(0)\rangle\right) \\ & - 2ig \langle \sigma_-(t)a(0)\rangle \left(\langle a^\dagger\sigma_+(t)a(0)\rangle + \langle a\sigma_+(t)a(0)\rangle\right) \\ & + ig \langle a^\dagger(t)a(0)\rangle (1 + 4 \langle \sigma_+(t)a(0)\rangle \langle \sigma_-(t)a(0)\rangle) \end{aligned}$$

$$\begin{aligned}
 & + ig \langle a(t)a(0) \rangle (1 + 4 \langle \sigma_+(t)a(0) \rangle \langle \sigma_-(t)a(0) \rangle), \\
 \frac{d \langle \sigma_+ \sigma_-(t)a(0) \rangle}{dt} & = -ig \left( \langle a^\dagger \sigma_+(t)a(0) \rangle + \langle a \sigma_+(t)a(0) \rangle - \langle a^\dagger \sigma_-(t)a(0) \rangle - \langle a \sigma_-(t)a(0) \rangle \right) \quad (\text{B.5})
 \end{aligned}$$

$$\begin{aligned}
 & - 2ig \Re(\alpha) (\langle \sigma_+(t)a(0) \rangle - \langle \sigma_-(t)a(0) \rangle) - \gamma (2N_q + 1) \langle \sigma_+ \sigma_-(t)a(0) \rangle, \\
 \frac{d \langle a^\dagger a(t)a(0) \rangle}{dt} & = -ig \left( \langle a^\dagger \sigma_+(t)a(0) \rangle + \langle a^\dagger \sigma_-(t)a(0) \rangle - \langle a \sigma_+(t)a(0) \rangle - \langle a \sigma_-(t)a(0) \rangle \right) \quad (\text{B.6})
 \end{aligned}$$

$$\begin{aligned}
 & + 2ig \Re(s_-) \left( \langle a^\dagger(t)a(0) \rangle - \langle a(t)a(0) \rangle \right) - \kappa \langle a^\dagger a(t)a(0) \rangle, \\
 \frac{d \langle a \sigma_-(t)a(0) \rangle}{dt} & = - \left\{ \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] + i(\omega_c + \omega_0) \right\} \langle a \sigma_-(t)a(0) \rangle \quad (\text{B.7})
 \end{aligned}$$

$$\begin{aligned}
 & - ig \left( \langle a^\dagger a(t)a(0) \rangle + \langle a a(t)a(0) \rangle - 2 \langle \sigma_-(t)a(0) \rangle \Re(s_-) \right) \\
 & + 2ig (\langle a a(t)a(0) \rangle \langle \sigma_+ \sigma_-(t)a(0) \rangle + 2 \langle a \sigma_+(t)a(0) \rangle \langle a \sigma_-(t)a(0) \rangle) \\
 & - 4ig \langle \sigma_-(t)a(0) \rangle \langle \sigma_+(t)a(0) \rangle \left( \langle a(t)a(0) \rangle^2 + \langle a^\dagger(t)a(0) \rangle \langle a(t)a(0) \rangle \right) \\
 & + ig \langle \sigma_+ \sigma_-(t)a(0) \rangle \left( 2 \langle a^\dagger a(t)a(0) \rangle + 1 \right) \\
 & + 2ig \left( \langle a \sigma_-(t)a(0) \rangle \langle a^\dagger \sigma_+(t)a(0) \rangle + \langle a^\dagger \sigma_-(t)a(0) \rangle \langle a \sigma_+(t)a(0) \rangle \right) \\
 & + 4ig \Re(\alpha) (\langle a(t)a(0) \rangle \langle \sigma_+ \sigma_-(t)a(0) \rangle + \langle \sigma_+(t)a(0) \rangle \langle a \sigma_-(t)a(0) \rangle) \\
 & + 4ig \Re(\alpha) \langle \sigma_-(t)a(0) \rangle \langle a \sigma_+(t)a(0) \rangle \\
 & - 2ig \Re(\alpha) (4 \langle a(t)a(0) \rangle \langle \sigma_+(t)a(0) \rangle \langle \sigma_-(t)a(0) \rangle + \langle a(t)a(0) \rangle), \\
 \frac{d \langle a^\dagger \sigma_+(t)a(0) \rangle}{dt} & = \left\{ i(\omega_c + \omega_0) - \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] \right\} \langle a^\dagger \sigma_+(t)a(0) \rangle \quad (\text{B.8}) \\
 & + ig \left( \langle a^\dagger a(t)a(0) \rangle + \langle a^\dagger a^\dagger(t)a(0) \rangle - 2 \langle \sigma_+(t)a(0) \rangle \Re(s_-) \right) \\
 & - 2ig \left( \langle a^\dagger a^\dagger(t)a(0) \rangle \langle \sigma_+ \sigma_-(t)a(0) \rangle + 2 \langle a^\dagger \sigma_-(t)a(0) \rangle \langle a^\dagger \sigma_+(t)a(0) \rangle \right) \\
 & + 4ig \langle \sigma_-(t)a(0) \rangle \langle \sigma_+(t)a(0) \rangle \left( \langle a^\dagger(t)a(0) \rangle^2 + \langle a^\dagger(t)a(0) \rangle \langle a(t)a(0) \rangle \right) \\
 & - ig \langle \sigma_+ \sigma_-(t)a(0) \rangle \left( 2 \langle a^\dagger a(t)a(0) \rangle + 1 \right) \\
 & - 2ig \left( \langle a \sigma_-(t)a(0) \rangle \langle a^\dagger \sigma_+(t)a(0) \rangle + \langle a^\dagger \sigma_-(t)a(0) \rangle \langle a \sigma_+(t)a(0) \rangle \right) \\
 & - 4ig \Re(\alpha) \left( \langle a^\dagger(t)a(0) \rangle \langle \sigma_+ \sigma_-(t)a(0) \rangle + \langle \sigma_-(t)a(0) \rangle \langle a^\dagger \sigma_+(t)a(0) \rangle \right) \\
 & + 4ig \Re(\alpha) \langle \sigma_+(t)a(0) \rangle \langle a^\dagger \sigma_-(t)a(0) \rangle \\
 & + 2ig \Re(\alpha) \left( 4 \langle a^\dagger(t)a(0) \rangle \langle \sigma_+(t)a(0) \rangle \langle \sigma_-(t)a(0) \rangle + \langle a^\dagger(t)a(0) \rangle \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{d\langle a^\dagger \sigma_-(t) a(0) \rangle}{dt} = & - \left\{ \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] + i(\omega_0 - \omega_c) \right\} \langle a^\dagger \sigma_-(t) a(0) \rangle \quad (\text{B.9}) \\
 & - ig \left( 2 \langle \sigma_-(t) a(0) \rangle \Re(s_-) + \langle a^\dagger a^\dagger(t) a(0) \rangle + \langle a^\dagger a(t) a(0) \rangle \right) \\
 & + 2ig \left( \langle a^\dagger a^\dagger(t) a(0) \rangle \langle \sigma_+ \sigma_-(t) a(0) \rangle + 2 \langle a^\dagger \sigma_+(t) a(0) \rangle \langle a^\dagger \sigma_-(t) a(0) \rangle \right) \\
 & - 4ig \langle \sigma_+(t) a(0) \rangle \langle \sigma_-(t) a(0) \rangle \left( \langle a^\dagger(t) a(0) \rangle^2 + \langle a^\dagger(t) a(0) \rangle \langle a(t) a(0) \rangle \right) \\
 & + ig \langle \sigma_+ \sigma_-(t) a(0) \rangle \left( 2 \langle a^\dagger a(t) a(0) \rangle + 1 \right) \\
 & + 2ig \left( \langle a \sigma_-(t) a(0) \rangle \langle a^\dagger \sigma_+(t) a(0) \rangle + \langle a^\dagger \sigma_-(t) a(0) \rangle \langle a \sigma_+(t) a(0) \rangle \right) \\
 & + 4ig \Re(\alpha) \left( \langle a^\dagger(t) a(0) \rangle \langle \sigma_+ \sigma_-(t) a(0) \rangle + \langle \sigma_+(t) a(0) \rangle \langle a^\dagger \sigma_-(t) a(0) \rangle \right) \\
 & + 4ig \Re(\alpha) \langle \sigma_-(t) a(0) \rangle \langle a^\dagger \sigma_+(t) a(0) \rangle \\
 & - 2ig \Re(\alpha) \left( 4 \langle a^\dagger(t) a(0) \rangle \langle \sigma_+(t) a(0) \rangle \langle \sigma_-(t) a(0) \rangle + \langle a^\dagger(t) a(0) \rangle \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{d\langle a \sigma_+(t) a(0) \rangle}{dt} = & \left\{ i(\omega_0 - \omega_c) - \left[ \frac{\kappa}{2} + \frac{\gamma}{2} (2N_q + 1) + \gamma_\phi \right] \right\} \langle a \sigma_+(t) a(0) \rangle \quad (\text{B.10}) \\
 & + ig \left( 2 \langle \sigma_+(t) a(0) \rangle \Re(s_-) + \langle a a(t) a(0) \rangle + \langle a^\dagger a(t) a(0) \rangle \right) \\
 & - 2ig \left( \langle a a(t) a(0) \rangle \langle \sigma_+ \sigma_-(t) a(0) \rangle + 2 \langle a \sigma_-(t) a(0) \rangle \langle a \sigma_+(t) a(0) \rangle \right) \\
 & + 4ig \langle \sigma_+(t) a(0) \rangle \langle \sigma_-(t) a(0) \rangle \left( \langle a(t) a(0) \rangle^2 + \langle a^\dagger(t) a(0) \rangle \langle a(t) a(0) \rangle \right) \\
 & - ig \langle \sigma_+ \sigma_-(t) a(0) \rangle \left( \langle a^\dagger a(t) a(0) \rangle + 1 \right) \\
 & - 2ig \left( \langle a \sigma_-(t) a(0) \rangle \langle a^\dagger \sigma_+(t) a(0) \rangle + \langle a^\dagger \sigma_-(t) a(0) \rangle \langle a \sigma_+(t) a(0) \rangle \right) \\
 & - 4ig(\alpha) \left( \langle a(t) a(0) \rangle \langle \sigma_+ \sigma_-(t) a(0) \rangle + \langle \sigma_-(t) a(0) \rangle \langle a \sigma_+(t) a(0) \rangle \right) \\
 & - 4ig(\alpha) \langle \sigma_+(t) a(0) \rangle \langle a \sigma_-(t) a(0) \rangle \\
 & + 2ig \Re(\alpha) \left( 4 \langle a(t) a(0) \rangle \langle \sigma_+(t) a(0) \rangle \langle \sigma_-(t) a(0) \rangle + \langle a(t) a(0) \rangle \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{d\langle a a(t) a(0) \rangle}{dt} = & -(\kappa + 2i\omega_c) \langle a a(t) a(0) \rangle \quad (\text{B.11}) \\
 & - 2ig \left( \langle a \sigma_+(t) a(0) \rangle + \langle a \sigma_-(t) a(0) \rangle - 2\Re(s_-) \langle a(t) a(0) \rangle \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{d\langle a^\dagger a^\dagger(t) a(0) \rangle}{dt} = & (2i\omega_c - \kappa) \langle a^\dagger a^\dagger(t) a(0) \rangle \quad (\text{B.12}) \\
 & + 2ig \left( \langle a^\dagger \sigma_-(t) a(0) \rangle + \langle a^\dagger \sigma_+(t) a(0) \rangle - 2\Re(s_-) \langle a^\dagger(t) a(0) \rangle \right).
 \end{aligned}$$

As one sees, this is a closed set of equations, from which the time-evolution can be obtained.

## Appendix C

# Equations of motion for the Jaynes-Cummings model

We list the equations of motion for the coupled two-level-system and driven harmonic oscillator, in the Jaynes-Cummings model. They are used to obtain the steady state behaviour, in RWA, for figures presented in [chapter 4](#). Derivation of this group of equations will not be presented here, for it is analogous to the derivation presented in [chapter 3](#) for the Rabi model.

$$\frac{d\alpha}{dt} = -\left(\frac{\kappa}{2} + i\omega_c\right)\alpha - ig s_- + \frac{A}{2}e^{-i\omega_d t}, \quad (\text{C.1})$$

$$\frac{ds_-}{dt} = -\left\{\left[\frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right] + i\omega_0\right\}s_- + ig\left[2s_- \langle a^\dagger \sigma_- \rangle^* + \alpha(2\langle \sigma_+ \sigma_- \rangle - 1)\right], \quad (\text{C.2})$$

$$\frac{d\langle \sigma_+ \sigma_- \rangle}{dt} = 2g\Im\left(\alpha s_-^* - \langle a^\dagger \sigma_- \rangle\right) - \gamma[(2N_q + 1)\langle \sigma_+ \sigma_- \rangle - N_q], \quad (\text{C.3})$$

$$\frac{d\langle a^\dagger a \rangle}{dt} = 2g\Im\left(\langle a^\dagger \sigma_- \rangle\right) - \kappa\left(\langle a^\dagger a \rangle - N_c\right), \quad (\text{C.4})$$

$$\begin{aligned} \frac{d\langle a^\dagger \sigma_- \rangle}{dt} = & -\left\{\left[\frac{\kappa}{2} + \frac{\gamma}{2}(2N_q + 1) + \gamma_\phi\right] + i(\omega_0 - \omega_c)\right\}\langle a^\dagger \sigma_- \rangle \\ & - ig\left[\langle a^\dagger a \rangle - \langle \sigma_+ \sigma_- \rangle\left(\langle a^\dagger a \rangle + 1\right) - 2\alpha s_-^* \langle a^\dagger \sigma_- \rangle - 2\left|\langle a^\dagger \sigma_- \rangle\right|^2 + |s_-|^2\right]. \end{aligned} \quad (\text{C.5})$$

When one uses the Jaynes-Cummings model, the resulting differential equation group is smaller than that needed for the Rabi case, this is because one can drop the counter rotating terms  $\langle a\sigma_- \rangle$  and  $\langle a^2 \rangle$  from the equations, and, thus, we do not have to derive equations for them. Still, these equations are too complicated to solve by hand, so we used numerical methods, explained in [chapter 4](#), to solve the approximate steady state.