

# Efimov states in ultracold gases

Mikko Savikko  
Master's Thesis  
University of Oulu  
Department of Physics  
Theoretical Physics

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# Abstract

This work will introduce the Efimov effect and the resonant and scaling limits and derive the formula for the binding energies of the Efimov states. We use the hyperspherical coordinates for the stationary wave function of three particles and solve the low-energy Faddeev equation with the hyperspherical expansion and use the expansion for solving the channel eigenvalues  $\lambda_n(R)$ . The channel eigenvalues are defined by a constant  $s_0$ , which is the solution of the resulting transcendental equation. We also solve the scaling-violation parameter  $\Lambda_0$  and finally compile all the results to derive the Efimov states. In the unitary limit we find infinitely many Efimov states, with an accumulation point at zero energy and an asymptotic discrete scaling symmetry with the discrete scaling factor  $e^{\frac{\pi}{s_0}} \approx 22.7$ .

In this work, we will also delve into effective field theories, which can be used to numerically analyze and solve Efimov states in different cases. We will first go through the two-body problem which is used as a simpler example on how to solve the three-body problem and to solve the two-body coupling constant  $g_2$ , which will also appear in the three-body problem. By using the diatom field trick introduced by Bedaque, Hammer and van Kolck, we derive the Skorniakov-Ter-Martirosian equation for the three-body problem.

Finally this work will take a quick look at the first experimental evidences for Efimov states that were found since 2006. In the experiment, proper Efimov resonances in measurements of three-body recombination have been observed.



# Preface

Finally it's complete! It has been almost two years since we first discussed about the contents in the office of docent Mikko Saarela and ever since then I have been slowly making progress and only recently got into high gear in finally putting this work to rest. I did say that I wanted a challenging subject and I believe I got what I asked for. My only regret is taking so long with this, however I am still quite satisfied in how it turned out.

I wish to thank docent Mikko Saarela for the continued support in the making of this work and making this possible, researcher Lassi Roininen for forcing me to learn Latex and giving me time during the 3,5 months of summer job to write a big part of this work. Also I want to thank my parents for being patient during not just this work but all these years of prolonged study I went through. I am finally done.



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# Chapter 1

## Introduction

In 1970, a Russian theoretical physicist named Vitaly N. Efimov showed that when scattering length,  $a$ , is large enough compared to the range of the potential, a sequence of 3-body bound states appears. The binding energies are spaced geometrically and differ by multiplicative factor of  $e^{\pi/s_0}$ , where  $s_0$  depends on statistics and the mass ratios of the particles, for example its numerical value is  $s_0 \approx 1.00624$ , so  $e^{\pi/s_0} \approx 22.7$  for identical bosons [1][2]. It took over 30 years to prove it experimentally. Ultracold gas of Caesium atoms was preferred due to its characteristics and evidence was found in 2006 [3] and later in 2011 [4][5]. In 2009 a team of physicists led by Randy Hulet of Rice University in Houston managed to create Efimov-states with lithium atoms [6]. They bound a set of lithium atoms together and then reproduced the bound set with a binding energy 515 times the first one, the characteristic energy level progression of Efimov-states.

The Efimov effect can occur if at least two of the three pairs of particles have a large scattering length. Scattering length,  $a$ , determines the scattering of two particles with short-range interactions at low energy. Low energy here means energy close to the scattering threshold for the two particles and we will have sufficiently low energy if the de Broglie wavelengths of the particles are large compared to the range of the interaction [1]. This applies to few-body and many-body systems as well. If all the particles in a few-body system have sufficiently low energy, scattering properties are primarily determined by  $a$  and for many-body systems we will also need large separations compared to the range of the interaction. The very low energy collisions usually have zero orbital angular momentum collisions as predominant collisions. These are called s-wave collisions [4].

The reason why the Efimov effect is important is that the sequence of 3-body bound states has universal properties that are insensitive to the details of the 2-body potential at short distances. This is important in the region called the universal regime, where the scattering length between two particles exceeds any other length scale of the two-body potential and here the physics

of the two-body system is determined only by scattering length [4].

Describing this work in brief, we start properly from chapter 2 where we introduce the Thomas catastrophe and how it connects to the Efimov effect, we define the resonant and scaling limits and derive the formula for Efimov states for the binding energy  $E_T^{(n)}$ . In chapter 3 we get to the effective field theories and solve the equations that describe the two-body problem and three-body problem through energies and momenta of the particles. In chapter 4 we will take a look at the evidence found for Efimov states so far.

# Chapter 2

## Efimov States

### 2.1 Thomas effect

In 1935 L. H. Thomas discovered that using a simple variational argument to particles interacting through a 2-body potential with depth  $V_0$  and range  $r_0$  that supported a single bound state with binding energy  $E_2$ , binding energy  $E_T$  of the deepest 3-body bound state goes to  $\infty$  in the zero-range limit [7]. This results in the spectrum of 3-body bound states being unbounded from below. He used the following setting: we have a positional part  $\psi(\mathbf{r}_1, \mathbf{r}_2)$  of a wave function for a neutron at  $\mathbf{r}_2$  and a proton at  $\mathbf{r}_1$  that satisfies

$$-\left(\frac{\hbar^2}{2m}\right)\{\nabla_1^2\psi + \nabla_2^2\psi\} + V(r_{12})\psi - E\psi = 0, \quad (2.1)$$

where potential  $V(r_{12})$  depends only on the distance between the particles and is a potential well so that  $V(r) < 0$ , when  $r < r_0$  and  $V(r) = 0$ , when  $r \geq r_0$ .  $E$  here is binding energy of the two-particle system in which  $V(r)$  supports a single bound state.

This produced a problematic conclusion that a 2-body potential that is attractive enough to support a single 2-body bound state could somehow produce 3-body bound states. The zero range limit used by Thomas, defined by  $r_0 \rightarrow 0$  and  $V_0 \rightarrow \infty$  with fixed binding energy, produces a very large scattering length,  $a$ , compared to the range  $r_0$  ( $a \gg r_0$ ). Variational arguments lead to a binding energy of the deepest 3-body bound state that scales like  $\frac{\hbar^2}{mr_0^2}$  and diverges as  $r_0 \rightarrow 0$ . Thomas effect does not take into account that the binding energy and other properties of the 3-body bound state may depend on the details of the interaction potential [1].

Efimov-effect is closely related to the Thomas effect since the deepest 3-body bound states found by Thomas' variational calculation are the deepest Efimov states [8]. When  $|a|$  becomes large enough compared to the range  $r_0$  of the potential, 3-body bound states start to appear and their binding energies are spaced geometrically up to  $\frac{\hbar^2}{ma^2}$ . These new bound states appear

in the spectrum at critical values of  $a$  that differ by multiplicative factors of  $e^{\pi/s_0}$ , where  $s_0$  depends on the statistics and mass ratios of the particles [1]. In the limit  $a \rightarrow \pm\infty$  we find infinitely many 3-body bound states, which accumulate to the 3-body scattering threshold.

## 2.2 The resonant and scaling limits

We can create two limits for the resonance, which we can approach. If we define a large scattering length to satisfy  $|a| \gg l$ , where  $l$  is the natural low-energy length scale, we get two limits which reduce the corrections to universal behaviour to zero. The corrections are suppressed by powers of  $l/|a|$ . The limits are the resonant limit and the scaling limit. In the resonant limit,  $l$  is fixed and  $a$  goes to  $\infty$  or  $-\infty$  and in the scaling limit  $a$  is fixed and  $l$  approaches 0.

### 2.2.1 The resonant limit

The resonant limit, also called the unitary limit, can often be approached by tuning a single parameter for example depth of the interatomic potential or a rescaling of the potential. With resonant limit, the parameter must be tuned to a critical value for which there is a 2-body bound state exactly at the 2-body threshold. Feshbach resonance, which is a 2-channel model with a closed and an open channel, is an excellent example. These Feshbach resonances appear when the total energy in an open channel matches the energy of a bound state in a closed channel. The closer the energy of the bound state is to the energy of the incoming particles in the open channels, the larger the effect on the scattering. The energies of these states depend on external parameters such as the magnetic field which can be tuned experimentally [9].

Natural low-energy length scale is only important in the 2-body case, however the Efimov effect reveals that there is another length scale in the 3-body case. In the resonant limit, there are infinitely many, arbitrarily-shallow 3-body bound states with a spectrum of the form

$$E_T^{(n)} \rightarrow (e^{-2\pi/s_0})^{n-n_*} \hbar^2 \kappa_*^2 / m, \quad (2.2)$$

as  $n \rightarrow +\infty$  with  $a = \pm\infty$ . Here  $n$  and  $n_*$  are some integers,  $m$  is the mass and parameter  $\kappa_*$  can be interpreted as the approximate binding wave number of the Efimov state labelled by the integer  $n_*$  [1].

### 2.2.2 The scaling limit

The scaling limit, or zero-range limit, can be defined by specifying the phase shifts for 2-body scattering. The s-wave phase shift  $\delta_0(k)$  has the following

form in the scaling limit

$$k \cot(\delta_0(k)) = -1/a, \quad (2.3)$$

while higher partial wave phase shifts  $\delta_L(k)$  disappear.

Scaling limit can be approached by tuning multiple parameters in the potential, such as the range of the potential and its depth. In this limit, the scattering length sets the scale for most low-energy observables. In the 2-body sector it is the only length scale but in 3-body sector observables can have logarithmic dependence on a second scale. In scaling limit, we can find infinitely many arbitrarily deep 3-body bound states with a spectrum of the form (2.2) but as  $n \rightarrow -\infty$  with  $l = 0$ . This spectrum is also characterised by parameter  $\kappa_*$  with dimensions of wave number [1].

Deep 3-body states do not have much effect on the low-energy physics we are interested in so we do not need to worry about the unboundedness from below of the spectrum form. We just have to remember that before taking the scaling limit, we had a natural low-energy length scale  $l$  and associated with this length scale is an energy scale we will refer to as the natural ultraviolet cutoff  $\hbar/ml^2$ . This means that any predictions involving energies  $|E|$  that are higher than the cutoff should be ignored at the scaling limit.

Scaling limit will be used here as a starting point for large scattering length atoms to describe them. Deviations from the scaling limit will be treated as perturbations. By taking numerical approximations, intricate correlations between 3-body observables associated with the Efimov effect can be easily lost when the scattering length is large. These correlations can be built exactly at high energy with this limit. These correlations are unphysical at high energy, but low-energy physics can be described accurately [1].

## 2.3 Hyperspherical coordinates

For the low-energy 3-body problem, we can use the hyperspherical formalism to derive the Efimov effect. The universal aspects of the 3-body problem are easiest to understand with hyperspherical coordinates which have a similar setup to Jacobi coordinates. A set of Jacobi coordinates consists of two vectors. The separation vector  $\mathbf{r}_{ij}$  between a pair of particles and another separation vector  $\mathbf{r}_{k,ij}$  which is between the third atom and the center-of-mass of the previous pair. For equal mass particles we have

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \quad (2.4)$$

and

$$\mathbf{r}_{k,ij} = \mathbf{r}_k - \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j). \quad (2.5)$$

For the hyperradius  $R$  which is the root-mean-square separation of the three atoms we have

$$R = \frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2) = \frac{1}{2}r_{ij}^2 + \frac{2}{3}r_{k,ij}^2. \quad (2.6)$$

We see that hyperradius is small if all the particles are close together. We will also need another coordinate since hyperradius can be the same for several different setups of particles. For this we have the Delves hyperangle,  $\alpha_k$ , defined by

$$\alpha_k = \arctan\left(\frac{\sqrt{3}r_{ij}}{2r_{k,ij}}\right). \quad (2.7)$$

The hyperangle is close to 0 when particle  $k$  is far from the center of mass of  $i$  and  $j$  and near  $\frac{\pi}{2}$  when  $k$  is close. We can express the separation vectors with hyperradius and hyperangle as

$$r_{ij} = \sqrt{2}R \sin \alpha_k, \quad (2.8)$$

and

$$r_{k,ij} = \sqrt{\frac{3}{2}}R \cos \alpha_k. \quad (2.9)$$

We also have the simple identity for hyperangle:

$$\sin^2 \alpha_1 + \sin^2 \alpha_2 + \sin^2 \alpha_3 = \frac{3}{2}. \quad (2.10)$$

The volume element for the Jacobi coordinates is

$$d^3r_{ij}d^3r_{k,ij} = \frac{3\sqrt{3}}{4}R^5 dR \sin^2(2\alpha_k) d\alpha_k d\Omega_{ij} d\Omega_{k,ij}, \quad (2.11)$$

where  $d\Omega_{ij}$  and  $d\Omega_{k,ij}$  are the differential solid angles for the unit vectors  $\hat{\mathbf{r}}_{ij}$  and  $\hat{\mathbf{r}}_{k,ij}$ .

## 2.4 The stationary wave function of three particles

We can write the Schrödinger equation for the stationary wave function  $\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  of three particles with mass  $m$  interacting through a potential  $V$

$$\left(-\frac{\hbar^2}{2m} \sum_{i=1}^3 \nabla_i^2 + V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\right) \Psi = E\Psi, \quad (2.12)$$

with the hyperradial coordinates as

$$\left(T_R + T_{\alpha_k} + \frac{\Lambda_{k,ij}^2}{2mR^2} + V(R, \Omega)\right) \Psi = E\Psi, \quad (2.13)$$



#### 2.4. THE STATIONARY WAVE FUNCTION OF THREE PARTICLES 7

where  $\Omega$  denotes the hyperangular variables: five dimensionless variables  $\alpha_k$ ,  $\hat{\mathbf{r}}_{ij}$  and  $\hat{\mathbf{r}}_{k,ij}$ . These five and the hyperradius  $R$  are chosen for the wave function which depends on 6 independent coordinates. Also if the potential  $V$  is translation invariant, it also depends only on 6 independent coordinates.

In the previous equation (2.13) we have three operators.  $T_R$ , which is the hyperradial kinetic energy operator

$$T_R = -\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial R^2} + \frac{5}{R} \frac{\partial}{\partial R} \right] = \frac{\hbar^2}{2m} R^{-5/2} \left[ -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right] R^{5/2}, \quad (2.14)$$

$T_{\alpha_k}$ , which is the hyperangle's kinetic energy operator

$$T_{\alpha_k} = -\frac{\hbar^2}{2mR^2} \left[ \frac{\partial^2}{\partial \alpha^2} + 4 \cot(2\alpha) \frac{\partial}{\partial \alpha} \right] = \frac{\hbar^2}{2mR^2} \frac{1}{\sin(2\alpha)} \left[ -\frac{\partial^2}{\partial \alpha^2} - 4 \right] \sin(2\alpha), \quad (2.15)$$

and  $\Lambda_{k,ij}^2$ , which is a generalized angular momentum operator with conventional angular momentum operators  $\mathbf{L}_{ij}$  and  $\mathbf{L}_{k,ij}$ :

$$\Lambda_{k,ij}^2 = \frac{\mathbf{L}_{ij}^2}{\sin^2(\alpha_k)} + \frac{\mathbf{L}_{k,ij}^2}{\cos^2(\alpha_k)}. \quad (2.16)$$

We can solve the Schrödinger equation (2.13) by using adiabatic hyperspherical representation, where for each value of  $R$ , the wave function  $\Psi(R, \Omega)$  is expanded in terms of a complete set of hyperangular functions  $\Phi_n(R, \Omega)$ :

$$\Psi(R, \Omega) = R^{-5/2} \sum_n f_n(R) \Phi_n(R, \Omega). \quad (2.17)$$

This gives us

$$\begin{aligned} & \left( T_R + T_{\alpha_k} + \frac{\Lambda_{k,ij}^2}{2mR^2} + V(R, \Omega) \right) R^{-5/2} \sum_n f_n(R) \Phi_n(R, \Omega) \\ &= ER^{-5/2} \sum_n f_n(R) \Phi_n(R, \Omega). \end{aligned} \quad (2.18)$$

Separating the radial operator from the rest, which are moved into the sum, we get

$$\begin{aligned} & T_R \left[ R^{-5/2} \sum_n f_n(R) \Phi_n(R, \Omega) \right] + R^{-5/2} \sum_n f_n(R) \left( T_{\alpha_k} + \frac{\Lambda_{k,ij}^2}{2mR^2} \right. \\ & \left. + V(R, \Omega) \right) \Phi_n(R, \Omega) = R^{-5/2} E \sum_n f_n(R) \Phi_n(R, \Omega). \end{aligned} \quad (2.19)$$

The non-radial operators can be simplified as the functions  $\Phi_n(R, \Omega)$  are solutions to a differential eigenvalue equation in the hyperangular variables.

$R$  is treated as a parameter and the eigenvalue  $V_n(R)$  is a function of that parameter. It can be interpreted as an effective potential for the channel associated with the hyperangular function  $\Phi_n$ :

$$\left[ T_{\alpha_k} + \frac{\Lambda_{k,ij}^2}{2mR^2} + V(R, \Omega) \right] \Phi_n(R, \Omega) = V_n(R) \Phi_n(R, \Omega). \quad (2.20)$$

Using this to equation (2.19) and replacing  $T_R$  with equation (2.14) in equation (2.19), we get

$$\begin{aligned} & \frac{\hbar^2}{2m} R^{-5/2} \left[ -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right] R^{5/2} \left[ R^{-5/2} \sum_n f_n(R) \Phi_n(R, \Omega) \right] \\ & + R^{-5/2} \sum_n f_n(R) V_n(R) \Phi_n(R, \Omega) = R^{-5/2} E \sum_n f_n(R) \Phi_n(R, \Omega). \end{aligned} \quad (2.21)$$

After some calculations, see A.1, this can be expressed as:

$$\begin{aligned} & \left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\ & - \frac{\hbar^2}{2m} \sum_m \left\{ 2 \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial}{\partial R} [\Phi_m(R, \Omega)] \frac{\partial}{\partial R} \right. \\ & \left. - \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial^2}{\partial R^2} [\Phi_m(R, \Omega)] \right\} f_m(R) \\ & = E f_n(R). \end{aligned} \quad (2.22)$$

Let us write the coupling potentials  $U_{nm}(R)$  and  $W_{nm}(R)$  as

$$U_{nm}(R) = -\frac{\hbar^2}{2m} \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial}{\partial R} [\Phi_m(R, \Omega)], \quad (2.23)$$

and

$$W_{nm}(R) = -\frac{\hbar^2}{2m} \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial^2}{\partial R^2} [\Phi_m(R, \Omega)]. \quad (2.24)$$

Using these we get

$$\begin{aligned} & \left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\ & + \sum_m \left[ 2U_{nm}(R) \frac{\partial}{\partial R} + W_{nm}(R) \right] f_m(R) = E f_n(R). \end{aligned} \quad (2.25)$$

The off-diagonal coupling potentials usually decrease much faster at large distances compared to the hyperspherical potentials [10]. In the low-energy limit, these off-diagonal terms in the above equation are small compared to

the diagonal terms. Here we can do the so called adiabatic hyperspherical approximation introduced by Macek in 1968 [11] and the equation reduces to

$$\left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) + 2U_{nn}(R) \frac{\partial}{\partial R} + W_{nn}(R) \right] f_n(R) = E f_n(R). \quad (2.26)$$

## 2.5 The Faddeev equations

To derive the Efimov states in the resonant limit, we have to take advantage of the simplifications that are associated with the 2-body cluster, that is well separated from the third atom. The 3-body Schrödinger equation does not do that, so we must look at the Faddeev equations which exploit these simplifications. We make approximations that can be justified at low energy and we also restrict the total angular momentum to zero [1].

We simplify the potential  $V_3$  as a sum of three 2-body potentials:

$$V_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V(r_{12}) + V(r_{23}) + V(r_{31}). \quad (2.27)$$

The Faddeev equations are a set of equations that generate solutions to the 3-body Schrödinger equation of the form [1]

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi^{(1)}(\mathbf{r}_{23}, \mathbf{r}_{1,23}) + \psi^{(2)}(\mathbf{r}_{31}, \mathbf{r}_{2,31}) + \psi^{(3)}(\mathbf{r}_{12}, \mathbf{r}_{3,12}), \quad (2.28)$$

and the Faddeev equations themselves are

$$\left( T_R + T_{\alpha_1} + \frac{\Lambda_{1,23}^2}{2mR^2} \right) \psi^{(1)} + V(r_{23})(\psi^{(1)} + \psi^{(2)} + \psi^{(3)}) = E\psi^{(1)}, \quad (2.29)$$

where  $\psi^{(n)}$  has been obtained with cyclic permutation of  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ .

We restrict the total angular momentum quantum number to  $L = 0$  and we also make an additional assumption that we can neglect any orbital angular momentum of the subsystems  $ij$  or  $k, ij$ . This reduces the Schrödinger equation wave function to

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi(R, \alpha_1) + \psi(R, \alpha_2) + \psi(R, \alpha_3), \quad (2.30)$$

and the Faddeev equations to

$$(T_R + T_{\alpha_1} - E) \psi(R, \alpha_1) + V(\sqrt{2}R \sin \alpha_1) [\psi(R, \alpha_1) + \psi(R, \alpha_2) + \psi(R, \alpha_3)] = 0. \quad (2.31)$$

These three equations can be reduced to a single equation, if we exploit the fact that the averages of  $\psi(R, \alpha_2)$  and  $\psi(R, \alpha_3)$  over the angular variables

$\hat{\mathbf{r}}_{23}$  and  $\hat{\mathbf{r}}_{1,23}$  can be expressed as an integral operator acting on  $\psi(R, \alpha_1)$ :

$$\langle \psi(R, \alpha_2) \rangle_{\hat{\mathbf{r}}_{23}, \hat{\mathbf{r}}_{1,23}} = \langle \psi(R, \alpha_3) \rangle_{\hat{\mathbf{r}}_{23}, \hat{\mathbf{r}}_{1,23}} = \frac{2}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha_1|}^{\frac{\pi}{2}-|\frac{\pi}{6}-\alpha_1|} \frac{\sin(2\alpha')}{\sin(2\alpha_1)} \psi(R, \alpha') d\alpha'. \quad (2.32)$$

This leads to the low-energy Faddeev equation.

### 2.5.1 The low-energy Faddeev equation

We start the derivation of Efimov states with the low-energy Faddeev equation, which is an integro-differential equation for  $\psi(R, \alpha)$ :

$$(T_R + T_\alpha - E)\psi(R, \alpha) = -V(\sqrt{2}R\sin\alpha) \left[ \psi(R, \alpha) + \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{\frac{\pi}{2}-|\frac{\pi}{6}-\alpha|} \frac{\sin(2\alpha')}{\sin(2\alpha)} \psi(R, \alpha') d\alpha' \right]. \quad (2.33)$$

Here we have assumed that we can neglect subsystem angular momentum. This is due to the general suppression of higher orbital angular momentum at low energies. This is a better approximation for the Faddeev equations than for the Schrödinger equation [1]. Similarly to the earlier solving of the Schrödinger equation, we use a hyperspherical expansion. For each value of  $R$ , the wave function  $\psi(R, \alpha)$  is expanded in a complete set of functions  $\phi_n(R, \alpha)$  of the hyperangle  $\alpha$ :

$$\psi(R, \alpha) = \frac{1}{R^{5/2}\sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha). \quad (2.34)$$

The divergence of  $1/\sin(2\alpha)$  at the endpoints  $\alpha = 0$  and  $\alpha = \frac{\pi}{2}$  imposes boundary conditions that  $\phi_n(R, \alpha)$  must vanish at the endpoints. Using this we can write (2.33) as

$$\begin{aligned} & (T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2}\sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\ = & -V(\sqrt{2}R\sin\alpha) \left[ \frac{1}{R^{5/2}\sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right. \\ & \left. + \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{\frac{\pi}{2}-|\frac{\pi}{6}-\alpha|} \frac{\sin(2\alpha')}{\sin(2\alpha)} \frac{1}{R^{5/2}\sin(2\alpha')} \sum_n f_n(R) \phi_n(R, \alpha') d\alpha' \right]. \quad (2.35) \end{aligned}$$

Cleaning up inside the integral and then moving the denominator and sum out of the brackets as common terms:

$$\begin{aligned}
& (T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
= & - V(\sqrt{2}R \sin \alpha) \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \left[ \phi_n(R, \alpha) \right. \\
& \left. + \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{|\frac{\pi}{2}-\frac{\pi}{6}-\alpha|} \phi_n(R, \alpha') d\alpha' \right]. \tag{2.36}
\end{aligned}$$

The functions  $\phi_n(R, \alpha)$  are solutions to an integro-differential eigenvalue equation in the single variable  $\alpha$ :

$$\begin{aligned}
\left[ -\frac{\partial^2}{\partial \alpha^2} - \lambda_n(R) \right] \phi_n(R, \alpha) = & - \frac{2mR^2}{\hbar^2} V(\sqrt{2}R \sin \alpha) \left[ \phi_n(R, \alpha) \right. \\
& \left. + \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{|\frac{\pi}{2}-\frac{\pi}{6}-\alpha|} \phi_n(R, \alpha') d\alpha' \right]. \tag{2.37}
\end{aligned}$$

Modifying equation (2.36) so that we can use the above equation we get

$$\begin{aligned}
& (T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
= & - \frac{\hbar^2}{2mR^2} \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \frac{2mR^2}{\hbar^2} V(\sqrt{2}R \sin \alpha) \left[ \phi_n(R, \alpha) \right. \\
& \left. + \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{|\frac{\pi}{2}-\frac{\pi}{6}-\alpha|} \phi_n(R, \alpha') d\alpha' \right]. \tag{2.38}
\end{aligned}$$

Now we use (2.37) and get

$$\begin{aligned}
& (T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
= & \frac{\hbar^2}{2mR^2} \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \left[ -\frac{\partial^2}{\partial \alpha^2} - \lambda_n(R) \right] \phi_n(R, \alpha). \tag{2.39}
\end{aligned}$$

The eigenvalues  $\lambda_n(R)$  in (2.36) define channel potentials for the hyperradial variable:

$$V_n(R) = [\lambda_n(R) - 4] \frac{\hbar^2}{2mR^2}. \tag{2.40}$$

Using this we get

$$(T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] = \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \left[ -\frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial \alpha^2} - V_n(R) - 4\frac{\hbar^2}{2mR^2} \right] \phi_n(R, \alpha). \quad (2.41)$$

This can be derived to be

$$\begin{aligned} & \left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\ & - \sum_m \left\{ 2\frac{\hbar^2}{2m} \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial}{\partial R} [\phi_m(R, \alpha)] \frac{\partial}{\partial R} \right. \\ & \left. + \frac{\hbar^2}{2m} \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial^2}{\partial R^2} [\phi_m(R, \alpha)] \right\} f_m(R) \\ & = E f_n(R), \end{aligned} \quad (2.42)$$

see A.2.

Let us write this with coupling potentials  $P_{nm}(R)$  and  $Q_{nm}(R)$  defined by

$$P_{nm}(R) = -\frac{\hbar^2}{2m} \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial}{\partial R} [\phi_m(R, \alpha)], \quad (2.43)$$

and

$$Q_{nm}(R) = -\frac{\hbar^2}{2m} \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial^2}{\partial R^2} [\phi_m(R, \alpha)]. \quad (2.44)$$

We finally get the coupled set of eigenvalue equations for the hyperradial wave functions  $f_n(R)$ :

$$\begin{aligned} & \left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\ & + \sum_m \left[ 2P_{nm}(R) \frac{\partial}{\partial R} + Q_{nm}(R) \right] f_m(R) = E f_n(R). \end{aligned} \quad (2.45)$$

The similarity with the radial equations for the 3-body Schrödinger equations is obvious, however the difference is that these channel potentials  $V_n(R)$  are obtained by solving integro-differential eigenvalue equations in only one variable, while for the 3-body Schrödinger equation they are obtained by solving differential eigenvalue equations in five hyperangular variables. This

difference in variables came from the assumption of neglecting subsystem angular momenta [1].

The adiabatic hyperspherical approximation consists of neglecting the off-diagonal terms in (2.45), in which case the eigenvalue equations decouple. If the diagonal coupling terms are also neglected, the equations reduce to radial Schrödinger equations for each of the hyperspherical potentials:

$$\left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \approx E f_n(R). \quad (2.46)$$

## 2.6 The channel eigenvalues $\lambda_n(R)$ and $s_0$

If the 2-body potential  $V(r)$  in equation (2.37) disappears, as a result we have a simple differential eigenvalue equation to solve:

$$\left[ -\frac{\partial^2}{\partial \alpha^2} - \lambda_n(R) \right] \phi_n(R, \alpha) = 0. \quad (2.47)$$

We solve this by using a characteristic equation of above:

$$-r^2 - \lambda_n(R) = 0. \quad (2.48)$$

Multiplying by -1 and then using the quadratic formula we get for  $r$

$$r = \frac{\pm \sqrt{-4\lambda_n(R)}}{2} = \frac{\pm 2i\sqrt{\lambda_n(R)}}{2} = \pm i\sqrt{\lambda_n(R)}. \quad (2.49)$$

This gives us for  $\phi_n(R, \alpha)$

$$\phi_n(R, \alpha) = C_1 \cos \left[ \sqrt{\lambda_n(R)} \alpha \right] + C_2 \sin \left[ \sqrt{\lambda_n(R)} \alpha \right]. \quad (2.50)$$

We can solve  $C_1$  and  $C_2$  with the endpoints  $\alpha = 0$  and  $\alpha = \frac{\pi}{2}$ , where  $\phi_n(R, \alpha)$  must vanish. For  $\alpha = 0$  we have

$$\phi_n(R, 0) = C_1 \cos 0 + C_2 \sin 0 = 0. \quad (2.51)$$

This leads to  $C_1 = 0$ . For  $\alpha = \frac{\pi}{2}$  we now have

$$\phi_n \left( R, \frac{\pi}{2} \right) = C_2 \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right] = 0. \quad (2.52)$$

Now  $C_2 \neq 0$  so the sine must be zero. This leads to

$$\sqrt{\lambda_n(R)} \frac{\pi}{2} = n\pi, \quad (2.53)$$

where  $n$  is a positive integer. We solve this for  $\lambda_n(R)$  and we get

$$\lambda_n(R) = 4n^2. \quad (2.54)$$

Due to the fact that  $\lambda_0(R)$ , the lowest channel eigenvalue, can not be zero, we write  $n \rightarrow n + 1$ ,  $n = 0, 1, 2, \dots$ :

$$\lambda_n(R) = 4(n + 1)^2. \quad (2.55)$$

This gives us the corresponding eigenfunctions, since we can choose  $C_2$  as we wish, except for 0. We choose  $C_2 = 1$  and we get

$$\phi_n(R, \alpha) = \sin [2(n + 1)\alpha]. \quad (2.56)$$

Using  $\lambda_n(R)$  we get the corresponding hyperspherical potentials from equation (2.40):

$$V_n(R) = [4(n + 1)^2 - 4] \frac{\hbar^2}{2mR^2}. \quad (2.57)$$

Moving 4 out as a common term and opening the brackets

$$V_n(R) = 4[n^2 + 2n + 1 - 1] \frac{\hbar^2}{2mR^2}, \quad (2.58)$$

and moving  $n$  out as a common term we get the final form for the potentials

$$V_n(R) = 4n(n + 2) \frac{\hbar^2}{2mR^2}. \quad (2.59)$$

### 2.6.1 Short-ranged potential

If we have a case where the 2-body potential  $V(r)$  is short-ranged, we get two regions in which the integro-differential eigenvalue equation (2.37) can be solved analytically for the angular function  $\Phi(R, \alpha)$ . First region, where  $R \sin \alpha$  is large enough that the  $V(R \sin \alpha)$ -term is small compared to  $\lambda_n(R)$ . In this region, we can dismiss the  $R^2V$  term and the equation reduces to

$$\left[ -\frac{\partial^2}{\partial \alpha^2} - \lambda_n(R) \right] \phi_n^{(\text{hi})}(R, \alpha) \approx 0. \quad (2.60)$$

This has a similar solution to (2.56), except now we have to take into account that  $\lambda_n(R)$  can not be expressed like before due to  $V(r)$  actually existing which means  $\lambda_n(R)$  will properly have a dependency of  $R$  in it. We can bypass the problem easily at the endpoint  $\alpha = \frac{\pi}{2}$  with solution

$$\phi_n^{(\text{hi})}(R, \alpha) \approx \sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \alpha \right) \right]. \quad (2.61)$$

The second region that can be solved analytically is when  $R \sin \alpha$  is small enough that  $\lambda_n(R)$  is small compared to the  $V(R \sin \alpha)$ -term. In this region



we can neglect  $\lambda_n$ . When  $\alpha \ll 1$ , which leads to  $\sin \alpha \approx \alpha$ , (2.37) reduces to

$$\begin{aligned} & -\frac{\partial^2}{\partial \alpha^2} \phi_n^{(\text{lo})}(R, \alpha) \approx -\frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \left[ \phi_n^{(\text{lo})}(R, \alpha) \right. \\ & \left. + \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{\frac{\pi}{2}-|\frac{\pi}{6}-\alpha|} \phi_n^{(\text{hi})}(R, \alpha') d\alpha' \right], \end{aligned} \quad (2.62)$$

where we have used  $\phi_n^{(\text{hi})}(R, \alpha)$  for the upper endpoint integral. Reorganizing this we get

$$\begin{aligned} & \left[ -\frac{\partial^2}{\partial \alpha^2} + \frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \right] \phi_n^{(\text{lo})}(R, \alpha) \\ & \approx -\frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{\frac{\pi}{2}-|\frac{\pi}{6}-\alpha|} \phi_n^{(\text{hi})}(R, \alpha') d\alpha'. \end{aligned} \quad (2.63)$$

From the integral, see A.3, we get

$$\frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right). \quad (2.64)$$

We put this back to equation (2.63) and we get

$$\left[ -\frac{\partial^2}{\partial \alpha^2} + \frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \right] \phi_n^{(\text{lo})}(R, \alpha) \approx -\frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right). \quad (2.65)$$

This has a solution combined from a particular solution and a general solution to the homogeneous equation. For the particular solution we try

$$\phi_n^{(\text{lo})}(R, \alpha) = C \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right). \quad (2.66)$$

We get

$$\begin{aligned} & \left[ -\frac{\partial^2}{\partial \alpha^2} + \frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \right] C \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right) \\ & = -\frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right), \end{aligned} \quad (2.67)$$

which quickly reduces to

$$-\frac{\partial^2}{\partial \alpha^2} (C) \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right) + C \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right) = -\frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right). \quad (2.68)$$

We see that if  $C = -\frac{8\alpha}{\sqrt{3}}$  the derivative part disappears and leaves us with the right side. So our particular solution is

$$\phi_n^{(\text{lo})}(R, \alpha) = -\frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right). \quad (2.69)$$

The homogeneous equation is of the form

$$\left[ -\frac{\partial^2}{\partial \alpha^2} + \frac{2mR^2}{\hbar^2} V(\sqrt{2}R\alpha) \right] \phi_n^{(\text{lo})}(R, \alpha) \approx 0, \quad (2.70)$$

which can be expressed in the form

$$\left[ -\frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial \alpha^2} + V(\sqrt{2}R\alpha) \right] \phi_n^{(\text{lo})}(R, \alpha) \approx 0. \quad (2.71)$$

This is identical to the radial Schrödinger equation for a pair of particles:

$$\left[ -\frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} + V(r) \right] \psi(r) = E\psi(r), \quad (2.72)$$

with zero energy interacting through the 2-body potential  $V(r)$ , where  $r = \sqrt{2}R\alpha$ . If we use  $\psi_0(r)$  for the zero-energy solution, the most general solution to equation (2.65) is the particular solution and homogeneous solution put together:

$$\phi_n^{(\text{lo})}(R, \alpha) \approx A(R)\psi_0(\sqrt{2}R\alpha) - \frac{8\alpha}{\sqrt{3}}\phi_n^{(\text{hi})}\left(R, \frac{\pi}{3}\right). \quad (2.73)$$

## 2.6.2 The consistency equation

We have only one assumption of the potential  $V_3$  so far, which is that the potential can be expressed as the sum of three 2-body potentials each of which depends only on the separation  $r_{ij}$  of a pair of atoms:

$$V_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V(r_{12}) + V(r_{23}) + V(r_{31}). \quad (2.74)$$

We now use the low-energy Faddeev equation (2.33) to the problem of a 2-body potential with large scattering length  $|a| \gg l$ . We will follow the derivation by Fedorov and Jensen [12]. We have the approximations for the hyperangular functions  $\Phi_n(R, \alpha)$  given in equations (2.61) and (2.73) in their respective regions of  $R$  and  $\alpha$ . When scattering length is large, the high- $\alpha$  solution in (2.61) holds for  $\alpha \ll 1$  and  $R\alpha \gg l$ . The zero-energy solution  $\psi_0(r)$  in (2.73) is the limit for  $R\alpha \gg l$  of the atom-atom scattering solution in equation

$$\psi_0(\mathbf{r}) = \frac{1}{k} \sin [kr + \delta_0(k)]. \quad (2.75)$$

At low energies  $\delta_0(k) = -ak$ :

$$\psi_0(\mathbf{r}) = \frac{1}{k} \sin (kr - ak), \quad (2.76)$$

and since  $k \rightarrow 0$  we have  $\sin(kr - ak) = (kr - ak)$  and we get

$$\psi_0(\mathbf{r}) = \frac{kr - ak}{k}, \quad (2.77)$$

which reduces to

$$\psi_0(\mathbf{r}) = r - a. \quad (2.78)$$

Now we insert this into equation (2.73) we get

$$\phi_n^{(lo)}(R, \alpha) \approx A(R) \left( \sqrt{2}R\alpha - a \right) - \frac{8\alpha}{\sqrt{3}} \phi_n^{(hi)} \left( R, \frac{\pi}{3} \right). \quad (2.79)$$

This is compatible with the high- $\alpha$  solution (2.61), so we can match their values at  $\alpha = 0$  to determine the unknown function  $A(R)$ :

$$A(R) (-a) = \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right], \quad (2.80)$$

which is

$$A(R) = -\frac{1}{a} \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right]. \quad (2.81)$$

We obtain the derivatives with respect to  $\alpha$  at  $\alpha = 0$  of both sides, starting with low- $\alpha$  for which we have:

$$\frac{\partial \phi_n^{(lo)}(R, \alpha)}{\partial \alpha} = -\frac{1}{a} \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right] \sqrt{2}R - \frac{8}{\sqrt{3}} \phi_n^{(hi)} \left( R, \frac{\pi}{3} \right). \quad (2.82)$$

After inserting  $\phi_n^{(hi)}$  into this we get

$$\frac{\partial \phi_n^{(lo)}(R, \alpha)}{\partial \alpha} = -\frac{\sqrt{2}R}{a} \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right] - \frac{8}{\sqrt{3}} \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{6} \right]. \quad (2.83)$$

Now for the high- $\alpha$  we have:

$$\frac{\partial \phi_n^{(hi)}(R, \alpha)}{\partial \alpha} = -\sqrt{\lambda_n(R)} \cos \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \alpha \right) \right]. \quad (2.84)$$

Inserting  $\alpha = 0$  we get

$$\frac{\partial \phi_n^{(hi)}(R, 0)}{\partial \alpha} = -\sqrt{\lambda_n(R)} \cos \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right]. \quad (2.85)$$

Now we equate the derivatives of the high- $\alpha$  and the low- $\alpha$ :

$$-\frac{\sqrt{2}R}{a} \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right] - \frac{8}{\sqrt{3}} \sin \left[ \sqrt{\lambda_n(R)} \frac{\pi}{6} \right] = -\sqrt{\lambda_n(R)} \cos \left[ \sqrt{\lambda_n(R)} \frac{\pi}{2} \right]. \quad (2.86)$$

Moving the first term to the right and the cosine to the left side, then dividing with  $\lambda_n(R)^{1/2}$  and writing the square roots as exponents we get the consistency equation that determines the channel eigenvalues  $\lambda_n(R)$ . This equation was first derived by Efimov [13] and it is

$$\cos\left(\lambda^{1/2}\frac{\pi}{2}\right) - \frac{8}{\sqrt{3}}\lambda^{-1/2}\sin\left(\lambda^{1/2}\frac{\pi}{6}\right) = \sqrt{2}\lambda^{-1/2}\sin\left(\lambda^{1/2}\frac{\pi}{2}\right)\frac{R}{a}. \quad (2.87)$$

There are infinitely many solutions for the channel eigenvalues  $\lambda_n(R)$  at each value of  $R$ . The corresponding hyperangular wave functions are

$$\phi_n(R, \alpha) = \sin\left[\lambda_n^{1/2}(R)\left(\frac{\pi}{2} - \alpha\right)\right]. \quad (2.88)$$

However this solution does not satisfy the boundary condition  $\phi_n(R, \alpha) \rightarrow 0$  as  $\alpha \rightarrow 0$ , because the zero-energy solution in (2.78) is accurate only for  $R\alpha \gg l$ . The above equation is not accurate around  $R\alpha \sim l$ . The equation can still be used to calculate quantities that are insensitive enough to short distances.

The consistency equation (2.87) has a constant solution at  $\lambda(R) = 16$ , however it is unphysical. The hyperangular wave function would be  $\phi(R, \alpha) = \sin(4\alpha)$  and the resulting Faddeev wave function would be proportional to  $\cos(2\alpha)$ . This solution to the low-energy Faddeev equation would correspond to Schrödinger wave function given by

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi(R, \alpha_1) + \psi(R, \alpha_2) + \psi(R, \alpha_3), \quad (2.89)$$

and would be the trivial solution  $\Psi = 0$  [1].

We will need to solve the eigenvalues  $\lambda_n(R)$  numerically from the consistency equation (2.87). The eigenvalues approach constants independent of  $a$ . The limiting behaviour of the lowest eigenvalue as  $R \rightarrow 0$  is

$$\lambda_0(R) \rightarrow -s_0^2 \left(1 + 1.897\frac{R}{a}\right), \quad (2.90)$$

where  $s_0 = 1.00624$  and is the numerical solution to the transcendental equation

$$s_0 \cosh \frac{\pi s_0}{2} = \frac{8}{\sqrt{3}} \sinh \frac{\pi s_0}{6}. \quad (2.91)$$

## 2.7 The scaling-violation parameter $\Lambda_0$

For  $R \ll |a|$ , the eigenvalues approach constant values  $\lambda_n(0)$ , so the adiabatic hyperspherical approximation in (2.46) can be made. This eigenvalue equation then reduces to

$$\frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial R^2} + \frac{\lambda_n(0) - \frac{1}{4}}{R^2} \right] f_n(R) = E f_n(R), \quad (2.92)$$

in the  $R \ll |a|$ -region. In this region, all the channel potentials with the exception of  $V_0(R)$  are repulsive  $1/R^2$ -potentials [1]. The hyperspherical wave functions  $f_n(R)$  for  $n \geq 1$  thus decrease exponentially as  $R \rightarrow 0$ , and no boundary conditions are required as no information is lost. However, the potential  $V_0(R)$  is an attractive potential for  $R \ll |a|$ :

$$V_0(R) \approx -(4 + s_0^2) \frac{\hbar^2}{2mR^2}. \quad (2.93)$$

This too simple potential causes (2.92) to not have a well-behaved solution as  $R \rightarrow 0$ . This results from the scaling limit. While we took the limit we have lost information about the boundary condition at  $R \rightarrow 0$  provided by the 2-body potential  $V(r_{ij})$  at short distances.

We can fix this problem at short distances by thinking it as equivalent to choosing a matching point  $R_0$  and specifying the dimensionless number  $R_0 f'(R_0)/f(R_0)$ , where  $R_0$  is a hyperradius where (2.92) is accurate and the logarithmic derivatives just mentioned match. We can choose this matching point to lie in the scale-invariant region  $l \ll R_0 \ll |a|$ . We write  $\lambda_0(n)$  as terms of  $s_0$  by solving it from

$$V_n(R) \rightarrow [\lambda_n(0) - 4] \frac{\hbar^2}{2mR^2}, \quad (2.94)$$

which leads to

$$\lambda_n(0) \approx \frac{2mR^2}{\hbar^2} V_n(R) + 4. \quad (2.95)$$

Now we impose that  $V_n(R)$  equals to  $V_0(R)$  and we get

$$\lambda_n(0) = \frac{2mR^2}{\hbar^2} (-4 - s_0^2) \frac{\hbar^2}{2mR^2} + 4 = -4 - s_0^2 + 4 = -s_0^2. \quad (2.96)$$

Inserting this to equation (2.92) and also choosing  $R_0 \ll (m|E|1/\hbar^2)^{-1/2}$ , energy eigenvalue  $E$  can be neglected relative to the channel potential:

$$\frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial R^2} - \frac{s_0^2 + \frac{1}{4}}{R^2} \right] f_0(R) \approx 0. \quad (2.97)$$

For this we have a general solution

$$f_0(R) \approx AR^{\frac{1}{2}+is_0} + BR^{\frac{1}{2}-is_0}, \quad R \ll |a|, \quad 1/\kappa, \quad (2.98)$$

where  $A$  and  $B$  are constants. This can be expressed in another form, see A.4:

$$f_0(R) \approx AR^{\frac{1}{2}} \sin [s_0 \ln (\kappa R) + \theta]. \quad (2.99)$$

The phase  $\theta$  is determined by matching to the solution of the problem at long distances  $R \sim |a|$ . It depends on  $a$  and on the energy  $E$  and since it's

dimensionless, it can depend only on the dimensionless combination  $\kappa a$ , and on the signs of  $E$  and  $a$ .

We prove that  $f_0(R)$  is a solution of (2.97). We start by calculating the derivative of  $f_0(R)$ :

$$\begin{aligned} \frac{\partial f_0(R)}{\partial R} &= \frac{1}{2}AR^{-\frac{1}{2}}\sin [s_0 \ln (\kappa R) + \theta] + AR^{\frac{1}{2}}\frac{s_0\kappa}{\kappa R}\cos [s_0 \ln (\kappa R) + \theta] \\ &= AR^{-\frac{1}{2}}\left(\frac{1}{2}\sin [s_0 \ln (\kappa R) + \theta] + s_0 \cos [s_0 \ln (\kappa R) + \theta]\right). \end{aligned} \quad (2.100)$$

The second derivative is:

$$\begin{aligned} \frac{\partial^2 f_0(R)}{\partial R^2} &= -\frac{1}{2}AR^{-\frac{3}{2}}\left(\frac{1}{2}\sin [s_0 \ln (\kappa R) + \theta] + s_0 \cos [s_0 \ln (\kappa R) + \theta]\right) \\ &\quad + AR^{-\frac{1}{2}}\left(\frac{1}{2}\frac{s_0\kappa}{\kappa R}\cos [s_0 \ln (\kappa R) + \theta] - \frac{s_0^2\kappa}{\kappa R}\sin [s_0 \ln (\kappa R) + \theta]\right) \\ &= AR^{-\frac{3}{2}}\left(-\frac{1}{4} - s_0^2\right)\sin [s_0 \ln (\kappa R) + \theta]. \end{aligned} \quad (2.101)$$

Next we calculate the second term inside the brackets of equation (2.97) with  $f_0(R)$ :

$$\begin{aligned} -\frac{s_0^2 + \frac{1}{4}}{R^2}f_0(R) &= -\frac{s_0^2 + \frac{1}{4}}{R^2}AR^{\frac{1}{2}}\sin [s_0 \ln (\kappa R) + \theta] \\ &= AR^{-\frac{3}{2}}\left(-\frac{1}{4} - s_0^2\right)\sin [s_0 \ln (\kappa R) + \theta], \end{aligned} \quad (2.102)$$

which is the same as the second derivative of  $f_0(R)$ .

### 2.7.1 The matching point problem

Now we calculate the logarithmic derivative at the matching point which was mentioned before:

$$R_0 \frac{f'_0(R_0)}{f_0(R_0)} = R_0 \frac{\frac{1}{2}AR_0^{-\frac{1}{2}}\sin [s_0 \ln (\kappa R_0) + \theta] + AR_0^{-\frac{1}{2}}s_0 \cos [s_0 \ln (\kappa R_0) + \theta]}{AR_0^{\frac{1}{2}}\sin [s_0 \ln (\kappa R_0) + \theta]}.$$

Combining  $R_0^{1/2}$  and  $R_0^{-1/2}$  and taking  $R_0$  out from the denominator and also calculating the first term we get

$$R_0 \frac{f'_0(R_0)}{f_0(R_0)} = R_0 \frac{1}{R_0} \left\{ \frac{1}{2} + s_0 \frac{\cos [s_0 \ln (\kappa R_0) + \theta]}{\sin [s_0 \ln (\kappa R_0) + \theta]} \right\}.$$

Cleaning it up we get

$$R_0 \frac{f'_0(R_0)}{f_0(R_0)} = \frac{1}{2} + s_0 \cot [s_0 \ln (\kappa R_0) + \theta]. \quad (2.103)$$

We can separate the dependence on  $R_0$  inside the cotangent from the dependence on  $E$  and  $a$ . It seems that the wave function at distances  $R \gg l$  depends only on a function of the matching point  $R_0$  and the logarithmic derivative [1]. We get it from the above equation as we solve it for  $\kappa$ . Moving the terms from the right side to the left and dividing both sides with  $s_0$ :

$$\cot [s_0 \ln (\kappa R_0) + \theta] = \frac{1}{s_0} \left( R_0 \frac{f'_0(R_0)}{f_0(R_0)} - \frac{1}{2} \right). \quad (2.104)$$

Taking arccot on both sides and then dividing again with  $s_0$ :

$$\ln (\kappa R_0) + \frac{\theta}{s_0} = \frac{1}{s_0} \operatorname{arccot} \left[ \frac{1}{s_0} \left( R_0 \frac{f'_0(R_0)}{f_0(R_0)} - \frac{1}{2} \right) \right]. \quad (2.105)$$

Then using them as powers of  $e$ :

$$e^{\frac{\theta}{s_0}} \kappa R_0 = \exp \left\{ \frac{1}{s_0} \operatorname{arccot} \left[ \frac{1}{s_0} \left( R_0 \frac{f'_0(R_0)}{f_0(R_0)} - \frac{1}{2} \right) \right] \right\}, \quad (2.106)$$

and dividing with  $R_0$  and we get  $\Lambda_0$ , which we define to be  $e^{\frac{\theta}{s_0}} \kappa$ :

$$\Lambda_0 = e^{\frac{\theta}{s_0}} \kappa = \frac{1}{R_0} \exp \left\{ \frac{1}{s_0} \operatorname{arccot} \left[ \frac{1}{s_0} \left( R_0 \frac{f'_0(R_0)}{f_0(R_0)} - \frac{1}{2} \right) \right] \right\}. \quad (2.107)$$

The parameter  $\Lambda_0$  is a scaling-violation parameter and has dimensions of wave number.

Logarithmic scaling violations can affect the low-energy observables only through their dependence on this parameter. With this we can name the matching point  $R_0$  as the short-distance cutoff. At longer distances, the effects of smaller  $R$  are taken into account through the value of  $R_0 f'(R_0)/f(R_0)$ . All low-energy observables in the 3-body sector are determined either by specifying the cutoff  $R_0$  and the number  $R_0 f'(R_0)/f(R_0)$  or by specifying the scaling-violation parameter  $\Lambda_0$ . This phenomenon of a dimensionless short-distance parameter and a short-distance cutoff being equivalent to a dimensional long-distance parameter is known as dimensional transmutation [1].

## 2.8 The Efimov energy states in the resonant limit

Now we derive the Efimov effect in the resonant limit  $a \rightarrow \pm\infty$ . This derivation of the Efimov effect within the hyperspherical formalism was first done by Macek [14] in 1986. In the resonant limit, the adiabatic hyperspherical approximation is accurate at all finite values of  $R$ . It is very accurate for

bound states for which hyperradial wave functions  $f_n(R)$  fall exponentially as  $R \rightarrow \infty$  [1]. The only channel to support bound states is the only attractive one, the  $n = 0$ -channel. This channel has the previously calculated eigenvalue  $\lambda_0(R) = -s_0^2$ . The Schrödinger wave function in the center-of-mass frame reduces to

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = R^{-5/2} f_0(R) \sum_{i=1}^3 \frac{\sinh [s_0(\frac{\pi}{2} - \alpha_i)]}{\sin(2\alpha_i)}. \quad (2.108)$$

We can justify the adiabatic hyperspherical approximation with the fact that the channel eigenvalue is a constant. The equation (2.92) reduces to something really close to equation (2.97) but has an energy term in it:

$$\frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial R^2} - \frac{s_0^2 + \frac{1}{4}}{R^2} \right] f_0(R) = E f_0(R). \quad (2.109)$$

Like before, the boundary condition at short distances is specified by a matching point  $R_0$  and the logarithmic derivative  $R_0 f'(R_0)/f(R_0)$  or if  $R_0$  is in the scale-invariant region, the boundary condition can be specified by the 3-body parameter  $\lambda_0$  defined by (2.107). If an Efimov state has a binding energy  $E_T$ , a binding wave number  $\kappa$  can be defined by

$$E_T = \frac{\hbar^2 \kappa^2}{m}. \quad (2.110)$$

The solution to hyperradial equation (2.46) that decreases exponentially as  $R \rightarrow \infty$  is

$$f_0(R) = R^{1/2} K_{is_0}(\sqrt{2}\kappa R), \quad (2.111)$$

where  $K_z$  is a Bessel function with an imaginary index. The boundary condition at short distances determines the discrete spectrum of binding energies  $E_T^{(n)}$ . In the region  $\kappa R \ll 1$  the solution in above equation reduces to

$$f_0(R) \rightarrow - \left( \frac{\pi}{s_0 \sinh(\pi s_0)} \right)^{\frac{1}{2}} R^{\frac{1}{2}} \sin [s_0 \ln(\kappa R) + \alpha_0], \quad (2.112)$$

where angle  $\alpha_0$  is

$$\alpha_0 = -\frac{1}{2} s_0 \ln 2 - \frac{1}{2} \arg \frac{\Gamma(1 + is_0)}{\Gamma(1 - is_0)}. \quad (2.113)$$

For insertion into (2.107), we calculate  $R_0 \frac{f'_0(R_0)}{f_0(R_0)}$  beforehand for convenience.  $f'_0(R)$  is

$$\begin{aligned} f'_0(R) = & - \frac{1}{2} \left( \frac{\pi}{s_0 \sinh(\pi s_0)} \right)^{\frac{1}{2}} R^{-\frac{1}{2}} \sin [s_0 \ln(\kappa R) + \alpha_0] \\ & - \left( \frac{\pi}{s_0 \sinh(\pi s_0)} \right)^{\frac{1}{2}} R^{\frac{1}{2}} \frac{s_0 \kappa}{\kappa R} \cos [s_0 \ln(\kappa R) + \alpha_0]. \end{aligned} \quad (2.114)$$



Taking similar terms and combining them gives us

$$f'_0(R) = - \left( \frac{\pi}{s_0 \sinh(\pi s_0)} \right)^{\frac{1}{2}} \times R^{-\frac{1}{2}} \left\{ \frac{1}{2} \sin [s_0 \ln(\kappa R) + \alpha_0] + s_0 \cos [s_0 \ln(\kappa R) + \alpha_0] \right\}. \quad (2.115)$$

Inserting this into  $R_0 \frac{f'_0(R_0)}{f_0(R_0)}$  we get

$$R_0 \frac{- \left( \frac{\pi}{s_0 \sinh(\pi s_0)} \right)^{\frac{1}{2}} R_0^{-\frac{1}{2}} \left\{ \frac{1}{2} \sin [s_0 \ln(\kappa R_0) + \alpha_0] + s_0 \cos [s_0 \ln(\kappa R_0) + \alpha_0] \right\}}{- \left( \frac{\pi}{s_0 \sinh(\pi s_0)} \right)^{\frac{1}{2}} R_0^{\frac{1}{2}} \sin [s_0 \ln(\kappa R_0) + \alpha_0]}. \quad (2.116)$$

Cleaning it up we get

$$R_0 \frac{f'_0(R_0)}{f_0(R_0)} = \frac{1}{2} + s_0 \cot [s_0 \ln(\kappa R_0) + \alpha_0]. \quad (2.117)$$

Inserting  $R_0 \frac{f'_0(R_0)}{f_0(R_0)}$  into (2.107) we get

$$\begin{aligned} \Lambda_0 &= e^{\frac{\alpha}{s_0} \kappa} \\ &= \frac{1}{R_0} \exp \left\{ \frac{1}{s_0} \operatorname{arccot} \left[ \frac{1}{s_0} \left( \frac{1}{2} + s_0 \cot [s_0 \ln(\kappa R_0) + \alpha_0] - \frac{1}{2} \right) \right] \right\} \\ &= \frac{1}{R_0} \exp \left\{ \frac{1}{s_0} \operatorname{arccot} \{ \cot [s_0 \ln(\kappa R_0) + \alpha_0] \} \right\}. \end{aligned} \quad (2.118)$$

We multiply both sides by  $R_0$ :

$$R_0 \Lambda_0 = \exp \left\{ \frac{1}{s_0} \operatorname{arccot} \{ \cot [s_0 \ln(\kappa R_0) + \alpha_0] \} \right\}, \quad (2.119)$$

take the natural logarithm from both sides

$$\ln(R_0 \Lambda_0) = \frac{1}{s_0} \operatorname{arccot} \{ \cot [s_0 \ln(\kappa R_0) + \alpha_0] \}, \quad (2.120)$$

and multiply by  $s_0$  and the equation for  $\Lambda_0$  has reduced to

$$s_0 \ln(R_0 \Lambda_0) = \operatorname{arccot} \{ \cot [s_0 \ln(\kappa R_0) + \alpha_0] \}. \quad (2.121)$$

We solve this equation for  $\kappa$ . We can get rid of  $\operatorname{arccot}(\cot)$  by defining  $-\frac{\pi}{2} + n_*\pi < s_0 \ln(\kappa R_0) + \alpha_0 < \frac{\pi}{2} + n_*\pi$  and adding  $n\pi$ , where  $n$  is an integer. We now have

$$s_0 \ln(R_0 \Lambda_0) + n_*\pi = [s_0 \ln(\kappa R_0) + \alpha_0] + n\pi. \quad (2.122)$$

Dividing both sides with  $-s_0$  and moving  $n_*$  term to the right and logarithm term to the left gives us

$$\ln(\kappa R_0) - \ln(R_0 \Lambda_0) = -\frac{\alpha_0}{s_0} - (n - n_*) \frac{\pi}{s_0}. \quad (2.123)$$

We combine the logarithms and get

$$\ln\left(\frac{\kappa}{\Lambda_0}\right) = -\frac{\alpha_0}{s_0} - (n - n_*) \frac{\pi}{s_0}. \quad (2.124)$$

Using both sides of above as the exponent of e and using  $e^{\ln x} = x$  we get

$$\frac{\kappa}{\Lambda_0} = e^{-\frac{\alpha_0}{s_0} - (n - n_*) \frac{\pi}{s_0}}. \quad (2.125)$$

Multiplying both sides with  $\Lambda_0$  and separating exponents of e. We also use  $\kappa^{(n)}$  for this:

$$\kappa^{(n)} = (e^{-\frac{\pi}{s_0}})^{n - n_*} e^{-\frac{\alpha_0}{s_0}} \Lambda_0. \quad (2.126)$$

We define  $\kappa_*$  as the binding wave number for the Efimov state labeled by  $n = n_*$ :

$$\kappa^{(n)} = (e^{-\frac{\pi}{s_0}})^{n - n_*} \kappa_*. \quad (2.127)$$

Going back to equation (2.110) with  $\kappa^{(n)}$  we have

$$E_T^{(n)} = \frac{\hbar^2 (\kappa^{(n)})^2}{m}, \quad (2.128)$$

into which we insert (2.127)

$$E_T^{(n)} = \frac{\hbar^2 \left[ (e^{-\frac{\pi}{s_0}})^{n - n_*} \kappa_* \right]^2}{m}. \quad (2.129)$$

We clean it up a bit and we get

$$E_T^{(n)} = (e^{-\frac{2\pi}{s_0}})^{n - n_*} \frac{\hbar^2 \kappa_*^2}{m}. \quad (2.130)$$

This spectrum is geometric, with binding energies of Efimov states having the ratio  $e^{\frac{2\pi}{s_0}} \approx 515.03$  between each other. The relation between  $\kappa_*$  and  $\Lambda_0$  can be expressed in another form, easily seen from comparing equations (2.123) and (2.127). It is

$$s_0 \ln(\kappa_*) = s_0 \ln(\Lambda_0) - \alpha_0, \text{ mod } \pi. \quad (2.131)$$

This relation is defined only up to a multiplicative factors of  $e^{\frac{\pi}{s_0}} \approx 22.7$ . This is due to the definitions that involve the integers  $n$  and  $n_*$ .

The spectrum in (2.130) is the result only if we take the scaling ( $l \rightarrow 0$ ) and resonant ( $a \rightarrow \pm\infty$ ) limits at the same time. If we stay in the resonant limit, but not in the scaling limit, the 3-body spectrum will be bounded from below [1]. The deepest Efimov state will then be at  $n = 0$  and its binding energy is comparable to the natural ultraviolet cutoff:

$$E_T^{(0)} \sim \frac{\hbar^2}{ml^2}. \quad (2.132)$$

Power-law scaling violations will also appear and give corrections to the binding energies. The corrections scale as  $\kappa^{(n)}l$ , where  $\kappa^{(n)}$  is the binding wave number of the Efimov state. As the binding energy  $E_T^{(n)}$  goes to zero, the corrections go to zero. This satisfies

$$E_T^{(n)} = (e^{-\frac{2\pi}{s_0}})^{n-n_*} \frac{\hbar^2 \kappa_*^2}{m} \text{ as } n \rightarrow \infty \text{ with } a = \pm\infty. \quad (2.133)$$

There are infinitely many Efimov states, with an accumulation point at zero energy and an asymptotic discrete scaling symmetry with discrete scaling factor  $e^{\frac{\pi}{s_0}} \approx 22.7$ .

### 2.8.1 Efimov states near the atom-dimer threshold

Resonant limit  $a = \pm\infty$  contains infinitely many Efimov trimers but for any finite value of  $a$  there are only finitely many trimers. These Efimov states near the atom-dimer threshold can be understood as 2-body systems, which contain an atom and a dimer. For almost all of these states,  $a$  needs a critical positive value at which the state appears below the atom-dimer threshold and a critical negative value at which it disappears through the 3-atom threshold as shown in fig. 3.1 [1].



# Chapter 3

## Effective field theories

We will now define and use effective field theories for numerical modelling of Efimov effect related problems. We will follow Eric Braaten's and H. -W. Hammer's derivation of the formulas [1].

### 3.1 Discrete scaling symmetry

There is a nontrivial discrete scaling symmetry in which  $\kappa_*$  is fixed and  $a$  and variables such as energy  $E$  are scaled by integer powers of the discrete scaling factor  $\lambda_0 = e^{\pi/s_0} \approx 22.7$ . This leads to:

$$\kappa_* \rightarrow \kappa_*, a \rightarrow \lambda_0^m a, E \rightarrow \lambda_0^{-2m} E, \quad (3.1)$$

where  $m$  is an integer. Dimensional analysis suggests that observables such as binding energies and cross sections, scale with the integer powers of  $\lambda_0$ . For 3-body problem, we will use an energy variable  $K$  defined by

$$K = \text{sign}(E)(m|E|/\hbar^2)^{1/2}. \quad (3.2)$$

The possible low-energy 3-body states in the scaling limit can be identified with points in the  $(a^{-1}, K)$  plane for a given  $\kappa_*$ . In fig. 3.1 we can see the plane for three identical bosons in the scaling limit and the three possible states: 3-atom scattering states, atom-dimer scattering states and Efimov trimers. Efimov trimers cross the vertical axis at the points  $K = -(e^{-\pi/s_0})^{n-n_*} \kappa_*$ . Any given physical system has a specific scattering length and is represented by a vertical line in the plane. The part above the scattering threshold is the continuum of points representing 3-atom and atom-dimer scattering states. Each crossing with a trimer-line means that there is an Efimov trimer. If we change  $a$ , it means that we sweep a vertical line across the field horizontally. Resonant limit would mean that we tune the vertical line to the vertical axis. Looking at fig. 3.1, if we change  $1/a$  continuously from large negative value to large positive one, we'll be sweeping across the horizontal axis. Trimers

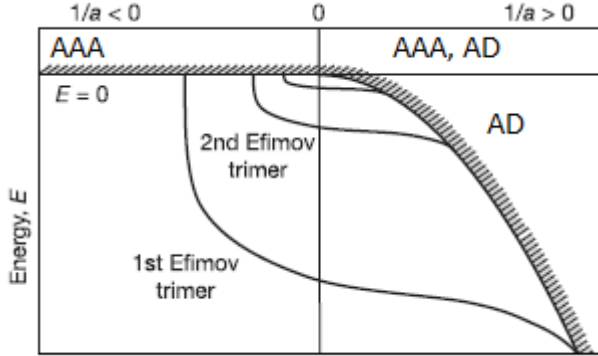


Figure 3.1: Efimov states for resonant two-body interaction. Here the binding energy is plotted as a function of the inverse two-body scattering length. AAA and AD mark the regions for 3-atom scattering states and atom-dimer scattering states. The cross-hatching is the threshold for scattering states. Here we depict a few trimer states while in reality there are infinitely many Efimov trimers as we approach the origo [3][1]

continue to appear at the atom-dimer threshold at positive critical values of  $1/a$ . This continues until there is infinitely many at  $1/a = 0$ . For negative values of  $a$  as  $1/a$  increases in magnitude, the trimers disappear one by one through the threshold at negative critical values of  $1/a$  that differ by the multiples of  $e^{\pi/s_0}$  [1].

## 3.2 Local effective field theory

We can use effective theories in order to approach low-energy physical systems. The one required here will be the effective field theory. This theory is very important in modern elementary particle physics and has two main branches of applications. First one involves the systematic development of various low-energy approximations to the Standard Model and the other one involves treating the Standard Model itself as a low-energy approximation to something like a unified field theory or string theory [1].

To use the theory we can simply replace  $V(\mathbf{r})$  by an effective potential  $V_{eff}(\mathbf{r}; c_1)$ . This effective potential needs to be identical to  $V(\mathbf{r})$  when  $r > r_0$  and its form for  $r < r_0$  involves an adjustable parameter  $c_1$ . The value of this parameter must be tuned to reproduce the scattering amplitude at threshold. This results in the Schrödinger equation with  $V_{eff}(\mathbf{r}; c_1)$  reproducing all the low-energy observables with errors that are linear in  $E$  [1].

The aspects of low-energy physics that can be represented as local quan-

tum field theories, like the Hamiltonian

$$\hat{H} = \int d^3r \frac{\hbar^2}{2m} \nabla \psi^\dagger \cdot \nabla \psi + \frac{1}{2} \int d^3r \int d^3r' \psi^\dagger(\mathbf{r}) V(|\mathbf{r} - \mathbf{r}'|) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}), \quad (3.3)$$

are most useful. This Hamiltonian which generates the time evolution of the quantum field can be expressed as the integral of a Hamiltonian density that depends only on the quantum field  $\psi$  and its gradients at the same point:

$$\hat{H}_{eff} = \int d^3r H_{eff}, \quad (3.4)$$

where

$$H_{eff} = \frac{\hbar^2}{2m} \nabla \psi^\dagger \cdot \nabla \psi + \mu \psi^\dagger \psi + \frac{g_2}{4} (\psi^\dagger \psi)^2 + \frac{h_2}{4} \nabla(\psi^\dagger \psi) \cdot \nabla(\psi^\dagger \psi) + \frac{g_3}{36} (\psi^\dagger \psi)^3 + \dots \quad (3.5)$$

There are infinitely many terms that can appear here and we can reduce the number of possible terms by introducing symmetries. If the  $\hat{H}$  has a symmetry, it can be imposed on the effective Hamiltonian. If we have a term with  $n$  factors of both  $\psi$  and  $\psi^\dagger$  is an  $n$ -body term. Only the two first terms of the effective Hamiltonian are not forbidden by a Galilean symmetry. Every other 2-body term is forbidden. For higher  $n$ -body terms we ignore Galilean symmetry as it is a much more complicated case. We can drop the term  $\psi^\dagger \nabla^2 \psi$  because it is equivalent to the term  $\nabla \psi^\dagger \cdot \nabla \psi$ , since terms that differ by integration by parts are equivalent since their difference integrates to a boundary term. Also terms with  $n$  factors of  $\psi^\dagger \psi$  affect only systems with  $n$  or more particles. In the equation above the written out terms are considering 3-body problem where we only need to consider 2-body and 3-body terms and the higher terms that will not have any effect have been truncated. Additional factors of  $\nabla$  will be suppressed by additional powers of the energy  $E$  thus we only need terms with up to  $2m$  factors of  $\nabla$ , which gives us predictions of the fundamental Hamiltonian up to errors that scale like  $E^{m+1}$ . The above equation with the written out terms is enough to describe 2-body observables unto errors that scale as  $E^2$  and 3-body observables unto errors that scale as  $E$  [1].

The coefficients  $g_2$ ,  $h_2$  and  $g_3$  in the equation (3.5) are coupling constants used as tuning parameters to reproduce low-energy observables.  $g_2$  can be tuned to reproduce the scattering length in the 2-body contact interaction,  $g_2$  and  $h_2$  can be used together to reproduce the scattering length and the effective range of 2-body potential.  $g_3$  might be necessary to tune for the 3-body contact interaction to reproduce low-energy 3-body scattering amplitudes or the desired value of the 3-body parameter  $\kappa_*$ . The advantage of using  $g_2$  and  $g_3$  is that the latter has no effect on the 2-body sector, thus it is possible to first tune  $g_2$  to get the desired  $a$  by calculating a 2-body observable and then tune  $g_3$  to get  $\kappa_*$  from a 3-body observable.

For the local quantum field theory to be well defined, we will need an ultraviolet cutoff. Usually this will be a cut off on the wave numbers of atoms that can appear in virtual states:  $|\mathbf{k}| < \Lambda$ . This cutoff will also affect the values of the coefficients. Now we do not need to be careful about specifying the ordering of the quantum field operators in the Hamiltonian density. A change in operator-ordering can be compensated by a change in the coupling constants.

### 3.3 Two-body problem

We will solve a 2-body problem with the local effective field theory to give insight for the 3-body problem and show the general gist of it. We will be using two identical bosons with large scattering length  $a$  which can be described by a local quantum field theory with just a 2-body contact interaction term. We will also use the Lagrangian instead of Hamiltonian due to convenience and we will also set  $\hbar = 1$  and  $m = 1$ . The Lagrangian can be simply acquired by a Legendre transformation from the Hamiltonian and the Lagrange density we will be using is

$$L = \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi - \frac{g_2}{4} (\psi^\dagger \psi)^2. \quad (3.6)$$

To get the field theory to describe atoms with scattering length  $a$ , the coupling constant  $g_2$  must be adjusted as a function of the ultraviolet cutoff  $\Lambda$ .

#### 3.3.1 The perturbation theory

Using perturbation theory and calculating the effects of the interaction term in equation (3.6) as a power series in  $g_2$ , we can get the effective field theory to describe scattering states of two atoms. Once renormalized, the scattering amplitude coincides with the expansion of the universal scattering amplitude in

$$f_k(\theta) = \frac{1}{-1/a - ik}, \quad (3.7)$$

in powers of  $ka$ . If we calculate the effects nonperturbatively, we get the same expression as before and on top of that the effective field theory also describes bound states with binding energy given by

$$E_D = \frac{\hbar^2}{ma^2}. \quad (3.8)$$

This means that this effective field theory reproduces all the universal low-energy observables of the 2-body problem with large scattering length [1].



### 3.3.2 The $2 \rightarrow 2$ off-shell amplitude

The physical information is encoded in the truncated connected Green's function in momentum space, denoted by  $iA$ . We can get this by subtracting the terms that have the factored form  $\langle 0 | \mathcal{T}(\psi\psi^\dagger) | 0 \rangle \langle 0 | \mathcal{T}(\psi\psi^\dagger) | 0 \rangle$ , Fourier transforming in all coordinates, factoring out an overall energy-momentum conserving delta function and also factoring out propagators associated with each of the four external legs. In the factored form above,  $|0\rangle$  is the vacuum state and  $\mathcal{T}$  represents time-ordering [1].

$A$  is called the  $2 \rightarrow 2$  off-shell amplitude. It depends on the energies and momenta of the four external lines and the reason for its name is that the energy  $p_0$  of an external line with momentum  $\mathbf{p}$  does not need to be equal to its physical value  $p^2/2$ . The T-matrix element  $T$  for a  $2 \rightarrow 2$  scattering process is obtained by evaluating  $A$  at the on-shell point where  $p_0$  is set to  $p^2/2$  for every external momentum  $\mathbf{p}$ . Looking at the center-of-mass frame, we have the 2 incoming momenta  $+\mathbf{k}$  and  $-\mathbf{k}$  and 2 outgoing momenta  $+\mathbf{p}$  and  $-\mathbf{p}$ . Now  $A$  will depend on  $\mathbf{k}$ ,  $\mathbf{p}$  and the 4 off-shell energies. We can express our case with Feynman diagrams, shown in figure 3.2.

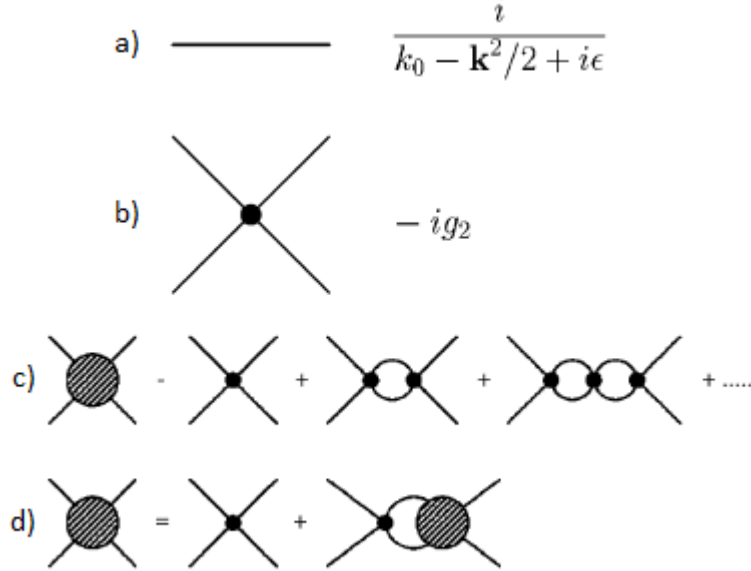


Figure 3.2: The Feynman rules and the diagrammatic equations for the  $2 \rightarrow 2$  off-shell amplitude: a) the propagator for an atom with energy  $k_0$  and momentum  $\mathbf{k}$ , b) the vertex for the 2-body contact interaction, c) the perturbative expansion in  $g_2$  and d) the integral equation [1]

When the only interaction is the 2-body contact interaction, the amplitude can depend only on the total momentum and the total off-shell energy  $E$ . In the center-of-mass frame, it is a function of  $E$  only, so we will denote

it by  $A(E)$ . On-shell point corresponds to setting  $E = 2(k^2/2) = 2(p^2/2)$ , which requires  $p = k$ . At low energies,  $A(E)$  encodes all information about the 2-body system [1]. We can get the T-matrix element for atoms of momenta  $\pm\mathbf{k}$  to scatter into atoms of momenta  $\pm\mathbf{k}'$  with  $|\mathbf{k}'| = |\mathbf{k}| = k$  and it is

$$T(k) = A(E = k^2). \quad (3.9)$$

The scattering amplitude  $f_k(\theta)$  for atoms with momenta  $\pm\mathbf{k}$  to scatter through an angle  $\theta$  is proportional to  $A(E)$  at the on-shell point and it is

$$f_k(\theta) = \frac{1}{8\pi} A(E = k^2). \quad (3.10)$$

As  $k \rightarrow 0$ , the limit of the scattering amplitude determines the scattering length  $a$  at  $A(0)$  as

$$a = -\frac{1}{8\pi} A(0). \quad (3.11)$$

Now the contact interaction in equation (3.6) is ill-defined and we will need an ultraviolet cutoff on the momenta. We can write down the off-shell amplitude for 2-body scattering at second order in perturbation theory to represent this:

$$A(E) \approx -g_2 - \frac{i}{2} g_2^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{dq_0}{2\pi} \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{1}{E - q_0 - q^2/2 + i\epsilon} + \dots \quad (3.12)$$

We can evaluate the integral over  $q_0$

$$\int \frac{dq_0}{2\pi} \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{1}{E - q_0 - q^2/2 + i\epsilon}, \quad (3.13)$$

by using contour integration, see A.5. This gives us the new form for equation (3.12):

$$A(E) \approx -g_2 - \frac{1}{2} g_2^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{E - q^2 + i2\epsilon} + \dots \quad (3.14)$$

This integral can be dimensionally regularised since the integral over  $\mathbf{q}$  is divergent[15]. Since we have a 3-dimensional system,  $D = 3$ . We can do the following:

$$\int \frac{d^Dq}{(2\pi)^D} = \frac{S_D}{(2\pi)^D} \int_0^\infty dq q^{D-1}, \quad (3.15)$$

where

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \quad (3.16)$$

so we get

$$S_3 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\Gamma(\frac{1}{2} + 1)} = \frac{2\pi\sqrt{\pi}}{\frac{2!}{4}\sqrt{\pi}} = 4\pi. \quad (3.17)$$

Using this to (3.14) we get

$$A(E) \approx -g_2 - \frac{1}{2}g_2^2 \frac{4\pi}{(2\pi)^3} \int_0^\infty dq \frac{q^2}{E - q^2 + i2\epsilon} + \dots, \quad (3.18)$$

which leads to

$$A(E) \approx -g_2 - \frac{1}{2}g_2^2 \frac{1}{2\pi^2} \int_0^\infty dq \frac{q^2}{-(q^2 - E - i2\epsilon)} + \dots. \quad (3.19)$$

Bringing -1 out from the denominator in the integral:

$$A(E) \approx -g_2 + \frac{1}{4\pi^2}g_2^2 \int_0^\infty dq \frac{q^2}{q^2 - E - i2\epsilon} + \dots. \quad (3.20)$$

We will use an ultraviolet cut-off,  $\Lambda$ , to prevent the integral from going into infinity. We will again use contour integration to solve the integral part of the equation, see A.6. Solving the integral gives us

$$A(E) \approx -g_2 + \frac{g_2^2}{4\pi^2} \frac{1}{2} \left( 2\Lambda - \pi\sqrt{-E - i\epsilon} \right) + \dots, \quad (3.21)$$

and cleaning it up we get

$$A(E) \approx -g_2 + \frac{g_2^2}{4\pi^2} \left( \Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon} \right) + \dots. \quad (3.22)$$

### 3.3.3 Renormalized $2 \rightarrow 2$ off-shell amplitude

For this, we will want to eliminate the parameter  $g_2$  in favour of scattering length  $a$  which we get with

$$a = \frac{1}{8\pi}A(0). \quad (3.23)$$

Inserting our approximation of  $A(E)$  with value  $E = 0$  we get

$$a \approx -\frac{1}{8\pi} \left[ -g_2 + \frac{g_2^2}{4\pi} \left( \Lambda - \frac{\pi}{2}\sqrt{-i\epsilon} \right) + \dots \right]. \quad (3.24)$$

Moving out the common term  $g_2$  we get

$$a \approx \frac{g_2}{8\pi} \left[ 1 - \frac{g_2}{4\pi}\Lambda + \frac{g_2}{8}\sqrt{-i\epsilon} + \dots \right]. \quad (3.25)$$

We set  $\epsilon \rightarrow 0$ , so the last term with  $\sqrt{-i\epsilon}$  in it vanishes and we get

$$a \approx \frac{g_2}{8\pi} \left( 1 - \frac{g_2\Lambda}{4\pi^2} + \dots \right). \quad (3.26)$$

We solve this for  $g_2$ . We start by multiplying both sides with  $8\pi$  and using Mclaurin series for the parenthesis. Simply written it says that for  $(1-x)^{-1}$  we have a series  $1+x+x^2+x^3+\dots$ , but we cut this series from the second order. Using this we get

$$8\pi a \approx g_2 \left(1 + \frac{g_2 \Lambda}{4\pi^2}\right)^{-1}. \quad (3.27)$$

Multiplying with the parenthesis:

$$8\pi a \left(1 + \frac{g_2 \Lambda}{4\pi^2}\right) \approx g_2. \quad (3.28)$$

Opening the parenthesis results in

$$8\pi a + \frac{2a\Lambda g_2}{\pi} \approx g_2. \quad (3.29)$$

Moving terms with  $g_2$  to the same side:

$$g_2 - \frac{2a\Lambda g_2}{\pi} \approx 8\pi a. \quad (3.30)$$

Taking  $g_2$  as the similar term and forming a parenthesis gives us

$$g_2 \left(1 - \frac{2a\Lambda}{\pi}\right) \approx 8\pi a. \quad (3.31)$$

Dividing with the parenthesis we get

$$g_2 \approx 8\pi a \left(1 - \frac{2a\Lambda}{\pi}\right)^{-1}, \quad (3.32)$$

to which we can again use the Mclaurin series. We end up with

$$g_2 \approx 8\pi a \left(1 + \frac{2a\Lambda}{\pi} + \dots\right). \quad (3.33)$$

We insert this into equation (3.22) and get

$$A(E) \approx -8\pi a - 16a^2\Lambda + \frac{1}{4\pi^2} (8\pi a + 16a^2\Lambda)^2 \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right) + \dots. \quad (3.34)$$

We now calculate the second power of the first parenthesis:

$$\begin{aligned} A(E) \approx & - 8\pi a - 16a^2\Lambda + \frac{1}{4\pi^2} (64\pi^2 a^2 \\ & + 256\pi a^3\Lambda + 256a^4\Lambda^2) \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right) + \dots, \end{aligned} \quad (3.35)$$

moving the  $\frac{1}{4\pi^2}$  into the parenthesis:

$$A(E) \approx -8\pi a - 16a^2\Lambda + \left(16a^2 + \frac{64}{\pi}a^3\Lambda + \frac{64}{\pi^2}a^4\Lambda^2\right) \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right) + \dots, \quad (3.36)$$

and getting rid of both parentheses we get

$$\begin{aligned} A(E) \approx & -8\pi a - 16a^2\Lambda + 16a^2\Lambda - 8\pi a^2\sqrt{-E - i\epsilon} + \frac{64}{\pi}a^3\Lambda^2, \\ & - 32a^3\Lambda\sqrt{-E - i\epsilon} + \frac{64}{\pi^2}a^4\Lambda^3 - \frac{32}{\pi}a^4\Lambda^2\sqrt{-E - i\epsilon} + \dots. \end{aligned} \quad (3.37)$$

We dismiss all the terms which have powers higher than 2 of  $a$ . This leads to

$$A(E) \approx -8\pi a - 8\pi a^2\sqrt{-E - i\epsilon} + \dots. \quad (3.38)$$

Finally we get the renormalised expression for the amplitude after taking the common term  $-8\pi a$  in front:

$$A(E) \approx -8\pi a \left(1 + a\sqrt{-E - i\epsilon} + \dots\right). \quad (3.39)$$

### 3.3.4 Nonperturbative $2 \rightarrow 2$ off-shell amplitude

If we are interested in observables involving energy  $E \sim 1/a^2$ , then we must solve the problem nonperturbatively [1]. We can solve the following integral equation to get the exact expression for the amplitude:

$$A(E) = -g_2 + \frac{i}{2}g_2 \int \frac{d^3q}{(2\pi)^3} \int \frac{dq_0}{2\pi} \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{1}{E - q_0 - q^2/2 + i\epsilon} A(E). \quad (3.40)$$

Now the function  $A(E)$  is independent of  $\mathbf{q}$  and  $q_0$  and can be taken out of the integrals. The integrals we already solved earlier so we can reduce the previous equation into form

$$A(E) = -g_2 - \frac{g_2}{4\pi^2} \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right) A(E), \quad (3.41)$$

placing both  $A(E)$ -terms to the left side and taking it as a common term we get

$$\left[1 + \frac{g_2}{4\pi^2} \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right)\right] A(E) = -g_2, \quad (3.42)$$

and dividing both sides with the  $A(E)$ 's multiplier gives us

$$A(E) = -g_2 \left[1 + \frac{g_2}{4\pi^2} \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right)\right]^{-1}. \quad (3.43)$$

### 3.3.5 Renormalized nonperturbative $2 \rightarrow 2$ off-shell amplitude

The equation for the nonperturbative  $2 \rightarrow 2$  off-shell amplitude (3.43) depends on  $g_2$  in the Lagrangian and on the ultraviolet cutoff  $\Lambda$ . We can renormalise this by removing  $g_2$  and replacing it with the scattering length  $a$ . Again using equation (3.23) we have

$$a = -\frac{1}{8\pi}A(0) = -\frac{1}{8\pi} \left\{ -g_2 \left[ 1 + \frac{g_2}{4\pi^2} \left( \Lambda - \frac{\pi}{2} \sqrt{-i\epsilon} \right) \right]^{-1} \right\}. \quad (3.44)$$

After opening the first and third set of brackets we get

$$a = \frac{g_2}{8\pi} \left( 1 + \frac{g_2}{4\pi^2} \Lambda - \frac{g_2}{8\pi} \sqrt{-i\epsilon} \right)^{-1}. \quad (3.45)$$

To simplify this, we again drop the term with  $\sqrt{-i\epsilon}$  and get the nonperturbative expression for the scattering length:

$$a = \frac{g_2}{8\pi} \left( 1 + \frac{g_2}{4\pi^2} \Lambda \right)^{-1}. \quad (3.46)$$

Equation (3.46) shows that the scattering length diverges when  $g_2$  is tuned to the critical value  $-4\pi^2/\Lambda$ . This comes from the involvement of virtual particles with wave numbers less than  $\Lambda$  [1]. This generates the term  $g_2\Lambda/4\pi$  in the denominator. The scattering length is not an analytic function of  $g_2$  at this point and a small change in  $g_2$  can produce a large change in  $a$ .

We solve (3.46) for  $g_2$ . We multiply both sides with the denominator:

$$\left( 1 + \frac{g_2}{4\pi^2} \Lambda \right) a = \frac{g_2}{8\pi}, \quad (3.47)$$

opening the parenthesis we get

$$a + \frac{g_2}{4\pi^2} \Lambda a - \frac{g_2}{8\pi} = 0, \quad (3.48)$$

taking the common term  $g_2$ :

$$\left( \frac{\Lambda a}{4\pi^2} - \frac{1}{8\pi} \right) g_2 = -a, \quad (3.49)$$

and dividing with the  $g_2$ 's multiplier and then cleaning up by taking out common terms we end up with

$$g_2 = -a \left( \frac{\Lambda a}{4\pi^2} - \frac{1}{8\pi} \right)^{-1} = 8\pi a \left( 1 - \frac{2a\Lambda}{\pi} \right)^{-1}. \quad (3.50)$$

For a fixed ultraviolet cutoff, this tells us how the parameter  $g_2$  must be tuned to get the correct scattering length  $a$ . For  $\Lambda \gg 1/|a|$  the coupling constant  $g_2$  is always negative not depending on  $a$ .

The formula in (3.50) shows that a large coupling constant is not necessarily such a bad thing in a nonperturbative field theory. The expression for  $g_2$  diverges as  $\Lambda \rightarrow \pi/2a$ . The effects of the arbitrarily large coupling constant  $g_2$  must be compensated by equally large effects from the involvement of virtual particles. This is because physical observables are independent of  $\Lambda$  [1].

Eliminating  $g_2$  from equation (3.43) with the tuning equation (3.50) we get

$$A(E) = -8\pi a \left(1 - \frac{2a\Lambda}{\pi}\right)^{-1} \times \left[1 + \frac{2a}{\pi} \left(1 - \frac{2a\Lambda}{\pi}\right)^{-1} \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right)\right]^{-1}. \quad (3.51)$$

We put the first fraction inside the brackets:

$$A(E) = -8\pi a \left[ \left(1 - \frac{2a\Lambda}{\pi}\right) + \left(1 - \frac{2a\Lambda}{\pi}\right) \frac{2a}{\pi} \left(1 - \frac{2a\Lambda}{\pi}\right)^{-1} \left(\Lambda - \frac{\pi}{2}\sqrt{-E - i\epsilon}\right) \right]^{-1}, \quad (3.52)$$

and cancel terms out and opening up parenthesis to get

$$A(E) = -8\pi a \left[1 - \frac{2a\Lambda}{\pi} + \frac{2a\Lambda}{\pi} - a\sqrt{-E - i\epsilon}\right]^{-1}. \quad (3.53)$$

Again terms cancel out each other and we write this in a proper fraction form:

$$A(E) = \frac{-8\pi a}{1 - a\sqrt{-E - i\epsilon}}, \quad (3.54)$$

which with a bit of organising signs gives the following for the nonperturbative off-shell amplitude:

$$A(E) = \frac{8\pi}{-\frac{1}{a} + \sqrt{-E - i\epsilon}}. \quad (3.55)$$

### 3.3.6 Dimensionless 2-body coupling constant

The field theory takes into account particles with wave numbers less than the ultraviolet cutoff  $\Lambda$ . We think of the cutoff as a fixed momentum scale

which is large enough compared to the wave numbers we're interested in. This however excludes particles with greater wave numbers. These wave numbers have effects in virtual 2-body states that are taken into account with parameter  $g_2$ , the strength of the 2-body contact interaction [1].

We take a look at the renormalisation of this field theory from a renormalisation group perspective. Considering a dimensionless combination of the coupling constant  $g_2$  and the ultraviolet cutoff  $\Lambda$ :

$$\hat{g}_2(\Lambda) = \frac{\Lambda g_2}{4\pi^2}. \quad (3.56)$$

As  $\Lambda$  is varied with  $a$  fixed, the expression for  $\hat{g}_2$  maps out a renormalizing group (RG) trajectory. With a given scattering length, the RG trajectories for these all represent the same physical theory.

Solving for  $g_2$  we get

$$g_2 = \frac{4\pi^2 \hat{g}_2(\Lambda)}{\Lambda}, \quad (3.57)$$

which we then place equal to the  $g_2$  in equation (3.50)

$$g_2 = \frac{4\pi^2 \hat{g}_2(\Lambda)}{\Lambda} = 8\pi a \left(1 - \frac{2a\Lambda}{\pi}\right)^{-1}. \quad (3.58)$$

We multiply both sides with  $\frac{4\pi^2}{\Lambda}$  and solving for  $\hat{g}_2(\Lambda)$  we get

$$\hat{g}_2(\Lambda) = \frac{2\Lambda a}{\pi} \frac{1}{1 - \frac{2\Lambda a}{\pi}} = \frac{2a\Lambda}{\pi - 2a\Lambda} = \frac{2a\Lambda}{-2(a\Lambda - \pi/2)}, \quad (3.59)$$

which leads to the dimensionless coupling constant

$$\hat{g}_2(\Lambda) = -\frac{a\Lambda}{a\Lambda - \pi/2}. \quad (3.60)$$

The dimensionless coupling constant flows towards an ultraviolet fixed point  $\hat{g}_2(\Lambda) \rightarrow -1$  as  $\Lambda \rightarrow \infty$ .

### 3.3.7 The fixed point theory

Using equation (3.46), we can identify the fixed-point theory as the 2-body problem in the resonant limit  $a \rightarrow \pm\infty$ . This fixed point corresponds to taking both limits, the resonant and scaling limits, at the same time. This is because the scaling limit  $l \rightarrow 0$  is implicit in our formulation. As  $\Lambda \rightarrow \infty$  all the RG trajectories are focused towards this point. This points to the fact that as energy scale becomes increasingly larger compared to  $1/a^2$ ,  $a$  will behave more like the resonant limit [1].



There is a natural ultraviolet cutoff  $\Lambda \sim 1/l$  for each physical system beyond which the physics can no longer be reproduced by a 2-body contact interaction. For  $\Lambda > 1/l$ , the behaviour of the system becomes more complicated and it no longer focuses towards the fixed point.

We take a deeper look into this problem by expressing the RG flow in terms of a differential equation for the  $\Lambda$ -dependence of  $\hat{g}_2$ . By differentiating both sides of equation, one can derive the differential RG equation:

$$\frac{d}{d\Lambda} \hat{g}_2(\Lambda) = \frac{d}{d\Lambda} \left( -\frac{a\Lambda}{a\Lambda - \pi/2} \right), \quad (3.61)$$

which by using the product rule becomes

$$\frac{d}{d\Lambda} \hat{g}_2 = -\frac{d(a\Lambda)}{d\Lambda} \frac{1}{a\Lambda - \pi} - a\Lambda \frac{d}{d\Lambda} \left( \frac{1}{a\Lambda - \pi/2} \right). \quad (3.62)$$

We carry out the derivations to get

$$\frac{d}{d\Lambda} \hat{g}_2 = -\frac{a}{a\Lambda - \pi} + \frac{a^2\Lambda}{(a\Lambda - \pi/2)^2}, \quad (3.63)$$

multiplying both sides with  $\Lambda$  and then inserting  $\hat{g}_2$  defined by (3.60) we get

$$\Lambda \frac{d}{d\Lambda} \hat{g}_2 = -\frac{a\Lambda}{a\Lambda - \pi} + \frac{a^2\Lambda^2}{(a\Lambda - \pi/2)^2} = \hat{g}_2 + \hat{g}_2^2, \quad (3.64)$$

and cleaning it up into its final form:

$$\Lambda \frac{d}{d\Lambda} \hat{g}_2 = \hat{g}_2(1 + \hat{g}_2). \quad (3.65)$$

We can see that this has two fixed points:  $\hat{g}_2 = -1$  and  $\hat{g}_2 = 0$ . The first point is the resonant limit  $a \rightarrow \infty$  and the second point  $\hat{g}_2 = 0$  corresponds to the noninteracting system with  $a = 0$ .

The 2-body system is scale invariant at these two fixed points. Usually continuous scaling symmetry defined by

$$a \rightarrow \lambda a, \quad E \rightarrow \lambda^{-2} E, \quad (3.66)$$

is a mapping of the theory onto another theory with a different scattering length. At the fixed points  $a = \pm\infty$  and  $0$ , the continuous scaling symmetry maps the theory onto itself. The case when the fixed point is at  $a = 0$  is trivial because there are no interactions. The  $a = \pm\infty$  fixed point has nontrivial scale invariance. The 2-body system at this point is invariant under a larger group of symmetry transformations so it's not affected by a transformation [1].

### 3.4 Three-body problem

After solving the 2-body problem, we move on to the 3-body problem. We again describe the problem of three identical bosons with large scattering length in the scaling limit with only interaction terms for 2-body and 3-body contact interactions. The Lagrangian density in the local quantum field theory for this is

$$L = \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi - \frac{g_2}{4} (\psi^\dagger \psi)^2 - \frac{g_3}{36} (\psi^\dagger \psi)^3. \quad (3.67)$$

In addition to the  $g_2$  we now have  $g_3$  for 3-body contact interactions. If the interaction terms in the above equation are treated perturbatively, the 3-body sector of the quantum field theory describes 3-atom scattering states.

We need to solve the quantum field theory nonperturbatively if we wish to handle observables involving energies of order  $1/ma^2$ . All information about physical observables in the 3-body sector is held in the 6-point Greens function  $\langle 0 | \mathcal{T}(\psi\psi\psi\psi^\dagger\psi^\dagger\psi^\dagger) | 0 \rangle$  [1]. It is kept in an even more compact form in the truncated connected 6-point Greens function in momentum space which we will denote by  $A$ . In the center of mass frame,  $A$  is a function of 4 momentum vectors and 5 off-shell energies. One can in principle solve the quantum field theory by solving an integral equation for  $A$ .

We can anticipate some of the features that would appear in the nonperturbative solution. There will be 3-atom scattering states, scattering states consisting of an atom and a dimer whose binding energy is  $E_D = 1/a^2$  and there must also be sequence of 3-body bound states with binding energies that range from order  $1/a^2$  to order  $\Lambda^2$ . The number of these Efimov states can be regulated with the ultraviolet cutoff imposed on the loop momenta, but it should be roughly  $\ln(|a|\Lambda)/\pi$  for asymptotically large  $\Lambda$ . These bound states must all appear from the nonperturbative effects held in the simple Lagrangian in equation [1].

#### 3.4.1 The diatom field trick

Bedaque, Hammer and van Kolck introduced an effective field theory that shows the connection between the 3-body problem with a large scattering length and renormalisation group limit cycles [16][17]. Using this effective field theory, it is very easy to derive the Skorniakov-Ter-Martirosian (STM) equation, a simple integral equation for the 3-body problem in the scaling limit. With this we can describe many of the aspects of the 3-body problem with a simple integral equation. There are two features Bedaque, Hammer and van Kolck exploited. First, the most interesting aspects of this problem appear in the sector with total angular momentum quantum number  $L = 0$

and second, the fact that the 2-body problem for this quantum field theory can be solved analytically.

The STM integral equation is not for the 6-point Green's function but instead it is an equation for the 4-point Green's function  $\langle 0 | \mathcal{T}(d\psi d^\dagger \psi^\dagger) | 0 \rangle$ , where the diatom field  $d$  is a local operator that replaces two operators for two atoms at a point [1]. The diatom field is basically just the quantum field operator  $\psi^2$ . Since we wish to get the simplest possible integral equation, we need to involve the field  $d$  and express a new formulation with it. The Lagrangian for the three-boson system used by Bedaque, Hammer and van Kolck is

$$L_{BHvK} = \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi + \frac{g_2}{4} d^\dagger d - \frac{g_2}{4} \left[ d^\dagger \psi^2 + (\psi^\dagger)^2 d \right] - \frac{g_3}{36} d^\dagger d \psi^\dagger \psi. \quad (3.68)$$

There are two important features in this Lagrangian. Firstly, there is no direct 2-body contact interaction term  $(\psi^\dagger \psi)^2$ . Secondly, there are also no time derivatives acting on the diatom field  $d$ , so it is not a dynamical field independent from  $\psi$ . Also the Lagrangian has only first and second order terms in  $d$ . We can eliminate  $d$  from the equation by varying  $d^\dagger$  and setting the resulting additional  $\sigma$ -terms to zero.

Starting from variation  $d^\dagger \rightarrow d^\dagger + \sigma$  we get

$$\psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi + \frac{g_2}{4} (d^\dagger + \sigma) d - \frac{g_2}{4} \left[ (d^\dagger + \sigma) \psi^2 + (\psi^\dagger)^2 d \right] - \frac{g_3}{36} (d^\dagger + \sigma) d \psi^\dagger \psi, \quad (3.69)$$

which after separating the  $\sigma$ -terms from the rest becomes

$$\begin{aligned} &= \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi + \frac{g_2}{4} d^\dagger d - \frac{g_2}{4} \left[ d^\dagger \psi^2 + (\psi^\dagger)^2 d \right] - \frac{g_3}{36} d^\dagger d \psi^\dagger \psi \\ &+ \sigma \left( \frac{g_2}{4} d - \frac{g_2}{4} \psi^2 - \frac{g_3}{36} d \psi^\dagger \psi \right). \end{aligned} \quad (3.70)$$

Setting the  $\sigma$ -term to zero we get

$$\frac{g_2}{4} d - \frac{g_2}{4} \psi^2 - \frac{g_3}{36} d \psi^\dagger \psi = 0. \quad (3.71)$$

Dividing this by  $\frac{g_2}{4}$  we get

$$d - \psi^2 - \frac{g_3}{9g_2} d \psi^\dagger \psi = 0. \quad (3.72)$$

Taking common terms of  $d$  and moving  $\psi^2$  to the right side we get

$$\left( 1 - \frac{g_3}{9g_2} \psi^\dagger \psi \right) d = \psi^2, \quad (3.73)$$

and dividing both sides with the common terms we get the solution for  $d$  we get

$$d = \frac{\psi^2}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi}, \quad (3.74)$$

and its conjugate:

$$d^\dagger = \frac{\psi^{\dagger 2}}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi}. \quad (3.75)$$

Inserting  $d$  and  $d^\dagger$  into equation (3.68), we get

$$\begin{aligned} L = & \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi + \frac{g_2}{4} \frac{\psi^{\dagger 2}}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi} \frac{\psi^2}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi} \\ & - \frac{g_2}{4} \left[ \frac{\psi^{\dagger 2}}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi} \psi^2 + (\psi^\dagger)^2 \frac{\psi^2}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi} \right] \\ & - \frac{g_3}{36} \frac{\psi^{\dagger 2}}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi} \frac{\psi^2}{1 - \frac{g_3}{9g_2}\psi^\dagger\psi} \psi^\dagger\psi. \end{aligned} \quad (3.76)$$

Expanding the terms inside the brackets and then combining all the terms over the same denominator gives us

$$\begin{aligned} L = & \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi \\ & + \frac{\frac{g_2}{4}(\psi^\dagger\psi)^2 - \frac{g_2}{4}(\psi^\dagger\psi)^2(1 - \frac{g_3}{9g_2}\psi^\dagger\psi) - \frac{g_2}{4}(1 - \frac{g_3}{9g_2}\psi^\dagger\psi)(\psi^\dagger\psi)^2 - \frac{g_3}{36}(\psi^\dagger\psi)^2\psi^\dagger\psi}{(1 - \frac{g_3}{9g_2}\psi^\dagger\psi)(1 - \frac{g_3}{9g_2}\psi^\dagger\psi)}. \end{aligned} \quad (3.77)$$

After some simple term matching, we obtain the following Lagrangian:

$$L = \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 \right) \psi - \frac{g_2}{4} \frac{(\psi^\dagger\psi)^2}{1 - (g_3/9g_2)\psi^\dagger\psi}. \quad (3.78)$$

### 3.4.2 STM integral equation

We will now derive the Skorniakov-Ter-Martirosian (STM) equation. The STM equation is an integral equation for the Fourier transform of the reconnected part of the Green's function  $\langle 0 | T(d\psi d^\dagger\psi^\dagger) | 0 \rangle$ , which we will denote by  $A$ .

In the centre-of-mass frame, we can take the external momenta of the atom and diatom to be  $-\mathbf{p}$  and  $+\mathbf{p}$  for the incoming lines and  $-\mathbf{k}$  and  $+\mathbf{k}$  for the outgoing lines. We take their energies to be  $E_A$  and  $E - E_A$  for the incoming lines and  $E'_A$  and  $E - E'_A$  for the outgoing lines. The amplitude  $A$  is then a function of the momenta  $\mathbf{p}$  and  $\mathbf{k}$  and the energies  $E$ ,  $E_A$  and  $E'_A$ .

The integral equation involves a loop over the momentum  $-\mathbf{q}$  and energy  $q_0$  of a virtual atom. We obtain

$$\begin{aligned}
A(\mathbf{p}, \mathbf{k}; E, E_A, E'_A) = & - \left[ \frac{g_2^2/4}{E - E_A - E'_A - (\mathbf{p} + \mathbf{k})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \\
& + \frac{32\pi i}{g_2^2} \int \frac{dq_0}{2\pi} \int \frac{d^3q}{(2\pi)^3} \left[ \frac{g_2^2/4}{E - E_A - q_0 - (\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \\
& \times \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, q_0, E'_A)}{1/a - \sqrt{-(E - q_0) + q^2/4 - i\epsilon}}. \tag{3.79}
\end{aligned}$$

Like in the 2-body case, we evaluate the integral over  $q_0$

$$\begin{aligned}
& \int \frac{dq_0}{2\pi} \left[ \frac{g_2^2/4}{E - E_A - q_0 - (\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \\
& \times \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, q_0, E'_A)}{1/a - \sqrt{-(E - q_0) + q^2/4 - i\epsilon}}, \tag{3.80}
\end{aligned}$$

by contour integration, see A.7. Solving the contour integral gives us

$$\begin{aligned}
A(\mathbf{p}, \mathbf{k}; E, E_A, E'_A) = & - \left[ \frac{g_2^2/4}{E - E_A - E'_A - (\mathbf{p} + \mathbf{k})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \\
& + 8\pi \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E - E_A - \frac{(\mathbf{p} + \mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} + \frac{g_3}{9g_2^2} \right] \\
& \times \frac{A(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{3.81}
\end{aligned}$$

We obtain a simpler integral equation if we set the energies of both the initial and final atoms in  $A$  on-shell:  $E_A = p^2/2$ ,  $E'_A = k^2/2$ . Thus only the diatom lines have energies that are off-shell. We get

$$\begin{aligned}
A\left(\mathbf{p}, \mathbf{k}; E, \frac{p^2}{2}, \frac{k^2}{2}\right) = & - \frac{g_2^2}{4} \left[ \frac{1}{E - \frac{p^2}{2} - \frac{k^2}{2} - (\mathbf{p} + \mathbf{k})^2/2 + i\epsilon} + \frac{g_3}{9g_2^2} \right] \\
& + 8\pi \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E - \frac{p^2}{2} - (\mathbf{p} + \mathbf{q})^2/2 - \frac{q^2}{2} + 2i\epsilon} \right. \\
& \left. + \frac{g_3}{9g_2^2} \right] \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{3.82}
\end{aligned}$$

After we open up the parentheses in the denominators we get

$$\begin{aligned}
& - \frac{g_2^2}{4} \left[ \frac{1}{E - \frac{p^2}{2} - \frac{k^2}{2} - \left( \frac{p^2}{2} + \mathbf{p} \cdot \mathbf{k} + \frac{k^2}{2} \right) + i\epsilon} + \frac{g_3}{9g_2^2} \right] \\
& + 8\pi \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E - \frac{p^2}{2} - \frac{q^2}{2} - \left( \frac{p^2}{2} + \mathbf{p} \cdot \mathbf{q} + \frac{q^2}{2} \right) + 2i\epsilon} + \frac{g_3}{9g_2^2} \right] \\
& \times \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{3.83}
\end{aligned}$$

We also change the sign in front of  $8\pi$  by taking  $-1$  from the denominator under  $A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)$ . Cleaning this up we finally get an integral equation with three integration variables for an amplitude  $A$  that depends explicitly on seven independent variables:

$$\begin{aligned}
A\left(\mathbf{p}, \mathbf{k}; E, \frac{p^2}{2}, \frac{k^2}{2}\right) = & - \frac{g_2^2}{4} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{k} + k^2) + i\epsilon} + \frac{g_3}{9g_2^2} \right] \\
& - 8\pi \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} + \frac{g_3}{9g_2^2} \right] \\
& \times \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{3.84}
\end{aligned}$$

There is also an extra variable from the ultraviolet cutoff  $|\mathbf{q}| < \Lambda$  on the loop momentum. If we set  $g_3 = 0$  and ignore the ultraviolet cutoff, the integral equation in (3.84) is equivalent to the Skorniakov-Ter-Martirosian (STM) equation, an integral equation for three particles interacting via zero-range 2-body forces derived by Skorniakov and Ter-Martirosian in 1957 [18]. We will call the 3-body STM equation with  $g_3$  STM3 equation.

### 3.4.3 Three-body problem with zero total orbital angular momentum

We can further simplify the integral equation by having the total orbital angular momentum  $L = 0$  in the 3-body problem. The projection onto  $L = 0$  can be done by averaging the integral equation over the cosine of the angle between  $\mathbf{p}$  and  $\mathbf{k}$ :

$$x = \mathbf{p} \cdot \mathbf{k} / (pk). \tag{3.85}$$

It is also convenient to multiply the amplitude  $A$  by the wave function renormalisation factor  $Z_D$ :

$$Z_D = \frac{64\pi}{ag_2^2}. \quad (3.86)$$

We will denote the resulting amplitude by  $A_S$ :

$$A_S(p, k, E) \equiv Z_D \int_{-1}^1 \frac{dx}{2} A\left(\mathbf{p}, \mathbf{k}; E, \frac{p^2}{2}, \frac{k^2}{2}\right). \quad (3.87)$$

It is also convenient to express the 3-body coupling constant in the form

$$g_3 = -\frac{9g_2^2}{\Lambda^2} H(\Lambda). \quad (3.88)$$

$H$  is dimensionless, so it can only be a function of the dimensionless variables  $a\Lambda$  and  $\Lambda/\kappa_*$ , where  $\kappa_*$  is the 3-body parameter defined by the spectrum of Efimov states in the resonant limit (2.2). We will see that  $H$  is a function of  $\Lambda/\kappa_*$  only [1].

We will now solve the STM3 integral equation into its simplest form. First we insert equation (3.88) into equation (3.84) and then solve the integral over  $q$ :

$$\begin{aligned} A\left(\mathbf{p}, \mathbf{k}; E, \frac{p^2}{2}, \frac{k^2}{2}\right) = & -\frac{g_2^2}{4} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{k} + k^2) + i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right] \\ & - 8\pi \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} \right. \\ & \left. - \frac{H(\Lambda)}{\Lambda^2} \right] \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{-1/a + \sqrt{-E + \frac{3q^2}{4}} - 2i\epsilon}. \end{aligned} \quad (3.89)$$

We can express the integral over  $q$  in form

$$-\frac{8\pi}{(2\pi)^3} \int d^3q = -\frac{8\pi}{(2\pi)^3} \int q^2 dq \int d\Omega_q, \quad (3.90)$$

where integral over  $\Omega_q$  can be expressed as

$$\int d\Omega_q = \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta = 2\pi \int_0^\pi \sin \theta d\theta. \quad (3.91)$$

If we set  $x = \cos \theta$  we get  $dx = -\sin \theta d\theta$ . The integral will be from -1 to 1. This is because inserting 0 and  $\pi$  to  $\theta$  we get integral from 1 to -1, however we must take into account the extra minus from  $dx$  which results in an integral from -1 to 1. We get

$$\int d\Omega_q = 2\pi \int_{-1}^1 dx. \quad (3.92)$$

Now inserting this back to (3.90) we get

$$-\frac{8\pi}{(2\pi)^3} \int d^3q = -\frac{16\pi^2}{(2\pi)^3} \int dq q^2 \int_{-1}^1 dx, \quad (3.93)$$

which reduces to

$$-\frac{8\pi}{(2\pi)^3} \int d^3q = -\frac{2}{\pi} \int dq q^2 \int_{-1}^1 dx. \quad (3.94)$$

We impose ultraviolet cutoff  $\Lambda$  and get

$$-\frac{2}{\pi} \int_0^\Lambda dq q^2 \int_{-1}^1 dx. \quad (3.95)$$

This form will come in handy once we take the averaging integral. We insert (3.90) into (3.89) and get

$$\begin{aligned} A\left(\mathbf{p}, \mathbf{k}; E, \frac{p^2}{2}, \frac{k^2}{2}\right) = & -\frac{g_2^2}{4} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{k} + k^2) + i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right] \\ & - \frac{8\pi}{(2\pi)^3} \int q^2 dq \int d\Omega_q \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} \right. \\ & \left. - \frac{H(\Lambda)}{\Lambda^2} \right] \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}, \end{aligned} \quad (3.96)$$

then we take the averaging integral over  $x_{\mathbf{k}} = \frac{\mathbf{p} \cdot \mathbf{k}}{pk}$  of this, similar to equation (3.87), and here we use the form (3.95) with  $x = x_{\mathbf{q}} = \frac{\mathbf{p} \cdot \mathbf{q}}{pq}$  for  $\int d\Omega_q$ :

$$\begin{aligned} A_S(p, k, E) = & Z_D \int_{-1}^1 \frac{dx_{\mathbf{k}}}{2} \left\{ -\frac{g_2^2}{4} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{k} + k^2) + i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right] \right\} \\ & - Z_D \frac{2}{\pi} \int_{-1}^1 \frac{dx_{\mathbf{k}}}{2} \int_0^\Lambda dq q^2 \int_{-1}^1 \frac{dx_{\mathbf{q}}}{2} \\ & \times \left\{ \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right] \right. \\ & \left. \times \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}} \right\}. \end{aligned} \quad (3.97)$$

The first term of this integral is

$$\frac{16\pi}{a} \left[ \frac{1}{2pk} \ln \frac{p^2 + pk + k^2 - E - i\epsilon}{p^2 - pk + k^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right], \quad (3.98)$$



see A.8, and the second term is

$$\frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \frac{A_S(q, k; E)}{-1/a + \sqrt{\frac{3q^2}{4} - E - 2i\epsilon}}, \quad (3.99)$$

see A.9.

Now combining these two terms (3.98) and (3.99), we get the STM3 equation in its simplest form:

$$\begin{aligned} A_S(p, k; E) = & \frac{16\pi}{a} \left[ \frac{1}{2pk} \ln \frac{p^2 + pk + k^2 - E - i\epsilon}{p^2 - pk + k^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\ & + \frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\ & \times \frac{A_S(q, k; E)}{-1/a + \sqrt{\frac{3q^2}{4} - E - 2i\epsilon}}, \end{aligned} \quad (3.100)$$

where the ultraviolet cutoff  $\Lambda$  on the integral over  $q$  is explicit.

### The function $H(\Lambda)$

We want to guarantee that  $A_S(p, k; E)$  has a well-behaved limit as  $\Lambda \rightarrow \infty$ , so  $H$  must be tuned as a function of  $\Lambda$  so that the cutoff dependence of the solution  $A_S(p, k; E)$  of equation (3.100) decreases as a power of  $\Lambda$ . Note that the  $H/\Lambda^2$  term in the in homogenous term of equation (3.100) can be omitted, since it goes to zero in the limit  $\Lambda \rightarrow \infty$ .

The function  $H$  in equation (3.100) must have the form [1]

$$H(\Lambda) = \frac{\cos[s_0 \ln(\Lambda/\Lambda_*) + \arctan s_0]}{\cos[s_0 \ln(\Lambda/\Lambda_*) - \arctan s_0]}. \quad (3.101)$$

This equation defines a 3-body scaling-violation parameter  $\Lambda_*$  with dimensions of momentum. Note that  $H$  is a periodic function of  $\Lambda/\Lambda_*$ , so  $\Lambda_*$  is defined up to a multiplicative factor of  $(e^{\pi/s_0})^n$ , where  $n$  is an integer. The relation between  $\Lambda_*$  and the 3-body parameter  $\kappa_*$  defined by the spectrum of Efimov states in the resonant limit (2.2) [1], can be written as

$$s_0 \ln(\kappa_*) \approx s_0 \ln(0.381\Lambda_*) \bmod \pi. \quad (3.102)$$

This can be obtained by using the STM3 integral equation (3.100) to calculate the binding energy of the Efimov trimers in the resonant limit  $a = \pm\infty$ .

### 3.4.4 The bound-state equation

The solution  $A_S(p, k; E)$  to the STM3 integral equation in (3.100) has all the information about 3-body observables with total orbital angular momentum

quantum number  $L = 0$ . This also gives us important information about the binding energies  $E_T^{(n)}$  of the Efimov states. The amplitude  $A_S(p, k; E)$  has a finite number of poles in  $E$  corresponding to the Efimov trimers whose binding energies are less than about  $\Lambda^2$  for a given ultraviolet cutoff  $\Lambda$ . As  $\Lambda$  increases, new poles appear corresponding to deeper Efimov trimers. In the limit  $\Lambda \rightarrow \infty$ , these poles approach the energies  $-E_T^{(n)}$  of the Efimov trimers [1].

The residues of the poles of  $A_S(p, k; E)$  factor into functions of  $p$  and functions of  $k$ :

$$A_S(p, k; E) \rightarrow \frac{B^{(n)}(p)B^{(n)}(k)}{E + E_T^{(n)}} \text{ as } E \rightarrow -E_T^{(n)}. \quad (3.103)$$

Inserting this into equation (3.100) gives us

$$\begin{aligned} \frac{B^{(n)}(p)B^{(n)}(k)}{E + E_T^{(n)}} &= \frac{16\pi}{a} \left[ \frac{1}{2pk} \ln \frac{p^2 + pk + k^2 - E - i\epsilon}{p^2 - pk + k^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\ &+ \frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\ &\times \frac{1}{-1/a + \sqrt{\frac{3q^2}{4} - E - 2i\epsilon}} \frac{B^{(n)}(q)B^{(n)}(k)}{E + E_T^{(n)}}, \end{aligned} \quad (3.104)$$

and matching the residues of the poles on both sides results in

$$\begin{aligned} B^{(n)}(p)B^{(n)}(k) &= \frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\ &\times \left[ -1/a + \sqrt{\frac{3q^2}{4} - E - 2i\epsilon} \right]^{-1} B^{(n)}(q)B^{(n)}(k). \end{aligned} \quad (3.105)$$

Dividing both sides with  $B^{(n)}(k)$  we obtain the bound-state equation

$$\begin{aligned} B^{(n)}(p) &= \frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\ &\times \left[ -1/a + \sqrt{\frac{3q^2}{4} - E - 2i\epsilon} \right]^{-1} B^{(n)}(q). \end{aligned} \quad (3.106)$$

The energies  $-E_T^{(n)}$  of the Efimov states are the values that are solutions to this integral equation. For a finite ultraviolet cutoff  $\Lambda$ , the spectrum of  $E_T^{(n)}$  is cut off around  $\Lambda^2$ , so the number of Efimov states is roughly  $\ln(|a|\Lambda)/\pi$ . We can find deeper Efimov states by increasing the cutoff.

The s-wave phase shifts for atom-dimer scattering can be determined from the solution  $A_S(p, k; E)$  to the STM3 integral equation in equation.

The T-matrix element for the elastic scattering of an atom and a dimer with momenta  $k$  is given by the amplitude  $A$  evaluated at the on-shell point  $p = k$  and  $E = -E_D + 3k^2/4$  and multiplied by a wave function renormalisation factor  $Z_D^{1/2}$  for each dimer in the initial or final state [1].

### 3.4.5 Dimensionless 3-body coupling constant

Finally we take a quick look at the dimensionless 3-body coupling constant,  $\hat{g}_3$ , since we introduced the same for 2-body coupling earlier. It is defined by

$$\hat{g}_3(\Lambda) = \frac{\Lambda^4 g_3}{144\pi^4}. \quad (3.107)$$

Inserting equation (3.88) for  $g_3$  we get

$$\hat{g}_3(\Lambda) = \frac{\Lambda^4 \left[ -\frac{9g_2^2}{\Lambda^2} H(\Lambda) \right]}{144\pi^4}. \quad (3.108)$$

Opening the parenthesis we get

$$\hat{g}_3(\Lambda) = -\frac{\Lambda^2 g_2^2 H(\Lambda)}{16\pi^4}. \quad (3.109)$$

Inserting equation (3.50) into  $g_2$  gives us

$$\hat{g}_3(\Lambda) = -\frac{\Lambda^2 \left[ 8\pi a \left( 1 - \frac{2a\Lambda}{\pi} \right)^{-1} \right]^2 H(\Lambda)}{16\pi^4}. \quad (3.110)$$

Bringing rest of the terms except  $H(\Lambda)$  inside the brackets:

$$\hat{g}_3(\Lambda) = -\left[ \frac{8\pi a \Lambda \left( 1 - \frac{2a\Lambda}{\pi} \right)^{-1}}{4\pi^2} \right]^2 H(\Lambda). \quad (3.111)$$

Moving terms around we get

$$\hat{g}_3(\Lambda) = -\left[ \frac{2a\Lambda}{\pi \left( 1 - \frac{2a\Lambda}{\pi} \right)} \right]^2 H(\Lambda). \quad (3.112)$$

Opening the parenthesis in the denominator results in

$$\hat{g}_3(\Lambda) = -\left( \frac{2a\Lambda}{\pi - 2a\Lambda} \right)^2 H(\Lambda), \quad (3.113)$$

which can be expressed after cleaning as

$$\hat{g}_3(\Lambda) = -\left( \frac{a\Lambda}{a\Lambda - \pi/2} \right)^2 H(\Lambda). \quad (3.114)$$

If we vary  $\Lambda$  while  $a$  and  $\Lambda_*$  are fixed, equation (3.107) maps out the RG trajectory. As  $\Lambda$  increases, the constant in equation (3.114) flows towards an ultraviolet limit cycle. It is

$$\hat{g}_3(\Lambda) \rightarrow -\frac{\cos [s_0 \ln (\Lambda/\Lambda_*) + \arctan s_0]}{\cos [s_0 \ln (\Lambda/\Lambda_*) - \arctan s_0]} \quad \text{as } \Lambda \rightarrow \infty. \quad (3.115)$$

This corresponds to resonant limit's  $a = \pm\infty$  theory [1].

# Chapter 4

## Conclusions

After deriving the Efimov states and using effective field theories to numerically approach them, we will take a look at the results that have recently appeared. Evidence for Efimov states have been found in 2006, by T. Kraemer et al [3]. They used ultra-cold caesium atoms in their energetically lowest state, in which their s-wave scattering length,  $a$ , varies strongly with the magnetic field. They used a fitting formula

$$a(B)/a_0 = (1722 + 1.52B/G) \left( 1 - \frac{28.72}{B/G + 11.74} \right), \quad (4.1)$$

for the dependence between 0 and 150 gauss, where it is well approximated. This gave them the s-wave scattering length range that they varied. It was between  $-2500a_0$  and  $1600a_0$ , where  $a_0$  is the Bohr's radius. We can find a link to the T-matrix of previous chapter through the three-body recombination.

Three-body recombination is the process in which two atoms form a bound state and the third one carries away the binding energy. For the physics of ultracold gases it is an important issue. In the formation of clusters of average size from individual atoms and bulk matter, three-body recombination represents the initial stage. Recombination to a weakly bound s level occurs in a collision between two atoms, when a third atom is located inside a sphere of radius  $l \sim a$  around the colliding pair. These locations of the third atom have a probability of order unity to form the weakly bound state with one of the colliding atoms [19].

The number of recombination events per unit time and unit volume is  $\nu_{rec} = \alpha_{rec}n^3$ , where  $n$  is the gas density and  $\alpha_{rec}$  is the recombination rate constant. When confined to three-body recombination of identical atoms at collision energies  $E \ll \epsilon_0$  to a weakly bound molecular s level, we can get the recombination rate constant from the equation

$$\nu_{rec} = \alpha_{rec}n^3 = \frac{2\pi}{\hbar} \int \frac{d^3k_f}{(2\pi)^3} |T_{if}|^2 \delta \left( \frac{3\hbar^2 k_f^2}{4m} - \epsilon_0 \right) \frac{n^3}{6}, \quad (4.2)$$

where  $\frac{n^3}{6}$  stands for the number of triples in the gas,  $\epsilon_0$  is the binding energy,  $k_f$  is the final-state momentum of the third atom relative to the center of mass of the molecule,

$$T_{if} = \int \psi_i \tilde{V} \psi_f^{(0)*} d^3x d^3x' \quad (4.3)$$

is the T-matrix element for three-body recombination, the coordinates  $(x, x')$  are the relative coordinates between two particles,  $x$  between particles 2 and 3 and  $x'$  between particles 1 and 3,  $\psi_i$  is the true wave function of the initial state of the triple, and  $\psi_f^{(0)}$  is the wave function of free motion of the third atom relative to the center of mass of the molecule formed in the recombination event [19].

Three-body recombination leads to losses from a trapped gas with rate proportional to the third power of atomic number density. T. Kraemer et al [3] used the loss rate coefficient  $L_3$  in the resonant case ( $|a| \gg l$ ). This coefficient was obtained by measuring three-body loss rates by recording the time evolution of the atom number  $N$  and the temperature  $T$ . The process of three-body recombination not only leads to a loss of atoms, but also induces anti-evaporation and recombination heating. The first effect occurs for any value of the scattering length  $a$  and the second effect happens for positive values of  $a$  when the products of the recombination remain trapped. The atom loss and temperature rise were modelled by a set of two coupled nonlinear differential equations. They used numerical solutions to this set of equations to fit their experimental data. From these fits including the measurements of the trapping parameters, they obtained the rate coefficient  $L_3$ . For the coefficient they used a convenient form:

$$L_3 = 3C(a) \frac{\hbar a^4}{m}, \quad (4.4)$$

where  $C(a)$  is an analytic expression for additional dependence. Efimov physics is reflected in a logarithmically period behaviour  $C(22.7a) = C(a)$ , which corresponds to the scaling of the infinite series of weakly bound trimer states [3].

The analytic expression  $C(a)$  T. Kraemer et al [3] used had two parts, when  $a < 0$ :

$$C(a) = 4590 \frac{\sinh(2\eta_-)}{\sin^2[s_0 \ln(|a|/a_-)] + \sinh^2\eta_-}, \quad (4.5)$$

and when  $a > 0$ :

$$C(a) = 67.1e^{-2\eta_+} \{ \cos^2[s_0 \ln(|a|/a_+)] + \sinh^2\eta_+ \} + 16.8(1 - e^{-4\eta_+}). \quad (4.6)$$

In these equations  $a_-$  is a free parameter for the resonance positions and depends on the short-range part of the effective three-body interaction.  $\eta_-$

is another free parameter in the first equation that describes the unknown decay rate of Efimov states into deeply bound dimer states and a free atom, thus characterizes the resonance width.  $a_+$  is a free parameter in the second equation which describes recombination into the weakly bound dimer state with an oscillatory behaviour that is due to an interference effect between two pathways and the last free parameter  $\eta_+$  results from decay into deeply bound states [3].

The data was presented in terms of a recombination length

$$\rho_3 = \left( \frac{2m}{\sqrt{3}\hbar} L_3 \right)^{\frac{1}{4}}, \quad (4.7)$$

which leads to simple relation:

$$\rho_3/a = 1.36C^{\frac{1}{4}}. \quad (4.8)$$

All measurements were performed with trapped thermal samples at temperatures ranging from 10 to 250 nK. Two different experimental set-ups were used, in first they used an essentially pure Bose-Einstein condensate and in the second set-up they used an optical surface trap. In this way they were able to obtain Efimov resonances in measurements of three-body recombination in figure 4.1.

When an Efimov state intersects with the continuum threshold at negative scattering lengths, three free atoms in the ultracold limit resonantly couple to a trimer. This results in a triatomic Efimov resonance seen in fig. 4.1. At positive  $a$ , we have the dimer-atom threshold which is another type of Efimov resonance [3].

Other results have also been found for different regions of magnetic field strengths [5] and a compilation of results from different published sources [4] displays proof of Efimov effect. So far caesium atoms have mostly been used but Efimov-states have been managed to be created with lithium atoms as well [6].

We've only just begun to shed light experimentally on the Efimov states and other related phenomena, which could be applied from few-body systems to many-body physics. Figuring out these kinds of phenomena is important for understanding and controlling the physics of quantum matter with strong interactions and gives us insight on understanding in depth quantum physics.

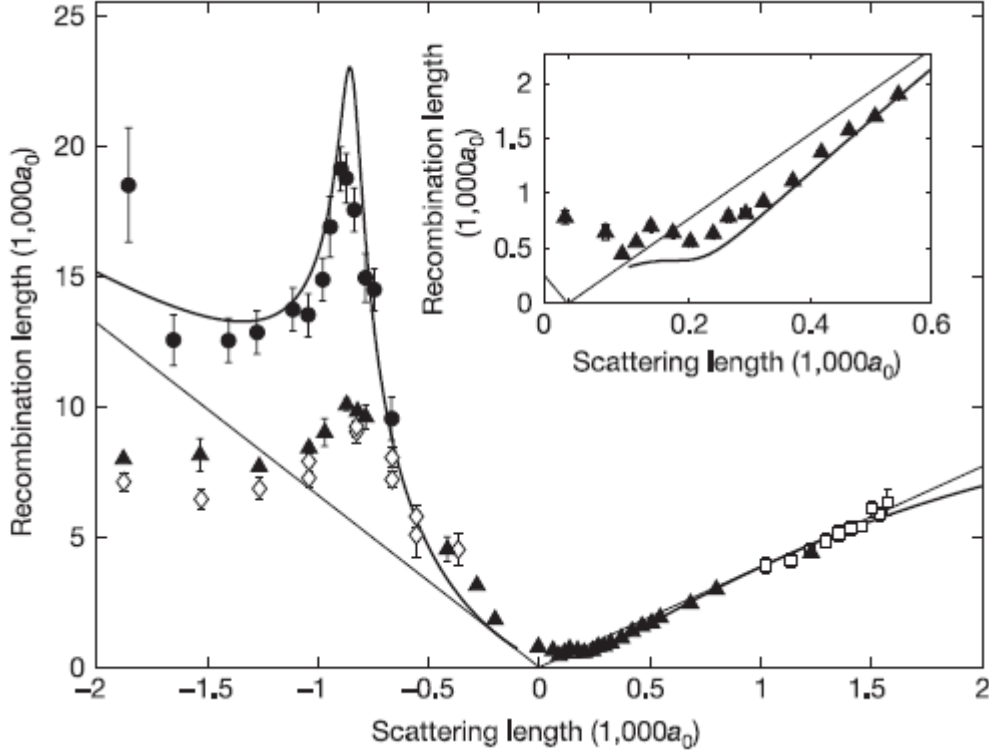


Figure 4.1: Efimov resonances observed in three-body recombination measurements. The recombination length  $\rho_3 \propto L_3^{1/4}$  is plotted as a function of scattering length  $a$ . In the figure  $a_0$  is Bohr's radius. For set-up A, the dots and filled triangles show the experimental data for 10 nK and 200 nK. For B set-up, the open diamonds are from temperatures of 250 nK. The open squares are from older data at temperatures between 250 and 450 nK[20]. The solid curve is there for the analytic model from effective field theory [1] with  $a_- = -850a_0$ ,  $a_+ = 1060a_0$ , and  $\eta_- = \eta_+ = 0.06$ . We get the straight line by setting the  $\sin^2$  and  $\cos^2$ -terms in the analytic theory to 1, which gives a lower recombination limit for  $a < 0$  and an upper limit for  $a > 0$ . The inset shows an expanded view for small positive scattering lengths with a minimum for the analytic expression  $C(a) \propto (\rho_3/a)^4$  near  $210a_0$ . The error bars refer only to statistical uncertainties. [3]



# Appendix A

## A.1 Equation (2.21)

Using equation (2.20) to (2.19) and  $T_R$  from (2.14), we get

$$\begin{aligned} & \frac{\hbar^2}{2m} R^{-5/2} \left[ -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right] R^{5/2} \left[ R^{-5/2} \sum_n f_n(R) \Phi_n(R, \Omega) \right] \\ & + R^{-5/2} \sum_n f_n(R) V_n(R) \Phi_n(R, \Omega) = R^{-5/2} E \sum_n f_n(R) \Phi_n(R, \Omega). \end{aligned} \quad (\text{A.1})$$

Dividing both sides with  $R^{-5/2}$  and opening the brackets for radial operator-term we get

$$\begin{aligned} & -\frac{\hbar^2}{2m} \sum_n \frac{\partial^2}{\partial R^2} [f_n(R)] \Phi_n(R, \Omega) - 2\frac{\hbar^2}{2m} \sum_n \frac{\partial}{\partial R} [f_n(R)] \frac{\partial}{\partial R} [\Phi_n(R, \Omega)] \\ & -\frac{\hbar^2}{2m} \sum_n f_n(R) \frac{\partial^2}{\partial R^2} [\Phi_n(R, \Omega)] + \sum_n f_n(R) V_n(R) \Phi_n(R, \Omega) \\ & + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_n f_n(R) \Phi_n(R, \Omega) = E \sum_n f_n(R) \Phi_n(R, \Omega). \end{aligned} \quad (\text{A.2})$$

The orthonormality condition on the hyperangular functions can be written as

$$\int d\Omega \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) = \delta_{nm}. \quad (\text{A.3})$$

To use this, we change the index  $n \Rightarrow m$  and then multiply both sides with  $\Phi_n^*(R, \Omega)$  from the left:

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_m \frac{\partial^2}{\partial R^2} [f_m(R)] \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) \\
& - 2 \frac{\hbar^2}{2m} \sum_m \frac{\partial}{\partial R} [f_m(R)] \Phi_n^*(R, \Omega) \frac{\partial}{\partial R} [\Phi_m(R, \Omega)] \\
& - \frac{\hbar^2}{2m} \sum_m f_m(R) \Phi_n^*(R, \Omega) \frac{\partial^2}{\partial R^2} [\Phi_m(R, \Omega)] \\
& + \sum_m f_m(R) V_m(R) \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) \\
& + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_m f_m(R) \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) \\
& = E \sum_m f_m(R) \Phi_n^*(R, \Omega) \Phi_m(R, \Omega). \tag{A.4}
\end{aligned}$$

Now we integrate both sides with  $\int d\Omega$  and after organizing it a bit, we get

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_m \frac{\partial^2}{\partial R^2} [f_m(R)] \int d\Omega \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) \\
& - 2 \frac{\hbar^2}{2m} \sum_m \frac{\partial}{\partial R} [f_m(R)] \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial}{\partial R} [\Phi_m(R, \Omega)] \\
& - \frac{\hbar^2}{2m} \sum_m f_m(R) \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial^2}{\partial R^2} [\Phi_m(R, \Omega)] \\
& + \sum_m f_m(R) V_m(R) \int d\Omega \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) \\
& + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_m f_m(R) \int d\Omega \Phi_n^*(R, \Omega) \Phi_m(R, \Omega) \\
& = E \sum_m f_m(R) \int d\Omega \Phi_n^*(R, \Omega) \Phi_m(R, \Omega). \tag{A.5}
\end{aligned}$$

Using the condition (A.3) on this, we get

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_m \frac{\partial^2}{\partial R^2} [f_m(R)] \delta_{nm} \\
& - 2 \frac{\hbar^2}{2m} \sum_m \frac{\partial}{\partial R} [f_m(R)] \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial}{\partial R} [\Phi_m(R, \Omega)] \\
& - \frac{\hbar^2}{2m} \sum_m f_m(R) \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial^2}{\partial R^2} [\Phi_m(R, \Omega)] \\
& + \sum_m f_m(R) V_m(R) \delta_{nm} + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_m f_m(R) \delta_{nm} \\
& = E \sum_m f_m(R) \delta_{nm}. \tag{A.6}
\end{aligned}$$

Sum over deltafunction reduces all but  $n = m$  to zero which leaves us with the following after a bit of organizing:

$$\begin{aligned}
& \left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\
& - \frac{\hbar^2}{2m} \sum_m \left\{ 2 \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial}{\partial R} [\Phi_m(R, \Omega)] \frac{\partial}{\partial R} \right. \\
& \left. - \int d\Omega \Phi_n^*(R, \Omega) \frac{\partial^2}{\partial R^2} [\Phi_m(R, \Omega)] \right\} f_m(R) \\
& = E f_n(R). \tag{A.7}
\end{aligned}$$

## A.2 Equation (2.41)

The eigenvalues  $\lambda_n(R)$  in (2.36) define channel potentials for the hyperradial variable:

$$V_n(R) = [\lambda_n(R) - 4] \frac{\hbar^2}{2mR^2}. \tag{A.8}$$

Using this we get

$$\begin{aligned}
& (T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] = \\
& \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \left[ -\frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial \alpha^2} - V_n(R) - 4 \frac{\hbar^2}{2mR^2} \right] \phi_n(R, \alpha). \tag{A.9}
\end{aligned}$$

Now we separate the terms in the brackets:

$$\begin{aligned}
& (T_R + T_\alpha - E) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
= & -\frac{1}{R^{5/2} \sin(2\alpha)} \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \frac{\partial^2}{\partial \alpha^2} [\phi_n(R, \alpha)] - \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \\
\times & V_n(R) \phi_n(R, \alpha) - \frac{1}{R^{5/2} \sin(2\alpha)} 4 \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \phi_n(R, \alpha). \quad (\text{A.10})
\end{aligned}$$

Opening the brackets and moving  $E$ -terms to the right side and  $T_\alpha$  and  $V_n(R)$  terms to the left side we get

$$\begin{aligned}
& T_R \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
+ & T_\alpha \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
+ & \frac{1}{R^{5/2} \sin(2\alpha)} \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \frac{\partial^2}{\partial \alpha^2} [\phi_n(R, \alpha)] \\
+ & \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) V_n(R) \phi_n(R, \alpha) \\
+ & \frac{1}{R^{5/2} \sin(2\alpha)} 4 \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \phi_n(R, \alpha) \\
= & E \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right]. \quad (\text{A.11})
\end{aligned}$$

We insert (2.14) and (2.15) for the operators and get

$$\begin{aligned}
& \frac{\hbar^2}{2m} R^{-5/2} \left[ -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right] R^{5/2} \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
& + \frac{\hbar^2}{2mR^2 \sin(2\alpha)} \left[ -\frac{\partial^2}{\partial \alpha^2} - 4 \right] \sin(2\alpha) \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
& + \frac{1}{R^{5/2} \sin(2\alpha)} \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \frac{\partial^2}{\partial \alpha^2} [\phi_n(R, \alpha)] \\
& + \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) V_n(R) \phi_n(R, \alpha) \\
& + \frac{1}{R^{5/2} \sin(2\alpha)} 4 \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \phi_n(R, \alpha) \\
& = E \left[ \frac{1}{R^{5/2} \sin(2\alpha)} \sum_n f_n(R) \phi_n(R, \alpha) \right]. \tag{A.12}
\end{aligned}$$

After we clean the above equation, we can divide both sides with  $(R^{5/2} \sin(2\alpha))^{-1}$  and it reduces to

$$\begin{aligned}
& \frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right] \left[ \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
& + \frac{\hbar^2}{2mR^2} \left[ -\frac{\partial^2}{\partial \alpha^2} - 4 \right] \left[ \sum_n f_n(R) \phi_n(R, \alpha) \right] \\
& + \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \frac{\partial^2}{\partial \alpha^2} [\phi_n(R, \alpha)] + \sum_n f_n(R) V_n(R) \phi_n(R, \alpha) \\
& + 4 \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \phi_n(R, \alpha) = E \sum_n f_n(R) \phi_n(R, \alpha). \tag{A.13}
\end{aligned}$$

Next we open the brackets containing the sum:

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_n \frac{\partial^2}{\partial R^2} [f_n(R)] \phi_n(R, \alpha) - 2 \frac{\hbar^2}{2m} \sum_n \frac{\partial}{\partial R} [f_n(R)] \frac{\partial}{\partial R} [\phi_n(R, \alpha)] \\
& - \frac{\hbar^2}{2m} \sum_n f_n(R) \frac{\partial^2}{\partial R^2} [\phi_n(R, \alpha)] + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_n f_n(R) \phi_n(R, \alpha) \\
& - \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \frac{\partial^2}{\partial \alpha^2} [\phi_n(R, \alpha)] - 4 \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \phi_n(R, \alpha) \\
& + \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \frac{\partial^2}{\partial \alpha^2} [\phi_n(R, \alpha)] + \sum_n f_n(R) V_n(R) \phi_n(R, \alpha) \\
& + 4 \frac{\hbar^2}{2mR^2} \sum_n f_n(R) \phi_n(R, \alpha) = E \sum_n f_n(R) \phi_n(R, \alpha). \tag{A.14}
\end{aligned}$$

Cleaning the terms up we get

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_n \frac{\partial^2}{\partial R^2} [f_n(R)] \phi_n(R, \alpha) - 2 \frac{\hbar^2}{2m} \sum_n \frac{\partial}{\partial R} [f_n(R)] \frac{\partial}{\partial R} [\phi_n(R, \alpha)] \\
& - \frac{\hbar^2}{2m} \sum_n f_n(R) \frac{\partial^2}{\partial R^2} [\phi_n(R, \alpha)] + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_n f_n(R) \phi_n(R, \alpha) \\
& + \sum_n f_n(R) V_n(R) \phi_n(R, \alpha) = E \sum_n f_n(R) \phi_n(R, \alpha). \tag{A.15}
\end{aligned}$$

Since the operator on the right side of equation (2.37) is not hermitian, hyperangular functions  $\phi_m(R, \alpha)$  and  $\phi_n(R, \alpha)$  with eigenvalues  $\lambda_m(R)$  and  $\lambda_n(R)$  do not need to be orthogonal functions of  $\alpha$ . We can define a matrix  $G_{nm}(R)$  with their inner products as

$$G_{nm}(R) = \int_0^{\pi/2} d\alpha \phi_n^*(R, \alpha) \phi_m(R, \alpha). \tag{A.16}$$

Again we change the current index  $n \rightarrow m$  and then multiply both sides from

the left with  $\phi_n^*(R, \alpha)$  :

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_m \frac{\partial^2}{\partial R^2} [f_m(R)] \phi_n^*(R, \alpha) \phi_m(R, \alpha) \\
& - 2 \frac{\hbar^2}{2m} \sum_m \frac{\partial}{\partial R} [f_m(R)] \phi_n^*(R, \alpha) \frac{\partial}{\partial R} [\phi_m(R, \alpha)] \\
& - \frac{\hbar^2}{2m} \sum_m f_m(R) \phi_n^*(R, \alpha) \frac{\partial^2}{\partial R^2} [\phi_m(R, \alpha)] \\
& + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_m f_m(R) \phi_n^*(R, \alpha) \phi_m(R, \alpha) \\
& + \sum_m f_m(R) V_m(R) \phi_n^*(R, \alpha) \phi_m(R, \alpha) \\
& = E \sum_m f_m(R) \phi_n^*(R, \alpha) \phi_m(R, \alpha). \tag{A.17}
\end{aligned}$$

Now we integrate both sides with  $\int_0^{\pi/2} d\alpha$  and use (A.16) to get

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_m \frac{\partial^2}{\partial R^2} [f_m(R)] G_{nm}(R) \\
& - 2 \frac{\hbar^2}{2m} \sum_m \frac{\partial}{\partial R} [f_m(R)] \int_0^{\pi/2} d\alpha \phi_n^*(R, \alpha) \frac{\partial}{\partial R} [\phi_m(R, \alpha)] \\
& - \frac{\hbar^2}{2m} \sum_m f_m(R) \int_0^{\pi/2} d\alpha \phi_n^*(R, \alpha) \frac{\partial^2}{\partial R^2} [\phi_m(R, \alpha)] \\
& + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_m f_m(R) G_{nm}(R) + \sum_m f_m(R) V_m(R) G_{nm}(R) \\
& = E \sum_m f_m(R) G_{nm}(R). \tag{A.18}
\end{aligned}$$

We multiply this by the inverse of the matrix  $G_{nm}(R)$  and get

$$\begin{aligned}
& - \frac{\hbar^2}{2m} \sum_k \frac{\partial^2}{\partial R^2} [f_k(R)] \delta_{nk} - 2 \frac{\hbar^2}{2m} \sum_m \frac{\partial}{\partial R} [f_m(R)] \sum_k G_{nk}^{-1}(R) \\
& \times \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial}{\partial R} [\phi_m(R, \alpha)] \\
& - \frac{\hbar^2}{2m} \sum_m f_m(R) \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial^2}{\partial R^2} [\phi_m(R, \alpha)] \\
& + \frac{\hbar^2}{2m} \frac{15}{4R^2} \sum_k f_k(R) \delta_{nk} + \sum_k f_k(R) V_k(R) \delta_{nk} = E \sum_k f_k(R) \delta_{nk}. \tag{A.19}
\end{aligned}$$

Sums leave only cases when  $k = n$  and organizing terms a bit we get

$$\begin{aligned}
& \left[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} + \frac{15}{4R^2} \right) + V_n(R) \right] f_n(R) \\
& - \sum_m \left\{ 2 \frac{\hbar^2}{2m} \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial}{\partial R} [\phi_m(R, \alpha)] \frac{\partial}{\partial R} \right. \\
& + \left. \frac{\hbar^2}{2m} \sum_k G_{nk}^{-1}(R) \int_0^{\pi/2} d\alpha \phi_k^*(R, \alpha) \frac{\partial^2}{\partial R^2} [\phi_m(R, \alpha)] \right\} f_m(R) \\
& = E f_n(R). \tag{A.20}
\end{aligned}$$

### A.3 The integral of equation (2.63)

We solve the integral of equation (2.63) . We insert  $\phi_n^{(\text{hi})}$  into the integral and we get

$$\frac{4}{\sqrt{3}} \int_{|\frac{\pi}{3}-\alpha|}^{\frac{\pi}{2}-|\frac{\pi}{6}-\alpha|} \sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \alpha' \right) \right] d\alpha'. \tag{A.21}$$

Since  $\alpha \ll 1$ , we can write the integral as going from  $\frac{\pi}{3}-\alpha$  to  $\frac{\pi}{2}-\frac{\pi}{6}+\alpha = \frac{\pi}{3}+\alpha$ :

$$\frac{4}{\sqrt{3}} \int_{\frac{\pi}{3}-\alpha}^{\frac{\pi}{3}+\alpha} \sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \alpha' \right) \right] d\alpha', \tag{A.22}$$

which we proceed to integrate to get

$$\frac{4}{\sqrt{3}} \frac{1}{\sqrt{\lambda_n(R)}} \Big|_{\frac{\pi}{3}-\alpha}^{\frac{\pi}{3}+\alpha} \cos \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \alpha' \right) \right]. \tag{A.23}$$

Inserting the values for  $\alpha'$  we get

$$\begin{aligned}
& \frac{4}{\sqrt{3}} \frac{1}{\sqrt{\lambda_n(R)}} \left\{ \cos \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} - \alpha \right) \right] \right. \\
& - \left. \cos \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} + \alpha \right) \right] \right\}. \tag{A.24}
\end{aligned}$$

Using identity  $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$  twice gives us

$$\begin{aligned}
& \frac{4}{\sqrt{3}} \frac{1}{\sqrt{\lambda_n(R)}} \left\{ \cos \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) \right] \cos \left[ \sqrt{\lambda_n(R)} \alpha \right] \right. \\
& + \sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) \right] \sin \left[ \sqrt{\lambda_n(R)} \alpha \right] \\
& - \cos \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) \right] \cos \left[ \sqrt{\lambda_n(R)} \alpha \right] \\
& + \left. \sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) \right] \sin \left[ \sqrt{\lambda_n(R)} \alpha \right] \right\}. \tag{A.25}
\end{aligned}$$



Simplifying this we get

$$\frac{8}{\sqrt{3}} \frac{1}{\sqrt{\lambda_n(R)}} \sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) \right] \sin \left[ \sqrt{\lambda_n(R)} \alpha \right]. \quad (\text{A.26})$$

Now we had  $\sin \alpha \approx \alpha$  if  $\alpha \ll 1$  and we can easily see that

$$\sin \left[ \sqrt{\lambda_n(R)} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) \right] = \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right), \quad (\text{A.27})$$

so we can write the equation (A.26) in form

$$\frac{8}{\sqrt{3}} \frac{1}{\sqrt{\lambda_n(R)}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right) \sqrt{\lambda_n(R)} \alpha, \quad (\text{A.28})$$

which after cleaning gives us

$$\frac{8\alpha}{\sqrt{3}} \phi_n^{(\text{hi})} \left( R, \frac{\pi}{3} \right). \quad (\text{A.29})$$

## A.4 Equation (2.98)

We have a general solution

$$f_0(R) \approx AR^{\frac{1}{2}+is_0} + BR^{\frac{1}{2}-is_0}, \quad R \ll |a|, \quad 1/\kappa, \quad (\text{A.30})$$

where  $A$  and  $B$  are constants. We can write them in a phase form:

$$A = |A|e^{ia}, \quad (\text{A.31})$$

and

$$B = |B|e^{i(b+\pi)}. \quad (\text{A.32})$$

In this case there are no deep bound states which leads to  $|A| = |B|$  [1], and we can write the solution in a simpler form:

$$f_0(R) \approx |A|R^{\frac{1}{2}} \left( R^{is_0} e^{ia} + R^{-is_0} e^{i(b+\pi)} \right). \quad (\text{A.33})$$

We insert factor  $\kappa = (m|E|/\hbar^2)^{1/2}$  to make  $\kappa R$  dimensionless:

$$f_0(R) \approx \frac{|A|}{\kappa} R^{\frac{1}{2}} \left( \kappa R^{is_0} e^{ia} + \kappa R^{-is_0} e^{ib+i\pi} \right). \quad (\text{A.34})$$

We write  $\kappa R$  as a logarithm exponent and write  $e^{i\pi}$  as -1, which gives us

$$f_0(R) \approx \frac{|A|}{\kappa} R^{\frac{1}{2}} \left( e^{ia} e^{\ln(\kappa R^{is_0})} - e^{ib} e^{\ln(\kappa R^{-is_0})} \right). \quad (\text{A.35})$$

Simplifying above, we get

$$f_0(R) \approx \frac{|A|}{\kappa} R^{\frac{1}{2}} \left( e^{\ln(\kappa R^{is_0})+ia} - e^{\ln(\kappa R^{-is_0})+ib} \right). \quad (\text{A.36})$$

We multiply by  $\frac{2i}{2i}$  and move the denominator into the parenthesis:

$$f_0(R) \approx 2i \frac{|A|}{\kappa} R^{\frac{1}{2}} \left( \frac{e^{\ln(\kappa R^{is_0})+ia} - e^{\ln(\kappa R^{-is_0})+ib}}{2i} \right). \quad (\text{A.37})$$

Now we must express  $ia$  and  $ib$  with a common term. We'll use a constant phase  $\theta$  which is the distance to a midpoint between the two, so that  $ia = i\theta + i\delta$  and  $ib = -i\theta + i\delta$ , where  $\delta$  is the midpoint:

$$f_0(R) \approx 2i \frac{|A|}{\kappa} R^{\frac{1}{2}} \left( \frac{e^{is_0 \ln \kappa R + i\theta + i\delta} - e^{-is_0 \ln \kappa R - i\theta + i\delta}}{2i} \right). \quad (\text{A.38})$$

Bringing the term  $e^{i\delta}$  to the front we get

$$f_0(R) \approx 2ie^{i\delta} \frac{|A|}{\kappa} R^{\frac{1}{2}} \left( \frac{e^{is_0 \ln \kappa R + \theta} - e^{-is_0 \ln \kappa R - \theta}}{2i} \right). \quad (\text{A.39})$$

Now we can write the parenthesis as a sine, also we write the constant  $2ie^{i\delta} \frac{|A|}{\kappa}$  as just a simple constant  $A$ . We finally get a simple form for  $f_0(R)$  :

$$f_0(R) \approx AR^{\frac{1}{2}} \sin [s_0 \ln(\kappa R) + \theta]. \quad (\text{A.40})$$

## A.5 Contour integral of equation (3.13)

We can evaluate the integral over  $q_0$

$$\int \frac{dq_0}{2\pi} \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{1}{E - q_0 - q^2/2 + i\epsilon}, \quad (\text{A.41})$$

by using contour integration. We start by creating  $q_1$  and  $q_2$  and then breaking the denominator into a simpler form:

$$q_1 = \frac{q^2}{2} - i\epsilon, \quad (\text{A.42})$$

$$q_2 = E - \frac{q^2}{2} + i\epsilon. \quad (\text{A.43})$$

With these we can express the integral (3.13) as

$$-\frac{1}{2\pi} \int dq_0 \frac{1}{q_0 - q_1} \frac{1}{q_0 - q_2}. \quad (\text{A.44})$$

Now we will break the denominator apart:

$$\begin{aligned}
& - \frac{1}{2\pi} \int dq_0 \frac{1}{q_0 - q_1} \frac{1}{q_0 - q_2} \\
&= - \frac{1}{2\pi} \int dq_0 \frac{Aq_0 - Aq_2 + Bq_0 - Bq_1}{(q_0 - q_1)(q_0 - q_2)} \\
&= - \frac{1}{2\pi} \int dq_0 \left( \frac{A}{q_0 - q_1} + \frac{B}{q_0 - q_2} \right), \tag{A.45}
\end{aligned}$$

and solve A and B:

$$\begin{cases} A + B = 0 \Leftrightarrow A = -B \\ -Aq_2 - Bq_1 = 1. \end{cases} \tag{A.46}$$

Inserting  $A = -B$  into the second equation gives us

$$\begin{cases} A = -B \\ Bq_2 - Bq_1 = 1 \Leftrightarrow B(q_2 - q_1) = 1 \Leftrightarrow B = \frac{1}{q_2 - q_1}. \end{cases} \tag{A.47}$$

Inserting B back to the first one, we get the coefficients:

$$\begin{cases} A = -\frac{1}{q_2 - q_1} = \frac{1}{q_1 - q_2} \\ B = \frac{1}{q_2 - q_1}. \end{cases} \tag{A.48}$$

Inserting A and B into (A.45), we get

$$\begin{aligned}
& - \frac{1}{2\pi} \int dq_0 \left( \frac{A}{q_0 - q_1} + \frac{B}{q_0 - q_2} \right) \\
&= - \frac{1}{2\pi} \int dq_0 \left( \frac{\frac{1}{q_1 - q_2}}{q_0 - q_1} + \frac{\frac{1}{q_2 - q_1}}{q_0 - q_2} \right) \\
&= - \frac{1}{2\pi} \int dq_0 \left[ \frac{1}{(q_0 - q_1)(q_1 - q_2)} + \frac{1}{(q_0 - q_2)(q_2 - q_1)} \right]. \tag{A.49}
\end{aligned}$$

Now that we have simplified a second power of  $q_0$  into two parts of single power terms, we can start the contour integration in proper. We set  $z = q_0$  on a complex plane. Now we create a function

$$f(z) = \frac{1}{(z - q_1)(q_1 - q_2)} + \frac{1}{(z - q_2)(q_2 - q_1)}. \tag{A.50}$$

We can see that there are poles at

$$z - q_1 = 0 \vee z - q_2 = 0,$$

or rather

$$z = q_1 \vee z = q_2. \tag{A.51}$$

We will be taking a path integral with  $f(z)$  around a contour  $C$  and using residues to calculate the result:

$$\oint_C f(z)dz = 2\pi i \sum \text{residues}. \quad (\text{A.52})$$

Now that we split (A.44) into two parts, we need to split the path integral as well:

$$\oint_C f(z)dz = \sum_i \oint_C f_i(z)dz = \oint_C f_1(z)dz + \oint_C f_2(z)dz, \quad (\text{A.53})$$

where  $f_1(z)$  is the first term of  $f(z)$  and  $f_2(z)$  is the second in equation (A.50). Now we split the summation of residues as well, we get

$$2\pi i \sum_i \text{residues} = 2\pi i \sum_i \text{residues}_1 + 2\pi i \sum_i \text{residues}_2. \quad (\text{A.54})$$

Inserting these two equations (A.53) and (A.54) into equation (A.52) and remembering that  $f_1(z)$  and  $f_2(z)$  both had only one pole, which allows us to simplify the final formula:

$$\oint_C f_1(z)dz + \oint_C f_2(z)dz = 2\pi i(\text{residue}_1) + 2\pi i(\text{residue}_2). \quad (\text{A.55})$$

Now we will need to solve the residues. First we decide on contour  $C$ .  $C$  will be a semicircle on a complex field. The line,  $C_1$ , will go along the Real-axis from  $-R$  to  $R$ , as  $R \rightarrow \infty$ , and the arc,  $C_2$ , will curve from 0 to  $\pi$  along the positive complex plane.  $\text{residue}_1$  will be 0, since the pole is left out from the encircled area due to the negative imaginary part  $-i\epsilon$ . Now we get  $\text{residue}_2$  with the simple pole residue formula:

$$\text{residue}(f, c) = \lim_{z \rightarrow c} (z - c)f(z). \quad (\text{A.56})$$

In our case  $c = q_2$  and  $f(z) = f_2(z)$ , we get

$$\text{residue}_2(f_2, q_2) = \lim_{z \rightarrow q_2} (z - q_2)f_2(z) = \lim_{z \rightarrow q_2} \frac{(z - q_2)}{(z - q_2)(q_2 - q_1)} = \frac{1}{q_2 - q_1}. \quad (\text{A.57})$$

Now we can calculate  $\oint_C f(z)dz$  with the residues:

$$\oint_C f(z)dz = 2\pi i(\text{residue}_1) + 2\pi i(\text{residue}_2). \quad (\text{A.58})$$

Inserting residue values and we get:

$$\oint_C f(z)dz = 0 + 2\pi i \frac{1}{q_2 - q_1} = \frac{2\pi i}{q_2 - q_1}. \quad (\text{A.59})$$

Now we need to solve the  $\oint_C f(z)dz$  without residuals. We will solve the integral in two parts, in the real  $C_1$ - and complex  $C_2$ -part. For the real part we set  $z = x$ ,  $dz = dx$  and we get

$$C_1 = \int_{-R}^R \frac{1}{(x - q_1)(x - q_2)} dx. \quad (\text{A.60})$$

For the imaginary part we set  $z = Re^{i\theta}$  and  $dz = iRe^{i\theta}d\theta$  and we get

$$C_2 = \int_0^\pi \frac{1}{(Re^{i\theta} - q_1)(Re^{i\theta} - q_2)} iRe^{i\theta} d\theta. \quad (\text{A.61})$$

As we set the real part to go towards infinity,  $R \rightarrow \infty$ , we notice that we have a situation where there is only a single power of  $R$  in the numerator and  $R^2$  in the denominator, which means that  $C_2$  reduces to a zero. Placing  $C_1$  into equation (A.59), we get

$$\int_{-\infty}^{\infty} \frac{1}{(x - q_1)(x - q_2)} dx = 2\pi i \frac{1}{q_2 - q_1}, \quad (\text{A.62})$$

which equals to the original integration equation (A.44) with  $q_0$ . Using the equation above we get

$$-\frac{1}{2\pi} \int dq_0 \frac{1}{q_0 - q_1} \frac{1}{q_0 - q_2} = -\frac{1}{2\pi} 2\pi i \frac{1}{q_2 - q_1} = -\frac{i}{q_2 - q_1}. \quad (\text{A.63})$$

Now we place back the proper values of  $q_1$  and  $q_2$ :

$$-\frac{i}{q_2 - q_1} = -\frac{i}{(E - \frac{q^2}{2} + i\epsilon) - (\frac{q^2}{2} - i\epsilon)} = -\frac{i}{E - q^2 + i2\epsilon}. \quad (\text{A.64})$$

Finally, with this result we can write equation (3.12) in it's new form of

$$A(E) \approx -g_2 - \frac{1}{2}g_2^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{E - q^2 + i2\epsilon} + \dots \quad (\text{A.65})$$

## A.6 Contour integral of equation (3.20)

$$A(E) \approx -g_2 + \frac{1}{4\pi^2}g_2^2 \int_0^\infty dq \frac{q^2}{q^2 - E - i2\epsilon} + \dots \quad (\text{A.66})$$

We use contour integration to solve the integral part of the equation. We set  $q = z$  and write the integral part as a complex integral:

$$f(z) = \int dz \frac{z^2}{z^2 - E - i2\epsilon}. \quad (\text{A.67})$$

This has a poles at

$$z^2 - E - i\epsilon = 0,$$

which after moving terms to right is

$$z^2 = E + i\epsilon. \quad (\text{A.68})$$

Taking a square root from both sides gives us

$$z = \pm\sqrt{E + i\epsilon}. \quad (\text{A.69})$$

Again we choose a semi-circle on the positive side of the complex plane. This eliminates  $z = -\sqrt{E + i\epsilon}$  from the residue calculations. Residue for  $z = \sqrt{E + i\epsilon}$  is

$$\text{residue}(f, \sqrt{E + i\epsilon}) = \lim_{z \rightarrow \sqrt{E + i\epsilon}} (z - \sqrt{E + i\epsilon}) \frac{z^2}{z^2 - E - i\epsilon}. \quad (\text{A.70})$$

Breaking the denominator into pole-form :

$$\text{residue}(f, \sqrt{E + i\epsilon}) = \lim_{z \rightarrow \sqrt{E + i\epsilon}} \frac{(z - \sqrt{E + i\epsilon})z^2}{(z - \sqrt{E + i\epsilon})(z + \sqrt{E + i\epsilon})}, \quad (\text{A.71})$$

cancelling out term  $(z - \sqrt{E + i\epsilon})$  in denominator and numerator we get

$$\text{residue}(f, \sqrt{E + i\epsilon}) = \lim_{z \rightarrow \sqrt{E + i\epsilon}} \frac{z^2}{z + \sqrt{E + i\epsilon}}. \quad (\text{A.72})$$

Now we take the limit and get

$$\text{residue}(f, \sqrt{E + i\epsilon}) = \frac{(\sqrt{E + i\epsilon})^2}{\sqrt{E + i\epsilon} + \sqrt{E + i\epsilon}}, \quad (\text{A.73})$$

and again cancelling term  $\sqrt{E + i\epsilon}$  we get the residue

$$\text{residue}(f, \sqrt{E + i\epsilon}) = \frac{1}{2}\sqrt{E + i\epsilon}. \quad (\text{A.74})$$

Now we take a look at the path integral which was defined by  $C_1$ , which was a line along real-axis from point  $-R$  to  $R$ , and by  $C_2$  which was the semi-circle in complex plane from point  $0$  to  $\pi$ . For  $C_1$  we have  $z = x$ ,  $dz = dx$  and

$$C_1 = \int_{-R}^R \frac{x^2}{x^2 - E - i\epsilon} dx. \quad (\text{A.75})$$

For  $C_2$  we have  $z = Re^{i\theta}$ ,  $dz = iRe^{i\theta}d\theta$  and

$$C_2 = \int_0^\pi \frac{(Re^{i\theta})^2}{(Re^{i\theta})^2 - E - i\epsilon} iRe^{i\theta} d\theta. \quad (\text{A.76})$$

Now since we can reduce the terms on denominator due to the fact that  $R \gg -E - i\epsilon$ :

$$C_2 = iR \int_0^\pi \frac{(Re^{i\theta})^2}{(Re^{i\theta})^2} e^{i\theta} d\theta = iR \int_0^\pi e^{i\theta} d\theta = iR \frac{1}{i} (e^{i\pi} - e^0) = R(-1 - 1),$$

which ends up being

$$C_2 = -2R. \quad (\text{A.77})$$

Now we need to impose an ultraviolet cutoff  $|\bar{\mathbf{q}}| < \Lambda$  which turns the  $R$  in  $C_2$  into  $\Lambda$ :

$$C_2 = -2R = -2\Lambda, \quad (\text{A.78})$$

which would've otherwise approached infinity. We can now write equation

$$\oint_C dz \frac{z^2}{z^2 - E - i\epsilon} = 2\pi i (\text{residue}), \quad (\text{A.79})$$

in a new form:

$$\int_{-R}^R \frac{x^2}{x^2 - E - i\epsilon} dx - 2\Lambda = 2\pi i \frac{1}{2} \sqrt{E + i\epsilon}. \quad (\text{A.80})$$

Solving for the integral we get

$$\begin{aligned} \int_{-R}^R \frac{x^2}{x^2 - E - i\epsilon} dx &= \pi i \sqrt{E + i\epsilon} + 2\Lambda, \\ \int_{-R}^R \frac{x^2}{x^2 - E - i\epsilon} dx &= 2\Lambda + \pi i \sqrt{-1(-E - i\epsilon)}. \end{aligned} \quad (\text{A.81})$$

We take out -1 from the square root to get rid of the  $i$  with  $i^2 = -1$ :

$$\begin{aligned} \int_{-R}^R \frac{x^2}{x^2 - E - i\epsilon} dx &= 2\Lambda + \pi i^2 \sqrt{-E - i\epsilon}, \\ \int_{-R}^R \frac{x^2}{x^2 - E - i\epsilon} dx &= 2\Lambda - \pi \sqrt{-E - i\epsilon}. \end{aligned} \quad (\text{A.82})$$

We set  $R \rightarrow \infty$  and we have an integral from  $-\infty$  to  $\infty$  while originally we needed an integral from 0 to  $\infty$ . However, we have an even function so we can set

$$\int_0^\infty dx = \frac{1}{2} \int_{-\infty}^\infty dx. \quad (\text{A.83})$$

Inserting this into (3.20) we get

$$A(E) \approx -g_2 + \frac{g_2^2}{4\pi^2} \frac{1}{2} \int_{-\infty}^\infty dq \frac{q^2}{q^2 - E - i2\epsilon} + \dots, \quad (\text{A.84})$$

to which we insert (A.82) and get

$$A(E) \approx -g_2 + \frac{g_2^2}{4\pi^2} \frac{1}{2} \left( 2\Lambda - \pi \sqrt{-E - i\epsilon} \right) + \dots. \quad (\text{A.85})$$

## A.7 Contour integral equation (3.80)

Like in the 2-body case, we evaluate the integral over  $q_0$  by contour integration:

$$\int \frac{dq_0}{2\pi} \left[ \frac{g_2^2/4}{E - E_A - q_0 - (\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \times \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, q_0, E'_A)}{1/a - \sqrt{-(E - q_0) + q^2/4 - i\epsilon}}. \quad (\text{A.86})$$

We divide the integral into two parts from the summation:

$$\frac{1}{2\pi} \left\{ \int dq_0 \left[ \frac{g_2^2/4}{E - E_A - q_0 - (\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} \frac{1}{q_0 - q^2/2 + i\epsilon} \times \frac{A(\mathbf{q}, \mathbf{k}; E, q_0, E'_A)}{1/a - \sqrt{-(E - q_0) + q^2/4 - i\epsilon}} \right] + \int dq_0 \left[ \frac{g_3}{36} \frac{1}{q_0 - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, q_0, E'_A)}{1/a - \sqrt{-(E - q_0) + q^2/4 - i\epsilon}} \right] \right\}. \quad (\text{A.87})$$

We set  $q_0 = z$  and denote the first integral term as  $f_1(z)$  and second  $f_2(z)$ . Since latter one is simpler, we start with its poles. The first pole is the same as in 2-body case in equation (3.13)  $z = \frac{q^2}{2} - i\epsilon$ . We can see that the second denominator has a branching point which resides on the positive complex plane. Due to this we will choose the contour semicircle on the negative complex plane to avoid the branching point. To have the integral along the real axis moving in the correct direction the contour must travel clockwise. This means it will be going in a negative direction which reverses the sign of the integral overall. The first pole which has term  $-i\epsilon$  will be used in the residue calculations.

We solve the residue for the first pole:

$$\text{Residue} \left( f_2(z), \frac{q^2}{2} - i\epsilon \right) = \lim_{z \rightarrow \frac{q^2}{2} - i\epsilon} \left( z - \frac{q^2}{2} + i\epsilon \right) f_2(z). \quad (\text{A.88})$$

Inserting  $f_2(z)$  we get

$$\text{Residue} \left( f_2(z), \frac{q^2}{2} - i\epsilon \right) = \lim_{z \rightarrow \frac{q^2}{2} - i\epsilon} \frac{g_3}{36} \left( z - \frac{q^2}{2} + i\epsilon \right) \frac{1}{z - \frac{q^2}{2} + i\epsilon} \times \frac{A(\mathbf{q}, \mathbf{k}; E, z, E'_A)}{1/a - \sqrt{-(E - z) + q^2/4 - i\epsilon}}, \quad (\text{A.89})$$

which after cleaning up some terms becomes

$$\text{Residue} \left( f_2(z), \frac{q^2}{2} - i\epsilon \right) = \lim_{z \rightarrow \frac{q^2}{2} - i\epsilon} \frac{g_3}{36} \frac{A(\mathbf{q}, \mathbf{k}; E, z, E'_A)}{1/a - \sqrt{-(E - z) + q^2/4 - i\epsilon}}. \quad (\text{A.90})$$



This sets  $z = q_0 = \frac{q^2}{2} - i\epsilon$  in amplitude  $A$  inside the integral, so the amplitude  $A$  inside the integral has the incoming atom on-shell. We continue by inserting the value for  $z$ :

$$\text{Residue} \left( f_2(z), \frac{q^2}{2} - i\epsilon \right) = \frac{g_3}{36} \frac{A \left( \mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A \right)}{1/a - \sqrt{- \left( E - \frac{q^2}{2} + i\epsilon \right) + q^2/4 - i\epsilon}}. \quad (\text{A.91})$$

Opening brackets in the denominator gives us

$$\text{Residue} \left( f_2(z), \frac{q^2}{2} - i\epsilon \right) = \frac{g_3}{36} \frac{A \left( \mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A \right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \quad (\text{A.92})$$

Poles for  $f_1(z)$  are the same including a third one:

$$E - E_A - z - \frac{(\mathbf{p} + \mathbf{q})^2}{2} + i\epsilon = 0, \quad (\text{A.93})$$

which we solve for  $z$  and get

$$z = E - E_A - \frac{(\mathbf{p} + \mathbf{q})^2}{2} + i\epsilon. \quad (\text{A.94})$$

However this one is also on the positive complex plane which we can ignore since we chose to take the semi-circle through the negative complex plane. Now we solve the residue for the same pole  $z = \frac{q^2}{2} - i\epsilon$ :

$$\text{Residue} \left( f_1(z), \frac{q^2}{2} - i\epsilon \right) = \lim_{z \rightarrow \frac{q^2}{2} - i\epsilon} \left( z - \frac{q^2}{2} + i\epsilon \right) f_1(z), \quad (\text{A.95})$$

to which we insert  $f_1(z)$  and get

$$\begin{aligned} & \lim_{z \rightarrow \frac{q^2}{2} - i\epsilon} \frac{g_2^2}{4} \frac{1}{E - E_A - z - \frac{(\mathbf{p} + \mathbf{q})^2}{2} + i\epsilon} \left( z - \frac{q^2}{2} + i\epsilon \right) \frac{1}{z - \frac{q^2}{2} + i\epsilon} \\ & \times \frac{A(\mathbf{q}, \mathbf{k}; E, z, E'_A)}{1/a - \sqrt{-(E - z) + q^2/4 - i\epsilon}}. \end{aligned} \quad (\text{A.96})$$

Through similarities we can quickly use equation (A.92) to solve this by replacing  $\frac{g_3}{36}$  with  $\frac{g_2^2}{4} \frac{1}{E - E_A - z - \frac{(\mathbf{p} + \mathbf{q})^2}{2} + i\epsilon}$ :

$$\begin{aligned} \text{Residue} \left( f_1(z), \frac{q^2}{2} - i\epsilon \right) &= \frac{g_2^2}{4} \frac{1}{E - E_A - \frac{(\mathbf{p} + \mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \\ & \times \frac{A \left( \mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A \right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \end{aligned} \quad (\text{A.97})$$

These residues combined form the residue for  $\oint_C f(z)$ :

$$\begin{aligned} & 2\pi i \sum \text{Residues} \\ &= 2\pi i \left[ \text{Residue} \left( f_1(z), \frac{q^2}{2} - i\epsilon \right) + \text{Residue} \left( f_2(z), \frac{q^2}{2} - i\epsilon \right) \right], \end{aligned} \quad (\text{A.98})$$

which after inserting both residues becomes

$$\begin{aligned} &= 2\pi i \left[ \frac{g_2^2}{4} \frac{1}{E - E_A - \frac{(\mathbf{p}+\mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}} \right. \\ & \left. + \frac{g_3}{36} \frac{A(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}} \right]. \end{aligned} \quad (\text{A.99})$$

Combining similar terms:

$$\begin{aligned} 2\pi i \sum \text{Residues} &= 2\pi i \left[ \frac{1}{E - E_A - \frac{(\mathbf{p}+\mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \frac{g_2^2}{4} + \frac{g_3}{36} \right] \\ & \times \frac{A(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \end{aligned} \quad (\text{A.100})$$

Now we have the right side of the equation

$$-\oint_C f(z) = -\oint_C f_1(z) - \oint_C f_2(z) = 2\pi i \sum \text{Residues}. \quad (\text{A.101})$$

Note the minuses on the left side in above equation. They come from the fact that the contour integral is going clockwise. We calculate the left side like before starting with  $\oint_C f_1(z)$ . Left side of the integral is defined by  $C$  which is divided into  $C_1$ , a line along real-axis from point  $-R$  to  $R$ , and  $C_2$  which is a semi-circle from point  $0$  to  $\pi$  along the negative complex plane. For  $C_1$  we have  $z = x$ ,  $dz = dx$  and we get

$$\begin{aligned} C_1 &= \int_{-R}^R \left[ \frac{g_2^2/4}{E - E_A - x - \frac{(\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} - x - q^2/2 + i\epsilon} \frac{1}{x - q^2/2 + i\epsilon} \right. \\ & \left. \times \frac{A(\mathbf{q}, \mathbf{k}; E, x, E'_A)}{1/a - \sqrt{-(E - x) + q^2/4 - i\epsilon}} \right] dx. \end{aligned} \quad (\text{A.102})$$

For  $C_2$  we have  $z = Re^{i\theta}$ ,  $dz = iRe^{i\theta} d\theta$ :

$$\begin{aligned} C_2 &= \int_0^\pi \left[ \frac{g_2^2/4}{E - E_A - Re^{i\theta} - \frac{(\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} - Re^{i\theta} - q^2/2 + i\epsilon} \frac{1}{Re^{i\theta} - q^2/2 + i\epsilon} \right. \\ & \left. \times \frac{A(\mathbf{q}, \mathbf{k}; E, \theta, E'_A)}{1/a - \sqrt{-(E - Re^{i\theta}) + q^2/4 - i\epsilon}} \right] iRe^{i\theta} d\theta. \end{aligned} \quad (\text{A.103})$$

Now since we can reduce the terms on denominator due to the fact that  $R \rightarrow \infty$ :

$$C_2 = \int_0^\pi \left[ \frac{g_2^2/4}{-Re^{i\theta}} \frac{1}{Re^{i\theta}} \frac{A(\mathbf{q}, \mathbf{k}; E, \theta, E'_A)}{-\sqrt{Re^{i\theta}}} \right] iRe^{i\theta} d\theta. \quad (\text{A.104})$$

Cleaning up a bit we get

$$C_2 = i \frac{g_2^2}{4} \int_0^\pi \frac{1}{(Re^{i\theta})^{3/2}} A(\mathbf{q}, \mathbf{k}; E, \theta, E'_A) d\theta. \quad (\text{A.105})$$

The term  $(Re^{i\theta})^{3/2}$  will overpower the  $A$  once we let  $R$  approach infinity. This leads to  $C_2 = 0$ . We do the same procedure for  $\oint_C f_2(z)$ . For  $C_1$  we have  $z = x$ ,  $dz = dx$  and we get

$$C_1 = \int_{-R}^R \left[ \frac{g_3}{36} \frac{1}{x - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, x, E'_A)}{1/a - \sqrt{-(E-x) + q^2/4 - i\epsilon}} \right] dx. \quad (\text{A.106})$$

For  $C_2$  we have  $z = Re^{i\theta}$ ,  $dz = iRe^{i\theta} d\theta$ :

$$C_2 = \int_0^\pi \left[ \frac{g_3}{36} \frac{1}{Re^{i\theta} - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, \theta, E'_A)}{1/a - \sqrt{-(E - Re^{i\theta}) + q^2/4 - i\epsilon}} \right] iRe^{i\theta} d\theta, \quad (\text{A.107})$$

which after cleaning up a bit becomes

$$C_2 = i \frac{g_2^2}{4} \int_0^\pi \frac{1}{\sqrt{Re^{i\theta}}} A(\mathbf{q}, \mathbf{k}; E, \theta, E'_A) d\theta. \quad (\text{A.108})$$

Yet again the term with  $R$ ,  $\sqrt{Re^{i\theta}}$ , will overpower the amplitude  $A$  once we let  $R$  approach infinity. This leads to  $C_2 = 0$  again. Compiling the previous results and inserting them into equation (A.101), we get

$$\begin{aligned} & - \int_{-R}^R \left[ \frac{g_2^2/4}{E - E_A - x - (\mathbf{p} + \mathbf{q})^2/2 + i\epsilon} \frac{1}{x - q^2/2 + i\epsilon} \right. \\ & \left. \times \frac{A(\mathbf{q}, \mathbf{k}; E, x, E'_A)}{1/a - \sqrt{-(E-x) + q^2/4 - i\epsilon}} \right] dx \\ & - \int_{-R}^R \left[ \frac{g_3}{36} \frac{1}{x - q^2/2 + i\epsilon} \frac{A(\mathbf{q}, \mathbf{k}; E, x, E'_A)}{1/a - \sqrt{-(E-x) + q^2/4 - i\epsilon}} \right] dx \\ & = 2\pi i \left[ \frac{1}{E - E_A - \frac{(\mathbf{p} + \mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \frac{g_2^2}{4} + \frac{g_3}{36} \right] \\ & \times \frac{A(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \end{aligned} \quad (\text{A.109})$$

We can see that the left side is comparable to the integral we wanted to solve originally, except for the minuses which we can take into account. We can insert the right side with -1 as multiplier into equation (A.87) and get

$$-\frac{1}{2\pi}2\pi i \left[ \frac{1}{E - E_A - \frac{(\mathbf{p}+\mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \frac{g_2^2}{4} + \frac{g_3}{36} \right] \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A\right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}, \quad (\text{A.110})$$

which reduces to

$$-i \left[ \frac{1}{E - E_A - \frac{(\mathbf{p}+\mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \frac{g_2^2}{4} + \frac{g_3}{36} \right] \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A\right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \quad (\text{A.111})$$

This result can be inserted into equation (3.79) to get

$$\begin{aligned} A(\mathbf{p}, \mathbf{k}; E, E_A, E'_A) = & - \left[ \frac{g_2^2/4}{E - E_A - E'_A - (\mathbf{p} + \mathbf{k})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \\ & + \frac{32\pi i}{g_2^2} \int \frac{d^3q}{(2\pi)^3} (-i) \left[ \frac{1}{E - E_A - \frac{(\mathbf{p}+\mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} \frac{g_2^2}{4} \right. \\ & \left. + \frac{g_3}{36} \right] \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A\right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \quad (\text{A.112}) \end{aligned}$$

Taking  $\frac{g_2^2}{4}$  as the common term and bringing it and  $-i$  out from the integral we get

$$\begin{aligned} A(\mathbf{p}, \mathbf{k}; E, E_A, E'_A) = & - \left[ \frac{g_2^2/4}{E - E_A - E'_A - (\mathbf{p} + \mathbf{k})^2/2 + i\epsilon} + \frac{g_3}{36} \right] \\ & + 8\pi \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E - E_A - \frac{(\mathbf{p}+\mathbf{q})^2}{2} - \frac{q^2}{2} + 2i\epsilon} + \frac{g_3}{9g_2^2} \right] \\ & \times \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, E'_A\right)}{1/a - \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \quad (\text{A.113}) \end{aligned}$$

## A.8 First term of equation (3.97)

We solve the following integral over  $x$ :

$$-\frac{g_2^2}{4} Z_D \int_{-1}^1 \frac{dx}{2} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{k} + k^2) + i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right], \quad (\text{A.114})$$

where we have set  $x_{\mathbf{k}} = x$ . We bring the integral inside the brackets:

$$-\frac{g_2^2}{4} Z_D \left[ \int_{-1}^1 \frac{dx}{2} \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{k} + k^2) + i\epsilon} - \int_{-1}^1 \frac{dx}{2} \frac{H(\Lambda)}{\Lambda^2} \right]. \quad (\text{A.115})$$

Now we solve the first integral by changing  $\mathbf{p} \cdot \mathbf{k} = pkx$ :

$$-\frac{g_2^2}{4} Z_D \left[ \int_{-1}^1 \frac{dx}{2} \frac{1}{E - (p^2 + pkx + k^2) + i\epsilon} - \int_{-1}^1 \frac{dx}{2} \frac{H(\Lambda)}{\Lambda^2} \right]. \quad (\text{A.116})$$

Performing the integrations we get

$$-\frac{g_2^2}{4} Z_D \left[ -\frac{1}{2} \Big|_{-1}^1 \frac{1}{pk} \ln(E - p^2 - pkx - k^2 + i\epsilon) - \frac{1}{2} \Big|_{-1}^1 \frac{H(\Lambda)}{\Lambda^2} x \right]. \quad (\text{A.117})$$

Inserting values into  $x$ :

$$\begin{aligned} & -\frac{g_2^2}{4} Z_D \left[ -\frac{1}{2pk} \ln(E - p^2 - pk - k^2 + i\epsilon) \right. \\ & \left. + \frac{1}{2pk} \ln(E - p^2 + pk - k^2 + i\epsilon) - \frac{1}{2} \frac{H(\Lambda)}{\Lambda^2} - \frac{1}{2} \frac{H(\Lambda)}{\Lambda^2} \right]. \quad (\text{A.118}) \end{aligned}$$

Combining terms we get

$$-\frac{g_2^2}{4} Z_D \left[ \frac{1}{2pk} \ln \frac{E - p^2 + pk - k^2 + i\epsilon}{E - p^2 - pk - k^2 + i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right]. \quad (\text{A.119})$$

Inserting  $Z_D = \frac{64\pi}{ag_2^2}$ , bringing minus inside the brackets and expanding the inside of the logarithm by -1 we get

$$\frac{g_2^2}{4} \frac{64\pi}{ag_2^2} \left[ -\frac{1}{2pk} \ln \frac{p^2 - pk + k^2 - E - i\epsilon}{p^2 + pk + k^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right]. \quad (\text{A.120})$$

Cleaning up and bringing the -1 inside the logarithm we get

$$\frac{16\pi}{a} \left[ \frac{1}{2pk} \ln \frac{p^2 + pk + k^2 - E - i\epsilon}{p^2 - pk + k^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right]. \quad (\text{A.121})$$

## A.9 Second term of equation (3.97)

We solve the following integral equation:

$$\begin{aligned} & -Z_D \frac{2}{\pi} \int_{-1}^1 \frac{dx_{\mathbf{k}}}{2} \int_0^\Lambda dq q^2 \int_{-1}^1 \frac{dx_{\mathbf{q}}}{2} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right] \\ & \times \frac{A\left(\mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2}\right)}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \quad (\text{A.122}) \end{aligned}$$

Rearranging this we get

$$\begin{aligned}
& - \frac{2}{\pi} \int_0^\Lambda dq q^2 Z_D \int_{-1}^1 \frac{dx_{\mathbf{q}}}{2} \int_{-1}^1 \frac{dx_{\mathbf{k}}}{2} A \left( \mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2} \right) \\
& \times \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} - \frac{H(\Lambda)}{\Lambda^2} \right] \\
& \times \frac{1}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{A.123}
\end{aligned}$$

By using equation (3.87), we can bring  $A \left( \mathbf{q}, \mathbf{k}; E, \frac{q^2}{2}, \frac{k^2}{2} \right)$  out from the integral in a new form:

$$\begin{aligned}
& - \frac{4}{\pi} \int_0^\Lambda dq q^2 A_S(q, k; E) \int_{-1}^1 \frac{dx_{\mathbf{q}}}{2} \left[ \frac{1}{E - (p^2 + \mathbf{p} \cdot \mathbf{q} + q^2) + 2i\epsilon} \right. \\
& \left. - \frac{H(\Lambda)}{\Lambda^2} \right] \frac{1}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{A.124}
\end{aligned}$$

The remaining part is similar to the first term so we can use this to quickly solve the integral by inserting the part inside brackets in equation (3.98) with  $q$  replacing  $k$ . Doing this gives us

$$\begin{aligned}
& - \frac{4}{\pi} \int_0^\Lambda dq q^2 A_S(q, k; E) \left[ -\frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \\
& \times \frac{1}{-1/a + \sqrt{-E + \frac{3q^2}{4} - 2i\epsilon}}. \tag{A.125}
\end{aligned}$$

Tidying up the above and we get the solved second term:

$$\frac{4}{\pi} \int_0^\Lambda dq q^2 \left[ \frac{1}{2pq} \ln \frac{p^2 + pq + q^2 - E - i\epsilon}{p^2 - pq + q^2 - E - i\epsilon} + \frac{H(\Lambda)}{\Lambda^2} \right] \frac{A_S(q, k; E)}{-1/a + \sqrt{\frac{3q^2}{4} - E - 2i\epsilon}}. \tag{A.126}$$

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