Representations of Locally Compact Groups

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Antti Rautio
Department of Mathematical Sciences
University of Oulu
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Contents

Introduction

The topic of this thesis is representation theory. The idea of representation theory is to represent an algebraic object, such as a locally compact group or an algebra, as a more concrete group or algebra consisting of matrices or operators. In this way we can study an algebraic object as collection of symmetries of a vector space. Hence we can apply the methods of linear algebra and functional analysis to the study of groups and algebras. Representation theory also provides a generalization of Fourier analysis to groups. The applications of representation theory are diverse, both within pure mathematics and outside of it. For example in the book [17] abstract harmonic analysis is applied to number theory. Outside of mathematics representation theory has been used in physics, chemistry and even engineering, for the latter see for instance [2].

The theory of representations of finite groups was initiated in the 1890's by people like Frobenius, Schur and Burnside. In the 1920's representations of arbitrary compact groups, and finite-dimensional (possibly nonunitary) representations of the classical matrix groups were investigated by Weyl and others. In the 1940's mathematicians such as Gelfand started to study (possibly infinite-dimensional) unitary representations of locally compact groups. Other important figures in representation theory include Harish-Chandra, Kirillov and Mackey. More on the history of representation theory can be found in [11].

Chapter 1 covers the results of Banach algebra and C*-algebra theory that we need for representation theory. The main theorem of this chapter is the spectral theorem for normal operators. In Chapter 2 we study locally compact groups and present basic results of Haar measures. Using the Haar measure we can define convolution of functions. The properties of this convolution are then investigated. We conclude the chapter with the construction of approximate identities.

In Chapter 3 we get to the main theme of the thesis, that is representation theory. We present the basic concepts of unitary representations of locally compact groups. The first important result is Schur's lemma, which describes irreducibility of a representation in terms of commuting operators. Then we describe the connection between unitary representations of a locally compact group and nondegenerate *-representations of the group algebra. In the last part of the chapter we study functions of positive type. We establish a correspondence between these functions and cyclic unitary representations. Then we can prove the last major result of the chapter, the Gelfand-Raikov theorem, which guarantees that locally compact groups have enough irreducible representations to separate points.

The representations of compact groups are particularly well behaved, which we shall show in Chapter 4. We summarize the results of this chapter in the Peter-Weyl theorem.

The main references used were [8] for Banach algebra theory, [17] for the spectral theorem and its application to Schur's lemma, and [5] for locally compact groups and representation theory.

Chapter 1

Banach Algebras

Banach and C*-algebras have an important role in the representation theory of locally compact groups. In this chapter we cover some of the basic theory of Banach algebras. Then we shall focus on commutative Banach algebras and the Gelfand theory of these algebras. We prove the Gelfand-Naimark theorem for commutative unital C*-algebras. We conclude the chapter by using the aforementioned theorem to prove the spectral theorem for normal operators, which will play a crucial role in representation theory.

1.1 Banach and C*-algebras

In this text the scalar field will be $\mathbb{C}$.

Definition 1.1.1. An *algebra* A is a vector space over $\mathbb{C}$ that is also a ring, with addition being the vector addition, and for every x and y in A and $\lambda, \mu \in \mathbb{C}$ the identity

$$(\lambda x)(\mu y) = \lambda\mu(xy)$$

holds. A *subalgebra* is a linear subspace of A that is also a subring of A.

We shall denote the Banach dual of a normed space A by A^*.

A normed linear space $(A, \|\cdot\|)$ that is also an algebra is called a *normed algebra* if

$$\|xy\| \leq \|x\|\|y\|$$

for every x and y in A. A normed algebra is a *Banach algebra* if it is also a Banach space.

An algebra is *commutative* if $xy = yx$ for all x and y in A. An algebra is *unital* if there exists an element $e \in A$ such that $ex = xe = x$ for all $x \in A$. We shall denote the identity element of a unital algebra by e. An element x of an unital algebra is *invertible* if there exists an element y such that $xy = yx = e$. Denote $y = x^{-1}$ and $A^{\times} = \{x \in A : x \text{ invertible in } A\}$.

Definition 1.1.2. Let A be an algebra over $\mathbb{C}$. An *involution* is a mapping $* : x \mapsto x^*$ from A to A such that

(a) $(x + y)^* = x^* + y^*$ and $(\lambda x)^* = \bar{\lambda} x^*$,

(b) $(xy)^* = y^* x^*$ and $(x^*)^* = x$

for all $x, y \in A$ and $\lambda \in \mathbb{C}$. This makes A into a **-algebra*. A normed algebra (Banach algebra) with an involution is called a *normed *-algebra* (*Banach *-algebra*) if the involution is isometric, that is if $\|x^*\| = \|x\|$ for all $x \in A$.

Some algebras do not have an identity. However an algebra A can always be embedded into an algebra with identity. Let $A_e = A \oplus \mathbb{C}$. With multiplication defined by $(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu)$, norm $\|(x, \lambda)\| = \|x\| + |\lambda|$, and involution $(x, \lambda)^* = (x^*, \overline{\lambda})$, the space A_e becomes a unital Banach *-algebra with identity $(0, 1)$. This is called *adjoining an identity* to A.

A Banach algebra A with involution $x \mapsto x^*$ is called a *C*-algebra*, if its norm satisfies the equation $\|x^* x\| = \|x\|^2$ for all $x \in A$. A closed subalgebra B of a C*-algebra A is called *C*-subalgebra* if $x^* \in B$ whenever $x \in B$. A C*-algebra is a Banach *-algebra since $\|x\|^2 = \|x^* x\| \leq \|x^*\|\|x\|$ implies $\|x\| \leq \|x^*\|$ and hence $\|x\| = \|x^*\|$ for every $x \in A$.

Example 1.1.3. Let X be locally compact Hausdorff space. We denote by $C^b(X)$ the set of bounded continuous complex valued functions on X. A continuous complex valued function *vanishes at infinity* if for every $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \varepsilon$ whenever $x \in X \setminus K$. Denote the set of all continuous functions that vanish at infinity by $C_0(X)$. The set $\mathrm{supp} f = \overline{\{x \in X : f(x) \neq 0\}}$ is the *support* of a function f. Denote by $C_c(X)$ the set of all continuous functions that have compact support. All of the sets $C^b(X)$, $C_0(X)$ and $C_c(X)$ are algebras with pointwise addition, multiplication and scalar multiplication. The norm is the supremum norm given by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

The involution for these spaces is the complex conjugation $f \mapsto \overline{f}$ given by $\overline{f}(x) = \overline{f(x)}$. With this norm $C^b(X)$ and $C_0(X)$ become commutative C*-algebras, whereas $C_c(X)$ is complete only when X is compact. If X is not compact, then only $C^b(X)$ is unital.

Example 1.1.4. Let $\mathcal{H}$ be a Hilbert space and denote the set of bounded operators on $\mathcal{H}$ by $L(\mathcal{H})$. Now $L(\mathcal{H})$ is a C*-algebra since if $T \in L\mathcal{H})$, then

$$\|T^*\| = \sup_{\|u\|=1} \|T^*u\| = \sup_{\|u\|=1} \sup_{\|v\|=1} |\langle T^*u, v\rangle| = \sup_{\|u\|=1} \sup_{\|v\|=1} |\langle u, Tv\rangle| \leq \|T\|,$$

so $\|T^*T\| \leq \|T\|^2$, and

$$\|T^*T\| = \sup_{\|u\|=1} \sup_{\|v\|=1} |\langle Tu, Tv\rangle| \geq \sup_{\|u\|=1} \|Tu\|^2 = \|T\|^2.$$

Therefore $\|T^*T\| = \|T\|^2$.

The modest looking equality that ties the multiplication, involution and norm of a C*-algebra turns out to have massive implications. Namely the (commutative) Gelfand-Naimark theorem, which we shall prove, states that every commutative C*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X, and more generally every C*-algebra is a C*-subalgebra of $L(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ by the (noncommutative) Gelfand-Naimark theorem.

The algebras we study in this chapter are all normed algebras.

When trying to understand an algebra, one natural question we may ask is, assuming the algebra is unital, what can we say about the invertible elements? If the algebra is a Banach algebra then the first nontrivial invertible element could be $e - x$ for some $\|x\| < 1$ since then the series $e + \sum_{n=1}^{\infty} x^n$ is convergent and is the inverse of $e - x$, which we will prove in a slightly more general form in Lemma 1.1.7. Modifying this example using scalar multiplication we obtain

$$\lambda^{-1}(e - \lambda^{-1}x)^{-1} = (\lambda e - x)^{-1}$$

if $\|x\| < \lambda$. This motivates our next definition.

Definition 1.1.5. For an element $x \in A$, the *spectrum* of x in A is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin A^\times\}.$$

The complement $\rho_A(x) = \mathbb{C} \setminus \sigma_A(x)$ is called the *resolvent set* of x. For $x \in A$, the number

$$r(x) = \inf\{\|x^n\|^{1/n} : n \in \mathbb{N}\}$$

is called the *spectral radius* of x.

Clearly $r(x) \leq \|x\|$. In the definition of spectral radius the infimum can in fact be replaced by a limit.

Lemma 1.1.6. *For every $x \in A$, $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.*

Proof. It is sufficient to show that for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|x^n\|^{1/n} < r(x) + \varepsilon$ for every $n \geq N(\varepsilon)$. Let $\varepsilon > 0$. Pick $k \in \mathbb{N}$ such that $\|x^k\|^{1/k} < r(x) + \varepsilon/2$. Any n can be expressed in the form $n = p(n)k + q(n)$, where $p(n) \in \mathbb{N}$, $0 \leq q(n) \leq k-1$. Therefore

$$\frac{p(n)}{n} = \frac{1}{k}\left(1 - \frac{q(n)}{n}\right) \to \frac{1}{k},$$

as $n \to \infty$. Hence $\|x^k\|^{p(n)/n}\|x\|^{q(n)/n} \to \|x^k\|^{1/k}$ as $n \to \infty$. Therefore there exists $n_k \in \mathbb{N}$ such that $\|x^k\|^{p(n)/n}\|x\|^{q(n)/n} < \|x^k\|^{1/k} + \varepsilon/2$ for all $n \geq n_k$. It follows that

$$\|x^n\|^{1/n} \leq \|x^k\|^{p(n)/n}\|x\|^{q(n)/n} < \|x^k\|^{1/k} + \varepsilon/2 < r(x) + \varepsilon$$

for all $n \geq n_k$. □

We already alluded to the following generalization of the geometric power series.

Lemma 1.1.7. *Let A be a Banach algebra and let $x \in A$ with $r(x) < 1$. Then $e - x$ is invertible in A and*

$$(e-x)^{-1} = e + \sum_{n=1}^{\infty} x^n.$$

Proof. Fix any η such that $r(x) < \eta < 1$. Then $\|x^n\|^{1/n} \leq \eta$ for all $n \geq N$ for some $N \in \mathbb{N}$. Then $\|x^n\| \leq \eta^n$ for all $n \geq N$, and since $\eta < 1$ the series $\sum_{n=1}^{\infty} \|x^n\|$ converges. Since A is complete, the sequence of partial sums $y_m = e + \sum_{n=1}^{m} x^n$, $m \in \mathbb{N}$ converges in A with limit $y = e + \sum_{n=1}^{\infty} x^n$. Indeed, $\|y - y_m\| \leq \sum_{n=m+1}^{\infty} \|x^n\|$. Now

$$(e-x)y_m = y_m(e-x) = e - x^{m+1}$$

for all m. Because $y_m \to y$ and $x^m \to 0$ as $m \to \infty$, we conclude that $(e-x)y = y(e-x) = e$. □

Note that if $\|x\| < 1$, then $r(x) < 1$ and the results of the above lemma hold.

As a corollary to the above construction we gain some insight to the topology of the set of invertible elements.

Lemma 1.1.8. *Let A be a normed unital algebra.*

(i) If $x, y \in A^\times$ are such that $\|y - x\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$, then

$$\|y^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|y - x\|.$$

Moreover $x \mapsto x^{-1}$ is a homeomorphism of $A^\times$.

(ii) *If A is complete, then $A^\times$ is open, and if $x \in A$ such that $\|x - e\| < 1$, then $x \in A^\times$.*

Proof. (i) If x and y are such that the inequality holds, then

$$\|y^{-1}\| - \|x^{-1}\| \leq \|y^{-1} - x^{-1}\| \leq \|y^{-1}\|\|x - y\|\|x^{-1}\| \leq \frac{1}{2}\|y^{-1}\|,$$

so $\|y^{-1}\| \leq 2\|x^{-1}\|$, and therefore

$$\|y^{-1} - x^{-1}\| \leq \|y^{-1}\|\|x - y\|\|x^{-1}\| \leq 2\|x^{-1}\|^2\|y - x\|.$$

Hence the bijection $x \mapsto x^{-1}$ of $A^\times$ is continuous, and since it is its own inverse, it is a homeomorphism.

(ii) If $\|x - e\| < 1$, then by Lemma 1.1.7 we have $e - (e - x) = x \in A^\times$. Now let x be any element of $A^\times$, and let $\|y - x\| < \|x^{-1}\|^{-1}$. Then

$$\|e - x^{-1}y\| \leq \|x^{-1}\|\|x - y\| < 1.$$

By what we have shown $x^{-1}y \in A^\times$, and hence $y \in A^\times$. Therefore $A^\times$ is open in A. □

The following theorem justifies the the name *spectral radius* for $r(x)$, and it is also one of the most fundamental results in the theory of Banach algebras.

Theorem 1.1.9. *Let A be a Banach algebra and $x \in A$. Then the spectrum $\sigma_A(x)$ is a non-empty compact subset of $\mathbb{C}$ and*

$$\max\{|\lambda| : \lambda \in \sigma_A(x)\} = r(x).$$

Proof. First note that $\sigma_A(x)$ is closed. This is true since $A^\times$ is open, and $\rho_A(x)$ is the inverse image of $A^\times$ with respect to the continuous function $\lambda \mapsto \lambda e - x$. Moreover $\sigma_A(x)$ is bounded, since if $|\lambda| > r(x)$, then $r((1/\lambda)x) < 1$ and hence by Lemma 1.1.7 $\lambda(e - (1/\lambda)x) = \lambda e - x \in A^\times$, so $\sigma_A(x) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r(x)\}$. Thus $\sigma_A(x)$ is compact.

Let us show next that $\sigma_A(x) \neq \emptyset$. Take any $l \in A^*$. We shall consider the function on $\rho_A(x)$ defined by

$$f(\lambda) = l((\lambda e - x)^{-1}).$$

If $\lambda, \mu \in \rho_A(x)$, then

$$(\lambda e - x)^{-1} = (\lambda e - x)^{-1}(\mu e - x)(\mu e - x)^{-1} = (\lambda e - x)^{-1}((\mu - \lambda)e + \lambda e - x)(\mu e - x)^{-1}$$

$$= ((\mu - \lambda)(\lambda e - x)^{-1} + e)(\mu e - x)^{-1} = (\mu - \lambda)(\lambda e - x)^{-1}(\mu e - x)^{-1} + (\mu e - x)^{-1}.$$

Now if $\lambda \neq \mu$, we have

$$\frac{f(\lambda)-f(\mu)}{\lambda-\mu} = -l((\lambda e - x)^{-1}(\mu e - x)^{-1}).$$

Since l is continuous and $y \mapsto y^{-1}$ is continuous on $A^\times$,

$$\lim_{\lambda\to\mu} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} = -l((\mu e - x)^{-2}),$$

so in particular the function f is analytic on $\rho_A(x)$. If $|\lambda| > \|x\|$, then

$$(\lambda e - x)^{-1} = \left(\lambda\left(e - \frac{1}{\lambda}x\right)\right)^{-1} = \frac{1}{\lambda}\left(e - \frac{1}{\lambda}x\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}x^n,$$

so

$$\|(\lambda e - x)^{-1}\| \leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\left(\frac{\|x\|}{|\lambda|}\right)^n = \frac{1}{|\lambda|}\frac{1}{1-|\lambda|^{-1}\|x\|},$$

which tends to zero as $|\lambda| \to \infty$. Thus $|f(\lambda)| \leq \|l\|\|(\lambda e - x)^{-1}\|$, so f vanishes at infinity.

Now assume $\sigma_A(x) = \emptyset$. Then clearly f is bounded on the closed disk $|\lambda| \leq \|x\|$ since it is continuous. It follows that f is bounded on the whole complex plane, and so by Liouville's theorem it is constant. Since f vanishes at infinity, we have $f = 0$. Because $l \in A^*$ was arbitrary, we get $l((\lambda e - x)^{-1}) = 0$ for each $\lambda \in \rho_A(x)$ and all $l \in A^*$, so by Hahn-Banach theorem $(\lambda e - x)^{-1} = 0$, which is a contradiction. Therefore $\sigma_A(x)$ is nonempty.

Let $s(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$. Now $s(x) \leq r(x)$, since $\sigma_A(x) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r(x)\}$. Assume that $s(x) < r(x)$. Then pick μ such that $s(x) < \mu < r(x)$. By what we have shown above, for $l \in A^*$ the function $f(\lambda) = l((\lambda e - x)^{-1})$ is analytic on $\rho(x)$, and in particular on the domain $U = \{\lambda : |\lambda| > s(x)\}$. Now for $|\lambda| > \|x\|$, we have

$$f(\lambda) = \sum_{n=0}^{\infty}\lambda^{-(n+1)}l(x^n).$$

This series is the Laurent series of f on the domain $|\lambda| > \|x\|$. Since f is analytic on U, the uniqueness of the Laurent series implies that

$$\sum_{n=0}^{\infty} l(x^n)\mu^{-(n+1)}$$

converges. Therefore $l(x^n)\mu^{-(n+1)} \to 0$ as $n \to \infty$. So for each $l \in A^*$ the set of complex numbers

$$\{l(x^n)\mu^{-(n+1)} : n \in \mathbb{N}\}$$

is bounded. Denote by $\widehat{y} \in A^{**}$ the functional $\widehat{y}(l) = l(y)$, where $y \in A$ and $l \in A^*$. Letting $y_n = \mu^{-(n+1)}x^n$ we see that $\sup_{n\in\mathbb{N}} \widehat{y_n}(l) < \infty$ for each $l \in A^*$, so by the Banach-Steinhaus theorem there exists $C > 0$ such that $\|\mu^{-(n+1)}x^n\| \leq C$ for all $n \in \mathbb{N}$. Hence $\|x^n\| \leq C\mu^{n+1}$, so

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} \leq \lim_{n\to\infty} (C\mu^{n+1})^{1/n} = \mu.$$

This is a contradiction, so $s(x) = r(x)$. □

It turns out that the Banach algebras (over the complex field) that do not have non-invertible elements other than zero are rather simple to describe.

Theorem 1.1.10 (Gelfand-Mazur theorem)**.** *Let A be a Banach algebra, and suppose each nonzero element is invertible. Then A is isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$. Since $\sigma_A(a)$ is nonempty, pick $\lambda \in \sigma_A(a)$. Now $\lambda e - a \notin A^\times$, so by assumption $\lambda e - a = 0$. Hence $a = \lambda e$, so every element is a scalar multiple of the identity, so A is isomorphic to $\mathbb{C}$. □

By the above we should have a look at the non-invertible elements of an algebra. It follows from the defintion that if x is not invertible, then $e \notin xA$ or $e \notin Ax$, so one of the sets xA or Ax is a proper subset of A. In fact such a set has some algebraic structure.

Definition 1.1.11. A subset I of an algebra A is an *ideal* if I is a subspace of A and $aI \subset I$ and $Ia \subset I$ for all $a \in A$. An ideal I is called proper if $I \neq A$. An ideal M is called *maximal* if it is proper and if I is an ideal of A such that $M \subset I$ and $M \neq I$, then $I = A$.

Every proper ideal of a unital algebra is in fact contained in a maximal ideal.

Lemma 1.1.12. *Let I be a proper ideal of a unital algebra A. Then I is contained in some maximal ideal M.*

Proof. Let I be a proper ideal of A. Let $\mathcal{L}$ be the set of all ideals L of A such that $I \subset L$ and $e \notin L$. Now $\mathcal{L}$ is nonempty since $I \in \mathcal{L}$. The set $\mathcal{L}$ is an ordered set with the inclusion order. We shall show that $\mathcal{L}$ satisfies the hypothesis of Zorn's lemma. Let $\mathcal{K}$ be a totally ordered subset of $\mathcal{L}$ and put $L = \bigcup\{K : K \in \mathcal{K}\}$. Then $e \notin L$ and L is an ideal since $\mathcal{K}$ is totally ordered. So $L \in \mathcal{L}$ and L is an upper bound for $\mathcal{K}$. Hence, by Zorn's lemma $\mathcal{L}$ has a maximal element M. If J is a proper ideal containing M, then by maximality $J = M$, so M is a maximal ideal. □

Remark 1.1.13. Suppose A is a commutative. Then an element $x \in A$ is invertible if and only if $x \notin M$ for every maximal ideal M. Indeed, $x \notin A^\times$ if and only if xA is a proper ideal, which is equivalent to $xA \subset M$ for some maximal ideal M by the previous lemma.

Recall that if I is a closed subspace of A, then the quotient norm on A/I is defined by

$$\|x + I\| = \inf_{a \in I} \|x + a\|.$$

The quotient of a normed (Banach) algebra by a closed ideal is again a normed (Banach) algebra.

Lemma 1.1.14. *Assume I is a closed ideal of a normed algebra A. Then A/I, equipped with the quotient norm, is a normed algebra. If A is a Banach algebra, then so is A/I.*

Proof. Since I is a closed subspace, we have that A/I is a Banach space if A is complete, so all we need to check is that the inequality $\|ab\| \le \|a\|\|b\|$ holds for the quotient norm. Now for any $x, y \in A$,

$$\|(x + I)(y + I)\| = \|xy + I\| = \inf_{z \in I} \|xy + z\| \le \inf_{a,b \in I} \|(x + a)(y + b)\|$$

$$\le \inf_{a,b \in I} \|x + a\|\|y + b\| = \|x + I\|\|y + I\|,$$

so the proof is complete. □

Since our algebras have topological structure, closed ideals are of particular interest.

Lemma 1.1.15. *Let A be a Banach algebra and $I \subset A$ be a proper ideal. Then*

$$I \cap \{x \in A : \|x - e\| < 1\} = \emptyset.$$

In particular $\overline{I}$ is also a proper ideal and every maximal ideal is closed in A.

Proof. If $x \in A$ is such that $\|x - e\| < 1$, then $e - (e - x) = x \in A^\times$, so $x \notin I$.

Now clearly $\overline{I}$ is a subspace, and if $x \in \overline{I}$ and $a \in A$, then

$$ax = a(\lim_n x_n) = \lim_n ax_n \in \overline{I},$$

where $x_n \in I$ for every $n \in \mathbb{N}$ and $\lim_n x_n = x$, so $\overline{I}$ is an ideal. By the first part of the lemma, $\overline{I}$ does not contain e, so $\overline{I}$ is a proper ideal.

If M is a maximal ideal, then $M \subset \overline{M} \subset A$, so $M = \overline{M}$. □

1.2 Gelfand Theory

In this subsection we study commutative Banach algebras. The main tool is the set of nonzero multiplicative functionals.

Definition 1.2.1. A linear functional φ on an algebra A is *multiplicative* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$. In other words φ is an algebra homomorphism between A and $\mathbb{C}$. We denote the set of all non-zero multiplicative functionals on a Banach algebra A by $\Delta(A)$.

Let us prove some basic properties of multiplicative functionals.

Lemma 1.2.2. *Let A be a Banach algebra with identity e. Suppose $\varphi \in \Delta(A)$. Then $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every invertible element $x \in A$.*

Proof. Pick $x \in A$ such that $\varphi(x) \neq 0$. Then $\varphi(x)\varphi(e) = \varphi(xe) = \varphi(x)$ so dividing by $\varphi(x)$ yields $\varphi(e) = 1$. If $x \in A$ is invertible, then $\varphi(x^{-1})\varphi(x) = \varphi(x^{-1}x) = \varphi(e) = 1$. Hence $\varphi(x) \neq 0$. □

Remark 1.2.3. Because $\psi(e) = 1$ for every $\psi \in \Delta(A_e)$, each $\varphi \in \Delta(A)$ has a unique extension $\widetilde{\varphi} \in \Delta(A_e)$ given by

$$\widetilde{\varphi}(x + \lambda e) = \varphi(x) + \lambda, \quad x \in A, \quad \lambda \in \mathbb{C}.$$

Let $\widetilde{\Delta}(A) = \{\widetilde{\varphi} : \varphi \in \Delta(A)\}$. Morever, let φ_∞ denote the homomorphism from A_e to $\mathbb{C}$ with kernel A, that is, $\varphi_\infty(x + \lambda e) = \lambda$. Then

$$\Delta(A_e) = \widetilde{\Delta}(A) \cup \{\varphi_\infty\}.$$

To see this, let $\psi \in \Delta(A_e)$ and $\psi \neq \varphi_\infty$. Then $\psi|_A \in \Delta(A)$ since ψ is nonzero. Hence $\psi = \widetilde{\psi|_A}$. Identifying $\Delta(A)$ with $\widetilde{\Delta(A)} \subset \Delta(A_e)$ we always regard $\Delta(A)$ as a subset of $\Delta(A_e)$. In this sense, $\Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$.

It is worth noting that we do not need to assume that a multiplicative functional is continuous. These mappings are in fact automatically continuous.

Lemma 1.2.4. *Let A be a Banach algebra. Every $\varphi \in \Delta(A)$ is a bounded linear functional on A. In particular, $\|\varphi\| \leq 1$ and $\|\varphi\| = 1$ if A is unital.*

Proof. If $|\lambda| > \|x\|$ then $\lambda e - x$ is invertible, so $\lambda - \varphi(x) = \varphi(\lambda e - x) \neq 0$. In other words $\varphi(x) \neq \lambda$ so $|\varphi(x)| \leq \|x\|$. Therefore $\|\varphi\| \leq 1$ and $\|\varphi\| = 1$ if A is unital, since $\varphi(e) = 1$. □

We will prove another automatic continuity result in Lemma 3.2.14. Automatic continuity of various maps, such as homomorphisms and derivations, is a field of research in its own right, see for instance [4] and [19].

Observe that the above theorem says that $\Delta(A)$ is a subset of the closed unit ball of A^* or the unit sphere if A is unital.

It should be noted that an algebra does not necessarily have any nonzero multiplicative functionals. For instance if $\mathcal{H}$ is a Hilbert space of dimension greater than one, then $\Delta(L(\mathcal{H})) = \emptyset$. We give a sketch of a proof. Note first that if $\varphi \in \Delta(A)$ and x is a *nilpotent* element, that is $x^n = 0$ for some n, then $\varphi(x)^n = \varphi(x^n) = 0$, so $\varphi(x) = 0$. Now let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis for $\mathcal{H}$. Assume first that $\dim \mathcal{H}$ is even or infinite. Then there exists a partition of Λ to disjoint sets Λ_1 and Λ_2 with the same cardinality. Let $\beta : \Lambda_1 \to \Lambda_2$ be a bijection. Now define operators $A(\sum_{\lambda\in C} \alpha_\lambda e_\lambda) = \sum_{\lambda\in C\cap\Lambda_1} \alpha_\lambda e_{\beta(\lambda)}$ and $B(\sum_{\lambda\in C} \alpha_\lambda e_\lambda) = \sum_{\lambda\in C\cap\Lambda_2} \alpha_\lambda e_{\beta^{-1}(\lambda)}$. It is easy to verify that $A^2 = 0 = B^2$, and $(A + B)^2 = I$. Hence $\varphi(I) = \varphi((A + B)^2) = (\varphi(A) + \varphi(B))^2 = 0$, so $\varphi = 0$. If $\dim \mathcal{H} = n$ is odd (or more generally finite) and greater than one, then let $A(x_1, \ldots, x_{n-1}, x_n) = (e_2, \ldots, e_n, 0)$ and $B(e_1, \ldots, e_n) = (0, \ldots, 0, e_1)$. Then $A^n = 0$ and $B^2 = 0$, but $(A+B)^n = I$, so $\varphi(I) = 0$ for every multiplicative functional φ.

However a commutative unital Banach algebra has maximal ideals, and those ideals correspond to multiplicative functionals, which we prove in Theorem 1.2.8. Because of this the results of Gelfand theory are often stated for commutative algebras. The proofs of some of the following results do not seem to depend on commutativity, however these statements are quite meaningless if the spectrum is empty. Hence we shall assume that the spectrum is nonempty, which is true when the algebra is commutative.

The set $\Delta(A)$ becomes a topological space when we give it the relative weak* topology from A^*. The space $\Delta(A)$ is often called the *spectrum* of A. Reader may find it confusing to use the term spectrum in two different contexts. However the two notions are in fact related, as we shall see.

It is important to know what kind of space the spectrum $\Delta(A)$ is.

Theorem 1.2.5. *Let A be a Banach algebra. Then*

(i) $\Delta(A)$ *is compact Hausdorff if A has an identity;*

(ii) $\Delta(A)$ *is a locally compact Hausdorff space;*

(iii) $\Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$ *is the one-point compactification of* $\Delta(A)$.

Proof. (i) Since $\Delta(A)$ is subset of the unit ball, which is compact in the weak*-topology, it is sufficient to show that $\Delta(A)$ is closed in A^*. To see this, let (φ_λ) be a net in $\Delta(A)$ such that $\varphi_\lambda \to \varphi \in A^*$ in the weak*-topology, so in other words $\varphi_\lambda(x) \to \varphi(x)$ for every $x \in A$. Let $x, y \in A$. Now $\varphi(xy) = \lim_\lambda \varphi_\lambda(xy) = \lim_\lambda \varphi_\lambda(x)\varphi_\lambda(y) = \lim_\lambda \varphi_\lambda(x) \lim_\lambda \varphi_\lambda(y) = \varphi(x)\varphi(y)$. Therefore φ is multiplicative. On the other hand $\varphi(e) = \lim_\lambda \varphi_\lambda(e) = \lim_\lambda 1 = 1$ so φ is not zero and $\varphi \in \Delta(A)$. Hence $\Delta(A)$ is closed and is compact.

(ii) Now we assume A does not have an identity. We denote the basic neighborhoods of $\Delta(A)$ and $\Delta(A_e)$ by U and U_e, respectively. Then, for $\varphi \in \Delta(A)$, $\varepsilon > 0$ and a finite subset $F \subset A$,

$$U_e(\varphi, F, \varepsilon) = \begin{cases} U(\varphi, F, \varepsilon) \cup \{\varphi_\infty\} & \text{if } |\varphi(x)| < \varepsilon \text{ for all } x \in F, \\ U(\varphi, F, \varepsilon) & \text{otherwise.} \end{cases}$$

Therefore the topology on $\Delta(A)$ coincides with the relative topology of $\Delta(A_e)$. Now if $\varphi \in \Delta(A) \subset \Delta(A_e)$, we may find open disjoint neighborhoods U and V such that $\varphi \in U$ and $\varphi_\infty \in V$. Now $X \setminus V$ is compact in $\Delta(A_e)$, so it is also compact in $\Delta(A)$ and $\varphi \in U \subset X \setminus V$. Hence $\Delta(A)$ is locally compact.

(iii) Let $x \in A$ and $\varepsilon > 0$. Now

$$\begin{aligned} U_e(\varphi_\infty, x, \varepsilon) &= \{\varphi_\infty\} \cup \{\varphi \in \Delta(A) : |\varphi(x)| < \varepsilon\} \\ &= \Delta(A_e) \setminus \{\psi \in \Delta(A_e) : |\psi(x)| \geq \varepsilon\}. \end{aligned}$$

Now the sets $\{\psi \in \Delta(A_e) : |\psi(x)| \geq \varepsilon\}$, $x \in A$ are closed in $\Delta(A_e)$ and hence compact. Finite union of such sets is compact too. Therefore the complement of a basic neighborhood $U_e(\varphi_\infty, F, \varepsilon)$ is compact, so $\Delta(A_e)$ is the one-point compactification of $\Delta(A)$. □

Using the spectrum we get a rather natural representation for A.

Definition 1.2.6. For $x \in A$, we define $\widehat{x} : \Delta(A) \to \mathbb{C}$ by $\widehat{x}(\varphi) = \varphi(x)$. Then $\widehat{x}$ is a continuous, since if $\varphi_\lambda \to \varphi$, then $\widehat{x}(\varphi_\lambda) = \varphi_\lambda(x) \to \varphi(x) = \widehat{x}(\varphi)$. The function $\widehat{x}$ is called the *Gelfand transform* of x. The mapping

$$\Gamma_A : A \to C(\Delta(A)), \quad x \mapsto \widehat{x}$$

is called *Gelfand representation* of A.

It is easy to verify that the Gelfand representation is an algebra homomorphism. We will prove some important results for the Gelfand representation.

Theorem 1.2.7. *Let A be a Banach algebra and Γ be the Gelfand representation of A.*

(i) Γ maps A into $C_0(\Delta(A))$ and is norm decreasing;

(ii) $\Gamma(A)$ separates the points of $\Delta(A)$.

Proof. (i) If A is unital, then $\Delta(A)$ is compact, so $C_0(\Delta(A)) = C(\Delta(A))$. Assume now that A does not have an identity. Then $\Delta(A_e)$ is the one-point compactification of $\Delta(A)$ and $\widehat{x}(\varphi_\infty) = 0$ for $x \in A$, so $\widehat{x} \in C_0(\Delta(A))$. Also

$$\|\Gamma(x)\|_\infty = \|\widehat{x}\|_\infty = \sup_{\varphi \in \Delta(A)} |\widehat{x}(\varphi)| = \sup_{\varphi \in \Delta(A)} |\varphi(x)| \leq \sup_{\varphi \in \Delta(A)} \|\varphi\|\|x\| \leq \|x\|.$$

(ii) If $\varphi_1 \neq \varphi_2$, then necessarily there exists $x \in A$ such that $\varphi_1(x) \neq \varphi_2(x)$. Therefore $\widehat{x}(\varphi_1) \neq \widehat{x}(\varphi_2)$ so $\Gamma(A)$ separates points of $\Delta(A)$. □

The spectrum of a commutative Banach algebra is sometimes called the *maximal ideal space* of the algebra, which is an appropriate name by the following theorem.

Theorem 1.2.8. *For a commutative unital Banach algebra A, the map*

$$\varphi \mapsto \ker \varphi = \{x \in A : \varphi(x) = 0\}$$

is a bijection between $\Delta(A)$ and the set of maximal ideals of A.

Proof. If $\varphi \in \Delta(A)$, then $\ker \varphi$ is a maximal ideal. Let $\varphi_1, \varphi_2 \in \Delta(A)$ and assume now that $\ker \varphi_1 = \ker \varphi_2$, and denote this ideal by I. Since $e \notin I$ and I is maximal, we can express any $x \in A$ uniquely as

$$x = \lambda e + y, y \in I, \lambda \in \mathbb{C}.$$

Now since $\varphi(e) = 1$ for any $\varphi \in \Delta(A)$, we get

$$\varphi_1(x) = \lambda\varphi_1(e) + \varphi_1(y) = \lambda = \lambda\varphi_2(e) + \varphi_2(y) = \varphi_2(x)$$

for every $x \in A$, so $\varphi_1 = \varphi_2$ and $\varphi \mapsto \ker \varphi$ is injective.

Let M be a maximal ideal of A. Now M is closed in A, so A/M is a Banach algebra. We shall show that if $x + M \neq M$ for some $x \in A$, then $x + M \in (A/M)^\times$. First if $x + M \neq M$ for some $x \in A$, then $x \in A \setminus M$.

Let $K = \{m + ax : m \in M, a \in A\} \subset A$. Now K is in fact an ideal in A, since if $m_1, m_2 \in M$, $a_1, a_2 \in A$ and $\lambda \in \mathbb{C}$, then

$$m_1 + a_1x + m_2 + a_2x = m_1 + m_2 + (a_1 + a_2)x \in K$$

and by commutativity

$$(m_1 + a_1x)a_2 = m_1a_2 + (a_1a_2)x \in K.$$

Also $K \neq M$ since $x = 0 + ex \in K$.

Since $M \subset K$ and $M \neq K$, we have $K = A$ due to maximality. Therefore there exists $m_0 \in M$ and $a_0 \in A$ such that $e = m_0 + a_0 x$, so $e - a_0 x \in M$. Therefore $e + M = e + (a_0 x - e) + M = a_0 x + M = (a_0 + M)(x + M)$, so $(x+M)^{-1} = a_0 + M \in A/M$. By the Gelfand-Mazur theorem A/M is isomorphic to $\mathbb{C}$. If we denote the quotient map from A to A/M by q and the isomorphism from A/M to $\mathbb{C}$ by i, then $M = \ker i \circ q$. □

As we promised earlier, the spectrum of an element and the spectrum of an algebra are indeed related.

Theorem 1.2.9. *Let A be a commutative unital Banach algebra. For each $x \in A$ $\widehat{x}(\Delta(A)) = \sigma_A(x)$.*

Proof. If $\lambda \in \rho_A(x)$, then $0 \neq \varphi(x - \lambda e) = \varphi(x) - \lambda$, so $\varphi(x) \in \mathbb{C} \setminus \rho_A(x) = \sigma_A(x)$. Therefore $\varphi(x) \in \sigma_A(x)$ for every $\varphi \in \Delta(A)$, so $\widehat{x}(\Delta(A)) \subset \sigma(x)$.

Conversely if $\lambda \in \sigma_A(x)$, then $I = (\lambda e - x)A$ is a proper ideal in A and hence it is contained in some $\ker \varphi$ for some $\varphi \in \Delta(A)$. It follows that $\lambda \in \widehat{x}(\Delta(A))$. □

A stronger relation between the spectrum of an element and an algebra will be contained in the proof of the spectral theorem.

Now we turn our attention to C*-algebras.

Lemma 1.2.10. *Let A be a C*-algebra. Then the Gelfand homomorphism is a *-homomorphism; that is, $\widehat{x^*} = \overline{\widehat{x}}$.*

Proof. We have to show that $\varphi(x^*) = \overline{\varphi(x)}$ for $\varphi \in \Delta(A)$ and $x \in A$. We may assume that A has identity A. Let

$$\varphi(x) = \alpha + i\beta \text{ and } \varphi(x^*) = \gamma + i\delta,$$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Towards a contradiction, assume that $\beta + \delta \neq 0$ and let

$$y = (\beta + \delta)^{-1}(x + x^* - (\alpha + \gamma)e) \in A.$$

Now since $e = e^{**} = (e^* e)^* = e^* e = e^*$, we have $y^* = y$ and

$$\varphi(y) = (\beta + \delta)^{-1}(\alpha + i\beta + \gamma + i\delta - (\alpha + \gamma)) = i.$$

Therefore for all $t \in \mathbb{R}$,

$$\varphi(y + tie) = \varphi(y) + ti = (t+1)i,$$

so $|t+1| = |\varphi(y + tie)| \leq \sup\{|l(y + tie)| : l \in A^*, \|l\| \leq 1\} = \|y + tie\|$. Since $y = y^*$, the C*-norm property gives

$$\begin{aligned}(t+1)^2 &\leq \|y + tie\|^2 = \|(y + tie)(y + tie)^*\| = \|(y + tie)(y - tie)\| \\ &= \|y^2 + t^2 e\| \leq \|y^2\| + t^2.\end{aligned}$$

It follows that $2t+1 \leq \|y^2\|$ for every $t \in \mathbb{R}$, which is impossible (take for instance $t = \|y^2\|$). This shows that $\beta + \delta = 0$, so $\delta = -\beta$. Hence

$$\varphi((ix)^*) = \varphi(-ix^*) = -i\varphi(x^*) = -i(\gamma + i\delta) = -\beta - i\gamma.$$

On the other hand $\varphi(ix) = i(\alpha + i\beta) = -\beta + i\alpha$. Going through exactly the same arguments with ix in place of x we obtain $\alpha + (-\gamma) = 0$, so $\gamma = \alpha$. This shows that $\varphi(x^*) = \overline{\varphi(x)}$. □

For commutative C*-algebras the spectral radius coincides with the norm of the element.

Lemma 1.2.11. *Let A be a commutative C*-algebra and $x \in A$. Then $r(x) = \|x\|$.*

Proof. First note that if $y^* = y$, then $\|y^2\| = \|y\|^2$, so

$$\|y^{2^n}\| = \|y\|^{2^n}$$

for all $n \geq 0$.

Now $(xx^*)^m = x^m(x^*)^m$ for all nonnegative integers m. Hence

$$\|x\|^{2^n} = \|xx^*\|^{2^{n-1}} = \|x^{2^n}(x^*)^{2^n}\|^{1/2} = \|x^{2^n}(x^{2^n})^*\|^{1/2} = \|x^{2^n}\|.$$

Therefore

$$\|x^{2^n}\|^{2^{-n}} = \|x\|$$

for all n, so $r(x) = \|x\|$. □

The above argument also works for a normal operator $T \in L(\mathcal{H})$. That is if $T \in L(\mathcal{H})$ such that $T^*T = TT^*$, then $r(T) = \|T\|$.

Now we can present the main theorem of this subsection.

Theorem 1.2.12 (Gelfand-Naimark Theorem)**.** *For a commutative unital C*-algebra A the Gelfand homomorphism is an isometric *-isomorphism from A onto $C(\Delta(A))$.*

Proof. It is sufficient to show that Γ is an isometry and surjective. If $x \in A$, then

$$\|\widehat{x}\|_\infty = \sup_{\varphi \in \Delta(A)} |\widehat{x}(\varphi)| = \sup_{\varphi \in \Delta(A)} |\varphi(x)| = \max_{\lambda \in \sigma_A(x)} |\lambda| = r(x) = \|x\|.$$

Now since Γ is an isometry and A is complete, we have that $\Gamma(A)$ is closed in $C(\Delta(A))$. On the other hand $\Gamma(A)$ separates points and contains constants, so by Stone-Weierstrass we have $\Gamma(A) = C(\Delta(A))$. □

The above theorem can also be stated and proved for commutative C*-algebras without an identity element. Then the Gelfand homomorphism is an isometric *-isomorphism from A onto $C_0(\Delta(A))$. However we only need the unital case to prove the spectral theorem.

1.3 The Spectral Theorem

In linear algebra there are many useful results that are called spectral theorems that describe how a matrix can be diagonalized. For example a square matrix T is normal if and only if there exists a unitary matrix U such that $T = UDU^*$ where D is a diagonal matrix. In this chapter we shall give a generalization of this theorem for (possibly infinite-dimensional) operators.

Assume that T is normal. Then we denote the smallest, closed, self-adjoint, unital subalgebra containing T by A_T. This is the closure of the algebra generated by T, T^* and I, and is commutative because T is normal. It is not difficult to see that if $T = UDU^*$ is a normal matrix, then A_T is isomorphic to $\mathbb{C}^n$ with pointwise operations, where n is the number of distinct eigenvalues of T. Hence A_T "diagonalizes" to $\mathbb{C}^n$. This is the form of the spectral theorem which we shall generalize.

In the next theorem we denote $\sigma(T)$ to be the spectrum with respect to $L(\mathcal{H})$ and $\sigma_A(T)$ to be the spectrum with respect to the subalgebra A_T.

Theorem 1.3.1 (Spectral Theorem)**.** *Let $T \in L(\mathcal{H})$ be a normal operator. Then there exists an isometric *-isomorphism $\Phi : C(\sigma(T)) \to A_T$ such that $\Phi(i_{\sigma(T)}) = T$, where $i_{\sigma(T)} : \sigma(T) \to \mathbb{C}$ is the inclusion mapping $i_{\sigma(T)}(z) = z$.*

Proof. Consider the Gelfand transform of T, that is,

$$\begin{aligned} \widehat{T} \ : \ & \Delta(A_T) \to \mathbb{C} \\ & \gamma \mapsto \gamma(T). \end{aligned}$$

Now $\widehat{T}$ is continuous. Moreover, if $\widehat{T}(\gamma_1) = \widehat{T}(\gamma_2)$, then we have

$$\gamma_1(T^*) = \overline{\gamma_1(T)} = \overline{\gamma_2(T)} = \gamma_2(T^*).$$

Thus γ_1 and γ_2 agree on a unital subalgebra of $L(\mathcal{H})$ generated by T and T^*, and by continuity they agree on A_T, so $\gamma_1 = \gamma_2$. Therefore $\widehat{T}$ is injective. Now since $\Delta(A_T)$ is compact and $\widehat{T}$ is continuous and injective, we have that $\Delta(A_T)$ is homeomorphic to its image, which in fact is $\sigma_A(T)$. That is

$$\widehat{T} : \Delta(A_T) \to \sigma_A(T)$$

is a homeomorphism.

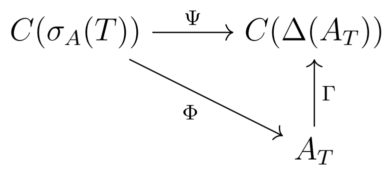
Next consider the map

$$\begin{aligned} \Psi : C(\sigma_A(T)) &\to C(\Delta(A_T)) \\ f &\mapsto f \circ \widehat{T}. \end{aligned}$$

The map Ψ is an isometric *-isomorphism. Now

$$\bar{f} \circ \widehat{T}(\gamma) = \bar{f}(\widehat{T}(\gamma)) = \overline{f(\widehat{T}(\gamma))} = \overline{f \circ \widehat{T}}(\gamma),$$

so Ψ is an isometric *-isomorphism. We now define $\Phi = \Gamma^{-1}\Psi$, so that the diagram



commutes. Being a composition of isometric *-isomorphisms, Φ is also an isometric *-isomorphism. We consider the effect of Φ on a function $f \in C(\sigma_A(T))$. First note that $\Psi(f)(\gamma) = f(\widehat{T}(\gamma)) = f(\gamma(T))$. Now since the Gelfand transform Γ is an isomorphism, for every $f \in C(\sigma(T))$ there exists unique $P \in A_T$ such that $\Psi(f) = \Gamma(P)$, so $P = \Gamma^{-1}\Psi(f) = \Phi(f)$. So for every $\gamma \in \Delta(A_T)$

$$f(\gamma(T)) = \Psi(f)(\gamma) = \Gamma(P)(\gamma) = \gamma(P) = \gamma(\Phi(f)).$$

In particular $\gamma(\Phi(i_{\sigma(T)})) = i_{\sigma(T)}(\gamma(T)) = \gamma(T)$ and $\gamma(\Phi(1)) = 1 = \gamma(I)$ for every $\gamma \in \Delta(A_T)$, so $\widehat{\Phi(i_{\sigma(T)})} = \widehat{T}$ and $\widehat{\Phi(1)} = \widehat{I}$, which implies $\Phi(i_{\sigma(T)}) = T$ and $\Phi(1) = I$. It remains to show that $\sigma_A(T) = \sigma(T)$. Clearly $\sigma(T) \subset \sigma_A(T)$, since if $\lambda I - T$ is invertible in A_T then it is clearly invertible in $L(\mathcal{H})$ as well.

Now let $\lambda \in \sigma_A(T)$, $\varepsilon > 0$ be arbitrary and choose $f \in C(\sigma_A(T))$ such that $\|f\|_\infty \leq 1$, $f(\lambda) = 1$ and $f(\mu) = 0$ whenever $|\lambda - \mu| \geq \varepsilon$. Let $P = \Phi(f)$. Since Φ is an isometry and f is zero outside a ball centered at λ, we have

$$\|(T - \lambda I)P\| = \|\Phi^{-1}((T - \lambda I)P)\|_\infty = \|(i_{\sigma_A(T)} - \lambda)f\|_\infty \leq \varepsilon.$$

Thus if $T - \lambda I$ is invertible, it would follow that

$$\begin{aligned} 1 &= \|f\|_\infty = \|P\| = \|(T - \lambda I)^{-1}(T - \lambda I)P\| \\ &\leq \|(T - \lambda I)^{-1}\|\|(T - \lambda I)P\| \leq \|(T - \lambda I)^{-1}\|\varepsilon. \end{aligned}$$

Since ε was arbitrary, this forces $\|(T - \lambda I)^{-1}\|$ to infinity. Hence $T - \lambda I$ is not invertible, so indeed $\lambda \in \sigma(T)$. □

From the spectral theorem we get a useful corollary.

Lemma 1.3.2. *Let T be a normal operator on a complex Hilbert space. The following are equivalent:*

(i) $\sigma(T)$ is a point.

(ii) T *is a scalar multiple of the identity operator.*

(iii) $A_T = \mathbb{C}$.

Proof. (i) implies (ii): Now

$$T = \Phi(i_{\sigma(T)}) = \Phi(i_{\{\lambda\}}) = \Phi(\lambda\chi_{\{\lambda\}}) = \lambda\Phi(\chi_{\{\lambda\}}) = \lambda I.$$

(ii) implies (iii): Since $T = \lambda I$, we have $A_T = \mathbb{C}I = \mathbb{C}$.

(iii) implies (i): If $\sigma(T)$ has more than one point, then there exists $f, g \in C(\sigma(T)) \setminus \{0\}$, such that f vanishes outside of an open ball centered at some $\lambda_1 \in \sigma(T)$ and g vanishes outside of an open ball centered at some different $\lambda_2 \in \sigma(T)$, and furthermore $fg = 0$. So $\Phi(f)\Phi(g) = \Phi(fg) = 0$, which is a contradiction, since in $A_T = \mathbb{C}$ all nonzero elements are invertible. □

Chapter 2

Locally Compact Groups

The principal objects on which abstract harmonic analysis takes place are the locally compact groups. The fundamental feature of a locally compact group, without which we could do little, is the existence and uniqueness of a translation invariant measure λ. Such a measure also gives the space $L^1(\lambda)$ the structure of a Banach *-algebra. These are the key ideas of this chapter. We conclude the chapter with the construction of approximate identities.

2.1 Haar measure

Definition 2.1.1. A *topological group* is a group G equipped with a topology such that the group operations are continuous, that is $(x, y) \mapsto xy$ is continuous from $G \times G$ to G and $x \mapsto x^{-1}$ is continuous from G to G.

We shall denote the unit of a topological group by e. If $A \subset G$ and $x \in G$, we define

$$Ax = \{yx : y \in A\}, \quad xA = \{xy : y \in A\}, \quad A^{-1} = \{y^{-1} : y \in A\},$$

and if $B \subset G$ then we define

$$AB = \{xy : x \in A, y \in B\}.$$

We say that A is symmetric if $A^{-1} = A$.

Theorem 2.1.2. *Let G be a topological group.*

(i) For every neighborhood U of e there is a symmetric neighborhood V of e such that $VV \subset U$.

(ii) If A and B are compact sets in G, so is AB.

Proof. (i) Since $(x, y) \mapsto xy$ is continuous at e it follows that for every neighborhood U of e there exists neighborhoods V_1 and V_2 of e with $V_1V_2 \subset U$. We can choose the desired set to be $V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}$, which is clearly symmetric and $VV \subset V_1V_2 \subset U$.

(ii) The set AB is the image of a compact set $A \times B$ under the continuous map $(x, y) \mapsto xy$, hence AB is compact. □

If f is a function on a topological group G and $y \in G$, we define the left and right translates of f through y by

$$L_y f(x) = f(y^{-1}x), \quad R_y f(x) = f(xy).$$

Here we use y^{-1} in L_y and y in R_y so that the maps $y \mapsto L_y$ and $y \mapsto R_y$ are group homomorphisms:

$$L_{yx} = L_y L_x, \quad R_{yz} = R_y R_z.$$

We say that a function f on G is *left* (respectively *right*) *uniformly continuous* if $\|L_y f - f\|_\infty \to 0$ (respectively $\|R_y f - f\|_\infty \to 0$) as $y \to e$. We shall denote the set of bounded left (right) uniformly continuous functions on G by $LUC(G)$ ($RUC(G)$).

Theorem 2.1.3. *If G is a topological group, then $C_c(G) \subset LUC(G) \cap RUC(G)$.*

Proof. We shall prove $f \in RUC(G)$. The argument for $f \in LUC(G)$ is similar. Let $f \in C_c(G)$ and $\varepsilon > 0$ and denote $K = \mathrm{supp} f$. For every $x \in K$ there exists a neighborhood U_x of e such that $|f(xy) - f(x)| < \frac{1}{2}\varepsilon$ for all $y \in U_x$, and there exists a symmetric neighborhood V_x of e such that $V_xV_x \subset U_x$. The family $\{xV_x\}_{x\in K}$ is an open cover of K, so there $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{k=1}^n x_kV_{x_k}$. Let $V = \bigcap_{k=1}^n V_{x_k}$. We claim that $\|R_y f - f\|_\infty < \varepsilon$ for $y \in V$.

If $x \in K$ then there exists k such that $x \in x_kV_{x_k}$, so $xy \in x_kV_{x_k}V_{x_k} \subset x_kU_{x_k}$. But then

$$|f(xy) - f(x)| \leq |f(xy) - f(x_k)| + |f(x_k) - f(x)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Similarly, if $xy \in K$, then $xy \in x_kV_{x_k}$ for some k and $x = xyy^{-1} \in x_kV_{x_k}V_{x_k} \subset x_kU_{x_k}$, so

$$|f(xy) - f(x)| \leq |f(xy) - f(x_k)| + |f(x_k) - f(x)| < \varepsilon.$$

If $x, xy \notin K$, then $f(x) = f(xy) = 0$. □

Definition 2.1.4. By a *locally compact group* we shall mean a topological group whose topology is locally compact and Hausdorff.

In this text the Borel sets of a topological space are generated by *open* sets.

Definition 2.1.5. A *left* (respectively *right*) *Haar measure* on G is a nonzero countably additive measure μ on G that satisfies the following properties:

(i) $\mu(xE) = \mu(E)$ $(\mu(Ex) = \mu(E))$ for every $x \in G$ and for every Borel set $E \subset G$;

(ii) $\mu(K) < \infty$ for every compact K;

(iii) $\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ open}\}$ for every Borel $E \subset G$;

(iv) $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ for every open $E \subset G$.

Theorem 2.1.6. *Let G be a locally compact group.*

(i) There exists a left Haar measure λ on G.

(ii) If λ is a left Haar measure on G, then $\lambda(U) > 0$ for every nonempty open set U, and $\int f d\lambda > 0$ for every $f \in C_c^+(G) = \{f \in C_c(G) : f \geq 0\} \setminus \{0\}$.

(iii) If λ and μ are left Haar measures on G, then there exists $c \in (0, \infty)$ such that $\mu = c\lambda$.

The proof can be found for instance in [5, p. 37, Theorem 2.10.]. In this book an invariant nonzero positive functional on $C_c(G)$ is constructed, and then by the Riesz representation theorem it is given by an appropriate measure. From now on we always assume that G is locally compact.

Example 2.1.7.

(1) $dx/|x|$ is a Haar measure on the multiplicative group $\mathbb{R} \setminus \{0\}$.

(2) The $ax+b$ group is the group of affine transformations $x \mapsto ax+b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$. On G $dadb/a^2$ is a left Haar measure and $dadb/a$ is a right Haar measure.

(3) Lebesgue measure $\prod_{i<j} d\alpha_{ij}$ is a left and right Haar measure on the group of $n \times n$ real matrices (α_{ij}) such that $\alpha_{ij} = 0$ for $i > j$ and $\alpha_{ii} = 1$ for $1 \leq i \leq n$. This is the group of upper triangular matrices of with diagonal entries all equal to 1. When $n = 3$ the group is often called the *Heisenberg group*.

(4) On the group $GL(n, \mathbb{R}) = \{T \in L(\mathbb{R}^n) : \det T \neq 0\}$ $|\det T|^{-n} dT$ is a left and right Haar measure, where dT is Lebesgue measure on $\mathbb{R}^{n^2}$, where we interpret $\mathbb{R}^{n^2}$ as the vector space of all real $n \times n$ matrices.

(5) It can be proved that the *special linear group* $SL(2,\mathbb{R}) = \{T \in GL(2,\mathbb{R}) : \det T = 1\}$ has the *Iwasawa decomposition*, that is

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : x, \theta \in \mathbb{R}, a > 0 \right\}.$$

Using this decomposition a left and right Haar measure is given by $\frac{d\theta}{2\pi}\frac{dxda}{a^2}$.

For the proof of (5) see [6, p. 255, 17.].

Definition 2.1.8. Let (X,μ) be a measure space and let $0 < p < \infty$. Let $L^p(\mu)$ denote the set of all μ-measurable functions f (or rather their equivalence classes) such that $|f|^p$ is μ-integrable. In particular $L^1(\mu)$ is the set μ-integrable functions. For $1 \leq p < \infty$ we set

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}, \quad f \in L^p(\mu).$$

Let $L^\infty(\mu)$ denote the set of all essentially bounded functions (or rather their equivalence classes), that is functions f that coincide with a bounded function almost everywhere with respect to μ. For $f \in L^\infty(\mu)$ we set

$$\|f\|_\infty = \inf\{a \in \mathbb{R} : \mu(\{x : f(x) > a\}) = 0\}.$$

When G is not σ-compact, the Haar measure is not σ-finite. This results in some technical complications in the measure theory. We will mention some of these problems and explain why they are not serious.

Firstly here is a useful lemma.

Lemma 2.1.9. *If G is a locally compact group, then G has an open, closed and σ-compact subgroup.*

Proof. Let U be a symmetric compact neighborhood of e. Then $H = \bigcup_{n=1}^\infty U^n$ is an open subgroup. Hence it is also closed since the cosets yH are also open and $X \setminus H = \bigcup_{y \notin H} yH$. □

Now let G be a non-σ-compact locally compact group, with left Haar measure λ. By the previous lemma there is a subgroup H that is open, closed and σ-compact. Let Y be a subset of G that contains exactly one element from each left coset of H, so that G is a disjoint union of the sets yH, $y \in Y$. It is not difficult to see that the restriction of λ to the Borel subsets of H is a left Haar measure on H. Moreover, this restriction determines λ entirely. First of all, it determines λ on the Borel subsets of each coset yH, since $\lambda(yE) = \lambda(E)$. One might then think that for every Borel $E \subset G$ one would have $\lambda(E) = \sum_{y\in Y} \lambda(E \cap yH)$. In fact what happens is the following.

Lemma 2.1.10. *Suppose $E \subset G$ is a Borel set. If $E \subset \bigcup_{j=1}^{\infty} y_j H$ for some countable set $\{y_j\} \subset Y$, then $\lambda(E) = \sum_{j=1}^{\infty} \lambda(E \cap y_j H)$. If $E \cap yH \neq 0$ for uncountably many y, then $\lambda(E) = \infty$.*

Proof. The first claim follows from the countable additivity of the Haar measure. By outer regularity it suffices to assume that E is open. In this case $\lambda(E \cap yH) > 0$ whenever $E \cap yH \neq 0$ since $E \cap yH$ is open. If this happens for uncountably many y, then if we write

$$\{y : \lambda(E \cap yH) > 0\} = \bigcup_{n=1}^{\infty} \left\{ y : \lambda(E \cap yH) > \frac{1}{n} \right\}$$

we see that for some n there are uncountably many y for which $\lambda(E \cap yH) > \frac{1}{n}$, and it follows that $\lambda(E) = \infty$. □

The above lemmas allow some theorems valid for σ-finite spaces to be generalized to general locally compact groups.

Here is an example that is useful to consider. Let $G = \mathbb{R} \times \mathbb{R}_d$, where $\mathbb{R}_d$ is $\mathbb{R}$ with discrete topology. We can take $H = \mathbb{R} \times \{0\}$ to be the subgroup as in the Lemma 2.1.9 and $Y = \{0\} \times \mathbb{R}_d$. To obtain Haar measure λ on G just take the Lebesgue measure on each horizontal line $\mathbb{R} \times \{y\}$ and add them together as in Lemma 2.1.10. In particular Y is closed and $\lambda(Y) = \infty$, but the intersection of Y with any coset of H, or with any compact set, has measure 0. Hence λ is not inner regular on Y. It also shows that λ is not quite the product of the Haar measures on $\mathbb{R}$ and $\mathbb{R}_d$. Indeed the Haar measure of $\mathbb{R}$ is the familiar Lebesgue measure μ and the Haar measure of $\mathbb{R}_d$ is the counting measure ν, so $(\mu \times \nu)(Y) = \mu(\{0\})\nu(\mathbb{R}) = 0 \cdot \infty = 0$ if we go by the convention $0 \cdot \infty = 0$.

We will need three fundamental theorems in measure theory that do not hold for general non-σ-compact spaces. These theorems are the Fubini's theorem, the Radon-Nikodym theorem, and the duality of $L^1(\mu)$ and $L^\infty(\mu)$. We will not give detailed explanations why we can use these. These matters are discussed in [5, p. 43-46]. Also the third chapter of [7] covers the measure theory necessary for integration on locally compact spaces.

We shall need Fubini's theorem to reverse the order of integration in double integrals $\int_G \int_G f(x, y) d\lambda(x) d\lambda(y)$. If the function f vanishes outside some σ-compact set $E \subset G \times G$, then there is no problem in doing this. Indeed the projections E_1 and E_2 of E onto the first and second factors are also σ-compact, and $E \subset E_1 \times E_2$. Therefore we can replace $G \times G$ by the σ-compact space $E_1 \times E_2$, and then we may apply Fubini's theorem. This hypothesis usually holds when f is constructed from functions on G that belong to $L^p(G)$ for some $p < \infty$, for such functions vanish outside some σ-compact set $\bigcup_{j=1}^{\infty} y_j H$ by Lemma 2.1.10. For instance when dealing with convolution we consider functions of the form $f(x, y) = g(x)h(x^{-1}y)$.

If g vanishes outside some σ-compact A and h vanish outside some σ-compact B, then f vanishes outside $A \times AB$, where AB is σ-compact.

Some kind of Radon-Nikodym theorem is necessary if we wish to obtain the aforementioned duality. A proof can be found in [7, Theorem 12.17.].

When μ is not σ-finite it is generally false that $L^\infty(\mu) = L^1(\mu)^*$ with the usual definition of $L^\infty(\mu)$. However we can modify the definition of $L^\infty(\mu)$ to make the duality hold in the case of Haar measure on a locally compact group. A set $E \subset X$ is *locally Borel* is $E \cap F$ is Borel whenever F is Borel and $\mu(F) < \infty$. A locally Borel set is *locally null* if $\mu(E \cap F) = 0$ whenever F is Borel and $\mu(F) < \infty$. An assertion about points of X is true *locally almost everywhere* if it is true except on a locally null set. A function $f : X \to \mathbb{C}$ is *locally measurable* if $f^{-1}(A)$ is locally Borel for every Borel set $A \subset \mathbb{C}$. We now (re-)define $L^\infty(\mu)$ to be the set of locally measurable functions that are bounded locally almost everywhere. Functions that agree locally almost everywhere are considered equivalent. The norm

$$\|f\|_\infty = \inf\{c : |f(x)| \leq c \text{ locally almost everywhere}\}$$

makes $L^\infty(\mu)$ a Banach space. Now $L^\infty(\mu) = L^1(\mu)^*$. In the case of Haar measure λ on a locally compact group the lemmas 2.1.9 and 2.1.10 can be used. The proof can also be found in [7, Theorem 12.18.].

Henceforth $L^\infty(\mu)$ will always denote the space defined above. When μ is σ-finite the definition coincides with the usual definition of $L^\infty(\mu)$.

The following approximation result will be needed.

Theorem 2.1.11. *Let $1 \leq p < \infty$. Then $C_c(G)$ is a dense subspace in $L^p(\mu)$.*

For the proof see [7, Theorem 12.10.].

Let G be a locally compact group with left Haar measure λ. If for every $x \in G$ we define $\lambda_x(E) = \lambda(Ex)$, then λ_x is again a left Haar measure. By the uniqueness of Haar measure, there exists a number $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$, and $\Delta(x)$ is independent of the original choice of λ. To see this, let μ and ν are left Haar measures, $c > 0$ such that $\nu = c\mu$, and $\Delta_1(x), \Delta_2(x) > 0$ such that $\mu(Ex) = \Delta_1(x)\mu(E)$ and $\nu(Ex) = \Delta_2(x)\nu(E)$. Then for measurable set $E \subset G$ with $0 < \mu(E), \nu(E) < \infty$ we have

$$\Delta_1(x)c\mu(E) = c\mu(Ex) = \nu(Ex) = \Delta_2(x)\nu(E) = \Delta_2(x)c\mu(E)$$

so dividing by $c\mu(E)$ we get $\Delta_1(x) = \Delta_2(x)$. The function $\Delta : G \to (0, \infty)$ is called the *modular function* of G. We shall denote the multiplicative group of positive real numbers by $\mathbb{R}_\times$.

Theorem 2.1.12. *The modular function Δ is a continuous homomorphism from G to $\mathbb{R}_\times$. Moreover, for any $f \in L^1(\lambda)$,*

$$\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda.$$

For the proof see [5, Proposition 2.24.].

A group is called *unimodular* if $\Delta = 1$. Compact groups are unimodular, since the only compact subgroup of $\mathbb{R}_\times$ is $\{1\}$.

The formula

$$d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x)$$

is useful when making substitutions in integrals.

2.2 Convolutions

From now on we shall assume that each locally compact group G is equipped with a fixed left Haar measure λ. We shall usually write dx for $d\lambda(x)$, $|E|$ for $\lambda(E)$, and $L^p(G)$ for $L^p(\lambda)$.

The group operation on G together with the Haar measure can be used to define another operation on $L^1(G)$.

Definition 2.2.1. If $f, g \in L^1(G)$, then the *convolution* of f and g is the function defined by

$$f * g(x) = \int f(y)g(y^{-1}x)dy.$$

Sometimes the functions can be also taken from spaces other than $L^1(G)$.

By applying Fubini's theorem we see that $f * g$ is integrable for almost every x and that $\|f * g\|_1 \leq \|f\|_1\|g\|_1$, for

$$\int\int |f(y)g(y^{-1}x)|dxdy = \int\int |f(y)g(x)|dxdy = \|f\|_1\|g\|_1$$

by the left invariance of the Haar measure.

The integral $f * g$ can be expressed in several different forms.

$$\begin{aligned} f * g(x) &= \int f(y)g(y^{-1}x)dy \\ &= \int f(xy)g(y^{-1})dy \\ &= \int f(y^{-1})g(yx)\Delta(y^{-1})dy \\ &= \int f(xy^{-1})g(y)\Delta(y^{-1})dy. \end{aligned}$$

The convolution product is associative, since if $f, g, h \in L^1(G)$, then

$$
\begin{aligned}
(f * g) * h(x) &= \int (f * g)(y) h(y^{-1}x) dy = \int \int f(z) g(z^{-1}) dz h(y^{-1}x) dy \\
&= \int f(z) \int g(z^{-1}y) h(y^{-1}x) dy dz = \int f(z) \int g(y) h(y^{-1}z^{-1}x) dy dz \\
&= \int f(z)(g * h)(z^{-1}x) dz = f * (g * h)(x).
\end{aligned}
$$

Remark 2.2.2. Convolution is commutative if and only if the underlying group G is abelian. Indeed if G is abelian, then

$$
f * g(x) = \int f(y) g(y^{-1}x) dy = \int g(xy) f(y^{-1}) dy = g * f(x).
$$

Before showing the converse, observe that $\text{supp}(f * g) \subset (\text{supp} f)(\text{supp} g)$. Indeed if $(f * g)(x) \neq 0$ for some $x \in G$, then there must be some $y \in G$ such that $f(xy)g(y^{-1}) \neq 0$, so $f(xy) \neq 0$ and $g(y^{-1}) \neq 0$. Therefore we have $x = xyy^{-1} \in (\text{supp} f)(\text{supp} g)$. Hence $\text{supp}(f * g) \subset (\text{supp} f)(\text{supp} g)$.

Now if G is nonabelian, then there exists $x, y \in G$ such that $xy \neq yx$. Since G is Hausdorff, there exists open disjoint neighborhoods W and W' of xy and yx respectively. Now by the joint continuity of the group operation there exists relative compact neighborhoods U_1 and U_2 of x and V_1 and V_2 of y such that $U_1V_1 \subset W$ and $V_2U_2 \subset W'$. Denoting $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$ we get $UV \subset W$ and $VU \subset W'$. Hence $\text{supp}(\chi_U * \chi_V) \subset UV \subset W$ and $\text{supp}(\chi_V * \chi_U) \subset W'$, so $\chi_U * \chi_V \neq \chi_V * \chi_U$.

We will need the following lemma for integration.

Lemma 2.2.3 (Minkowski's inequality for integrals)**.** *Let $1 \leq p < \infty$ and let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let ϕ be a complex valued $\mathcal{A} \times \mathcal{B}$ measurable function on the product $X \times Y$. Then*

$$
\left(\int \left| \int \phi(x, y) d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int \left(\int |\phi(x, y)|^p d\mu(x) \right)^{1/p} d\nu(y)
$$

in the sense that if the right side is finite, then the left side exists, and the inequality holds. The inequality can also be written as

$$
\left\| \int \phi(\cdot, y) d\nu(y) \right\|_p \leq \int \|\phi(\cdot, y)\|_p d\nu(y).
$$

Proof. When $p = 1$ the claim follows from Fubini's theorem. So assume $p > 1$ and let

$$C = \int \left(\int |\phi(x, y)|^p d\mu(x) \right)^{1/p} d\nu(y) < \infty.$$

It follows that $\int |\phi(x, y)|^p d\mu(x) < \infty$ for almost all y. If $q = p/(p-1)$ and $g \in L^q(\mu)$, then

$$\int |g(x)\phi(x, y)|dx \leq \|g\|_q \left(\int |\phi(x, y)|^p d\mu(x) \right)^{1/p}$$

by Hölder's inequality. Thus

$$\int \int |g(x)\phi(x, y)|d\mu(x)d\nu(y) \leq C\|g\|_q.$$

By Fubini's theorem it follows that

$$\int |g(x)\phi(x, y)|d\nu(y) < \infty$$

for almost all x. Since $g \in L^q(\mu)$ was arbitrary we see that

$$\int |\phi(x, y)|d\nu(y) < \infty$$

for almost all x and so $h(x) = \int \phi(x, y)d\nu(y)$ exists for almost all x. By Fubini's theorem

$$\left| \int g(x)h(x)\mu(x) \right| \leq \int \int |g(x)\phi(x, y)|d\nu(y)d\mu(x) \leq C\|g\|_q$$

Therefore there exists $h' \in L^p(\mu)$ with $\|h'\|_p \leq C$ such that

$$\int g(x)h(x)d\mu(x) = \int g(x)h'(x)d\mu(x)$$

for each $g \in L^q(\mu)$. □

Although we stated the previous theorem for σ-finite spaces, by what we have discussed we can also use it for locally compact groups that are not necessarily σ-compact.

We also need to know how convolution behaves when performed for functions other than $L^1(G)$.

Theorem 2.2.4. *Suppose* $1 \leq p \leq \infty$, $f \in L^1(G)$ *and* $g \in L^p(G)$. *Then we have* $f * g \in L^p(G)$ *and* $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Also if $f, g \in C_c(G)$, *then* $f * g \in C_c(G)$.

Proof. By Minkowski's inequality and left-invariance of the L^p norm for integrals we have

$$\|f * g\|_p = \left\| \int f(y) L_y g(\cdot) dy \right\|_p \leq \int |f(y)| \|L_y g(\cdot)\|_p dy = \|f\|_1 \|g\|_p$$

whenever $1 \leq p < \infty$. When $p = \infty$ we have $|f * g(x)| \leq \int |f(y) g(y^{-1}x)| dy \leq \|f\|_1 \|g\|_\infty$.

Let $f, g \in C_c(G)$, $x \in G$ and $\varepsilon > 0$. Denote $\int g(z^{-1}) dz = C$. Now f is uniformly left continuous, so there exists a neighborhood U of e such that $\|L_y f - f\|_\infty < \varepsilon / C$ whenever $y \in U$. Therefore

$$|f * g(x) - f * g(y)| = \left| \int [f(xz) - f(yz)] g(z^{-1}) dz \right|$$
$$\leq \int \|L_{x^{-1}} f - L_{y^{-1}} f\|_\infty |g(z^{-1})| dz = \|L_{yx^{-1}} f - f\|_\infty C < \varepsilon$$

whenever $y \in Ux$, proving that $f * g$ is continuous. On the other hand $\mathrm{supp}(f * g) \subset (\mathrm{supp} f)(\mathrm{supp} g)$ by Remark 2.2.2, so $f * g$ has compact support. □

When G is discrete, the function δ defined by $\delta(e) = 1$ and $\delta(x) = 0$ whenever $x \neq e$ satisfies $\delta * f = f * \delta = f$ for any f. Such function does not exist if G is not discrete. However there is a net of functions with this kind of property. But before we can prove that, let us show that the translation on $L^p(G)$ is continuous.

Theorem 2.2.5. *If* $1 \leq p < \infty$ *and* $f \in L^p(G)$, *then* $\|L_y f - f\|_p \to 0$ *and* $\|R_y f - f\|_p \to 0$ *as* $y \to e$.

Proof. First assume $g \in C_c(G)$ and that V is a fixed compact neighborhood of e. Now $K = (\mathrm{supp} g)V \cup V(\mathrm{supp} g)$ is a compact set, and $L_y g$ and $R_y g$ are supported in K whenever $y \in V$. Now

$$\|L_y g - g\|_p \leq \mu(K) \|L_y g - g\|_\infty \to 0$$

as $y \to e$, and similarly we get $\|R_y g - g\|_p \to 0$.

Now suppose $f \in L^p(G)$. Then if $\varepsilon > 0$ is arbitrary, then there exists $g \in C_c(G)$ such that $\|f - g\|_p < \varepsilon$. We have $\|L_y f\|_p = \|f\|_p$ and $\|R_y f\|_p = \Delta(y)^{-1/p} \|f\|_p \leq C \|f\|_p$ for $y \in V$ since V is compact. Hence

$$\|R_y f - f\|_p \leq \|R_y (f - g)\|_p + \|R_y g - g\|_p + \|g - f\|_p \leq (C + 1)\varepsilon + \|R_y g - g\|_p$$

where the term $\|R_y g - g\|_p \to 0$ when $y \to e$. The case for L_y goes the same way. □

Theorem 2.2.6. *Let $\mathcal{U}$ be a neighborhood base at e in G. For each $U \in \mathcal{U}$ let ψ_U be a function such that* $\mathrm{supp}\psi_U \subset U$, $\psi_U \geq 0$, $\psi_U(x) = \psi_U(x^{-1})$ *and* $\int \psi_U = 1$. *Then* $\|f * \psi_U - f\|_p \to 0$ *as* $U \to e$ *if* $1 \leq p < \infty$ *and* $f \in L^p(G)$ *or if* $p = \infty$ *and* $f \in RUC(G)$. *Also* $\|\psi_U * f - f\|_p \to 0$ *as* $U \to e$ *if* $1 \leq p < \infty$ *and* $f \in L^p(G)$ *or if* $p = \infty$ *and* $f \in LUC(G)$.

Proof. Since $\psi_U(x^{-1}) = \psi_U(x)$ and $\int \psi_U = 1$, we have

$$\begin{aligned} f * \psi_U(y) - f(y) &= \int f(yx)\psi_U(x^{-1})dx - f(y)\int \psi_U(x)dx \\ &= \int [R_x f(y) - f(y)]\psi_U(x)dx. \end{aligned}$$

Then by Minkowski's inequality for integrals we have

$$\|f * \psi_U - f\|_p \leq \int \|R_x f - f\|_p \psi_U(x)dx \leq \sup_{x\in U} \|R_x f - f\|_p.$$

Hence $\|f * \psi_U - f\|_p \to 0$ by the previous theorem or by right uniform continuity of f if $p = \infty$. The second claim follows in the same way, since

$$\begin{aligned} \psi_U * f(y) - f(y) &= \int \psi_U(x) f(x^{-1}y)dx - \int \psi_U(x)f(y)dx \\ &= \int [L_x f(y) - f(y)]\psi_U(x)dx. \end{aligned}$$

□

Remark 2.2.7. We do not need the symmetry of ψ_U when we prove that $\psi_U * f \to f$. This will be relevant later.

A family $\{\psi_U\}$ of functions as in the previous theorem is called an *approximate identity.* There are plenty of approximate identities. For instance, if we take the sets U to be compact and symmetric and then take $\psi_U = |U|^{-1}\chi_U$, or we could take the ψ_U's to be continuous, or even smooth in some circumstances.

Chapter 3

Representation Theory

In this chapter we present the basic concepts of the theory of unitary representations of locally compact groups. The main results that we prove are Schur's lemma and the Gelfand-Raikov theorem concerning the existence of irreducible representations. The key idea in the proof of the latter claim is the correspondence between cyclic representations and functions of positive type. We will also touch on the correspondence between unitary representations of groups and non-degenerate *-representations of the group algebra.

3.1 Hilbert Space Theory

In this section we recall some of results concerning Hilbert spaces necessary for representation theory.

Let $\mathcal{V}$ and $\mathcal{X}$ be complex vector spaces. A map $T : \mathcal{V} \to \mathcal{X}$ is *antilinear* if $T(\alpha u + \beta v) = \overline{\alpha} Tu + \overline{\beta} Tv$ for all $\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathcal{V}$. A map $B : \mathcal{V} \times \mathcal{V} \to \mathcal{X}$ is *sesquilinear* if $B(\cdot, v)$ is linear for every $v \in \mathcal{V}$ and $B(u, \cdot)$ is antilinear for every $u \in \mathcal{V}$. A sesquilinear map from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{C}$ is called a *sesquilinear form* on $\mathcal{V}$. Sesquilinear maps are completely determined by their values on the diagonal.

Lemma 3.1.1 (The Polarization Identity)**.** *Suppose $B : \mathcal{V}\times\mathcal{V} \to \mathcal{X}$ is sesquilinear, and let $Q(v) = B(v, v)$. Then for all $u, v \in \mathcal{V}$,*

$$B(u, v) = \frac{1}{4}[Q(u + v) - Q(u - v) + iQ(u + iv) - iQ(u - iv)].$$

Proof. Simply expand the expression on the right and collect the terms. □

A sesquilinear form B on $\mathcal{V}$ is called *Hermitian* if $B(v, u) = \overline{B(u, v)}$ for all $u, v \in \mathcal{V}$ and *positive* (*semi-definite*) if $B(u, u) \geq 0$ for all $u \in \mathcal{V}$. A sesquilinear form B on a normed space $\mathcal{V}$ is called *bounded* if there exists $M \geq 0$ such that $|B(u, v)| \leq M\|u\|\|v\|$ for every $u, v \in \mathcal{V}$.

Lemma 3.1.2. *A sesquilinear form B is Hermitian if and only if $B(u,u) \in \mathbb{R}$ for all $u \in \mathcal{V}$. Every positive form is Hermitian.*

Proof. If B is Hermitian, then $B(u,u) = \overline{B(u,u)} \in \mathbb{R}$. For any sesquilinear form we have $Q(au) = |a|^2 Q(u)$ whenever $a \in \mathbb{C}$, so if $B(u,u) \in \mathbb{R}$, then by the polarization identity

$$\begin{aligned}\overline{B(v,u)} &= \frac{1}{4}[Q(v+u) - Q(v-u) - iQ(v+iu) + iQ(v-iu)] \\ &= \frac{1}{4}[Q(u+v) - Q(u-v) - iQ(u-iv) + iQ(u+iv)] = B(u,v).\end{aligned}$$

The second assertion follows from the first one. □

Lemma 3.1.3 (The Schwarz and Minkowski Inequalities)**.** *Let B be a positive sesquilinear form on $\mathcal{V}$, and let $Q(u) = B(u,u)$. Then*

$$|B(u,v)|^2 \le Q(u)Q(v), \quad Q(u+v)^{1/2} \le Q(u)^{1/2} + Q(v)^{1/2}.$$

Proof. The usual proofs of these inequalities do not depend on the definiteness, so they apply for positive forms. □

An operator on a Hilbert space is called *unitary* if it is surjective and $\langle Tu, Tv\rangle = \langle u, v\rangle$. An operator is called *self-adjoint* if $T^* = T$. If $\langle Tu, u\rangle \ge 0$ then the operator T is *positive*. Every positive operator is self-adjoint since $\langle \cdot, \cdot \rangle_T = \langle T\cdot, \cdot\rangle$ is a positive form.

Theorem 3.1.4. *If $\mathcal{H}$ is a Hilbert space and $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a bounded Hermitian sesquilinear form, then there exists a bounded, self-adjoint operator $T \in L(\mathcal{H})$ such that $B(u,v) = \langle Tu, v\rangle$.*

Proof. The map $u \mapsto B(u,v)$ defines a bounded functional for every $v \in \mathcal{H}$, so by the Frechet-Riesz representation theorem for each $v \in \mathcal{H}$ there exists a vector $v_B \in \mathcal{H}$ such that $B(u,v) = \langle u, v_B\rangle$. Now the map $T(v) = v_B$ is linear and bounded, since

$$\begin{aligned}\langle u, T(\alpha v + v')\rangle &= B(u, \alpha v + v') = \overline{\alpha} B(u,v) + B(u,v') \\ &= \overline{\alpha}\langle u, T(v)\rangle + \langle u, T(v')\rangle = \langle u, \alpha T(v) + T(v')\rangle,\end{aligned}$$

so T is indeed linear and $\|Tv\| = \sup_{\|u\|=1} |\langle Tv, u\rangle| = \sup_{\|u\|=1} |B(u,v)| \le M\|v\|$. Also

$$\langle u, Tv\rangle = B(u,v) = \overline{B(v,u)} = \overline{\langle v, Tu\rangle} = \langle Tu, v\rangle$$

so T is self-adjoint. □

Recall that an operator $T \in L(\mathcal{H})$ is *compact* if the image of any bounded subset of $\mathcal{H}$ is relatively compact, that is, its closure is compact. It will be important to know the properties of compact operators when developing the representation theory of compact groups. The proofs of the following theorems concerning compact operators can be found in [10, Chapter 8].

Theorem 3.1.5. *Let $\mathcal{H}$ be a Hilbert space.*

(i) Every finite rank operator is compact.

(ii) The set of compact operators in $L(\mathcal{H})$ is closed in the operator topology.

Theorem 3.1.6. *If $T \in L(\mathcal{H})$ is self-adjoint and compact, there exists an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of T. Each eigenspace is finite dimensional.*

Let $\{H_\alpha\}_{\alpha\in A}$ be a family of Hilbert spaces. The *direct sum* $\bigoplus_{\alpha\in A}\mathcal{H}_\alpha$ is the set of all $v = (v_\alpha)_{\alpha\in A}$ in the Cartesian product $\prod_{\alpha\in A}\mathcal{H}_\alpha$ such that $\sum\|v_\alpha\|^2 < \infty$. This condition implies that $v_\alpha = 0$ for all but a countably many α. The space $\bigoplus_{\alpha\in A}\mathcal{H}_\alpha$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{\alpha\in A}\langle u_\alpha, v_\alpha\rangle,$$

and the summands $\mathcal{H}_\alpha$ are embedded in it as mutually orthogonal closed subspaces.

3.2 Unitary Representations

Definition 3.2.1. Let G be a locally compact group. A *unitary representation* of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on some Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology. In other words a map $\pi : G \to U(\mathcal{H}_\pi)$ such that $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and for which $x \mapsto \pi(x)u$ is continuous from G to $\mathcal{H}_\pi$ for any $u \in \mathcal{H}_\pi$. The space $\mathcal{H}_\pi$ is called the *representation space* of π and its dimension is called the *dimension* or *degree* of π.

In this thesis we are concerned almost exclusively with unitary representations.

It is worth noting that strong continuity is implied by the seemingly less restrictive condition of weak continuity, namely, that $x \mapsto \langle\pi(x)u, v\rangle$ should be continuous from G to $\mathbb{C}$ for every $u, v \in \mathcal{H}_\pi$. This is true since the strong and weak operator topologies coincide on $U(\mathcal{H}_\pi)$. Indeed, if (T_α) is a net of unitary operators converging to T in the weak operator topology of $U(\mathcal{H}_\pi)$, then for any $u \in \mathcal{H}_\pi$

$$\|(T_\alpha - T)u\|^2 = \|T_\alpha u\|^2 - 2\mathrm{Re}\langle T_\alpha u, Tu\rangle + \|Tu\|^2 = 2\|u\|^2 - 2\mathrm{Re}\langle T_\alpha u, Tu\rangle.$$

The last term converges to $2\|Tu\|^2 = 2\|u\|^2$, so $\|(T_\alpha - T)u\| \to 0$.

Example 3.2.2. Left translations yield the *left regular representation* π_L of G on $L^2(G)$, which is defined by

$$[\pi_L(x)f](y) = L_x f(y) = f(x^{-1}y).$$

Similarly one can define the *right regular representation* π_R on $L^2(G)$ (with *left* Haar measure)

$$[\pi_R(x)f](y) = \Delta(x)^{1/2} R_x f(y) = \Delta(x)^{1/2} f(yx).$$

In this text the regular representation that we treat is the left one. In fact these two are equivalent in a sense that will be explained below.

Any unitary representation π of G on $\mathcal{H}_\pi$ defines another representation $\overline{\pi}$ on the dual space $\mathcal{H}_\pi^*$ of $\mathcal{H}_\pi$, namely $\overline{\pi}(x) = \pi(x^{-1})'$ where the prime denotes the transpose. This representation $\overline{\pi}$ is called the *contragedient* of π. Let $v' = \langle \cdot, v\rangle$ and denote the inner product on the dual $\mathcal{H}_\pi^*$ by $\langle u', v'\rangle' = \langle v, u\rangle$. Now if $u, v \in \mathcal{H}_\pi$, then by the formula $T'v' = (T^*v)'$ we get

$$\langle \overline{\pi}(x)u', v'\rangle' = \langle \pi(x^{-1})'u', v'\rangle' = \langle (\pi(x)u)', v'\rangle' = \langle v, \pi(x)u\rangle = \overline{\langle \pi(x)u, v\rangle}.$$

Hence the contragedient of π is something like the "complex conjugate" of π.

Definition 3.2.3. If π_1 and π_2 are unitary representations of G, an *intertwining operator* for π_1 and π_2 is a bounded linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. The set of all intertwining operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$. Two representations π_1 and π_2 are (*unitarily*) *equivalent* if $\mathcal{C}(\pi_1, \pi_2)$ contains a unitary transformation $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$, so that $\pi_2(x) = U\pi_1(x)U^{-1}$. By unitary transformation we simply mean a linear surjective isometry.

Example 3.2.4. The left and right regular representations are unitarily equivalent, and the intertwining operator $T \in \mathcal{C}(\pi_L, \pi_R)$ is given by

$$Tf(y) = \Delta(y^{-1})^{1/2} f(y^{-1}).$$

We shall write $\mathcal{C}(\pi)$ for $\mathcal{C}(\pi, \pi)$. This is the space of bounded operators on $\mathcal{H}_\pi$ that commute with $\pi(x)$ for every $x \in G$. It is called the *commutator* or *centralizer* of π. The commutator is in fact a *-algebra that is closed in the weak operator topology, that is, a *von Neumann algebra*.

If a unitary representation π of G is of the form

$$\pi(x) = \begin{pmatrix} \pi_1(x) & 0 \\ 0 & \pi_2(x) \end{pmatrix} \tag{3.1}$$

where π_1 and π_2 are unitary representations of G, then sometimes it is better to analyze π_1 and π_2 if we wish to understand π.

Definition 3.2.5. A closed subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ is called an *invariant subspace* for π if $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in G$. If $\mathcal{M} \neq \{0\}$ is invariant, the restriction of π to $\mathcal{M}$,

$$\pi^{\mathcal{M}}(x) = \pi(x)|\mathcal{M},$$

defines a representation of G on $\mathcal{M}$, called a *subrepresentation* of π. If π admits a closed invariant subspace $\mathcal{M}$ that is nontrivial, that is $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}_\pi$, then π is called *reducible*, otherwise π is *irreducible*.

If $\{\pi_\alpha\}_{\alpha \in A}$ is a family of unitary representations, their *direct sum* $\bigoplus \pi_\alpha$ is the representation π on $\mathcal{H} = \bigoplus \mathcal{H}_{\pi_\alpha}$ defined by $\pi(x)(\sum v_\alpha) = \sum \pi_\alpha(x) v_\alpha$, where $v_\alpha \in \mathcal{H}_{\pi_\alpha}$. In this case the spaces $\mathcal{H}_{\pi_\alpha}$, as subspaces of $\mathcal{H}$, are invariant under π, and each π_α is a subrepresentation of π.

Theorem 3.2.6. *If $\mathcal{M}$ is invariant under π, then so is $\mathcal{M}^\perp$.*

Proof. If $u \in \mathcal{M}$ and $v \in \mathcal{M}^\perp$, then

$$\langle u, \pi(x)v\rangle = \langle \pi(x)^* u, v\rangle = \langle \pi(x^{-1})u, v\rangle = 0,$$

so $\pi(x)v \in \mathcal{M}^\perp$. □

As a corollary if π has a nontrivial invariant subspace $\mathcal{M}$, then π is the direct sum of $\pi^{\mathcal{M}}$ and $\pi^{\mathcal{M}^\perp}$. This result is false for non-unitary representations. For example, $\pi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ defines a representation of $\mathbb{R}$ on $\mathbb{C}^2$, and the only nontrivial invariant subspace is the one spanned by $(1, 0)$.

Definition 3.2.7. If π is a representation of G and $u \in \mathcal{H}_\pi$, the closed linear span $\mathcal{M}_u$ of $\{\pi(x)u : x \in G\}$ in $\mathcal{H}_\pi$ is called the *cyclic subspace* generated by u. Clearly $\mathcal{M}_u$ is invariant under π. If $\mathcal{M}_u = \mathcal{H}_\pi$, then u is called the *cyclic vector* for π. A representation π is called a *cyclic representation* if it has a cyclic vector.

Remark 3.2.8. Every irreducible representation is a cyclic representation. Furthermore every nonzero vector in the representation space is cyclic. To see this, pick any $u \neq 0$. Now $\mathcal{M}_u \neq \{0\}$, so by irreducibility it is the whole space.

However a cyclic representation is not necessarily irreducible. Consider the the representation ρ of $\mathbb{R}$ on $\mathbb{C}^2$ given by

$$\rho(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.$$

This is a unitary cyclic representation with cyclic vector $(1,0)$, since $\rho(\pi/2)(1,0) = (0,1)$. It is not irreducible, since

$$\rho(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

Here the invariant subspaces are

$$U\{(x,0) : x \in \mathbb{C}\} = \{(-x, ix) : x \in \mathbb{C}\}$$

and

$$U\{(0,y) : y \in \mathbb{C}\} = \{(iy, -y) : y \in \mathbb{C}\}.$$

Theorem 3.2.9. *Every unitary representation is a direct sum of cyclic representations.*

Proof. Let π be a representation on $\mathcal{H}_\pi$. By Zorn's lemma, there is a maximal collection $\{\mathcal{M}_\alpha\}_{\alpha \in A}$ of mutually orthogonal cyclic subspaces of $\mathcal{H}_\pi$. If there is a nonzero $u \in \mathcal{H}_\pi$ orthogonal to all the subspaces $\mathcal{M}_\alpha$, the cyclic subspace generated by u would also be orthogonal to the subspaces $\mathcal{M}_\alpha$, since $\langle \pi(x)u, m \rangle = \langle u, \pi(x^{-1})m \rangle = 0$ whenever $x \in G$ and $m \in \mathcal{M}_\alpha$. This contradicts maximality. Hence $\mathcal{H}_\pi = \bigoplus \mathcal{M}_\alpha$, and $\pi = \bigoplus \pi^{\mathcal{M}_\alpha}$. □

One may observe that if π is a unitary representation as in (3.1), then every $\pi(x)$ commutes with nontrivial elements $\begin{pmatrix} \lambda I & 0 \\ 0 & \mu I \end{pmatrix}$, where $\lambda, \mu \in \mathbb{C}$ may differ. This suggests a relationship between the reducibility of a representation and the intertwining operators.

Theorem 3.2.10. *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}_\pi$ and let P be the orthogonal projection onto $\mathcal{M}$. Then $\mathcal{M}$ is invariant under π if and only if $P \in \mathcal{C}(\pi)$.*

Proof. If $P \in \mathcal{C}(\pi)$ and $v \in \mathcal{M}$, then $\pi(x)v = \pi(x)Pv = P\pi(x)v \in \mathcal{M}$, so $\mathcal{M}$ is invariant under π. Conversely, if $\mathcal{M}$ is invariant, then $\pi(x)Pv = \pi(x)v = P\pi(x)v$ for $v \in \mathcal{M}$ and $\pi(x)Pv = 0 = P\pi(x)v$ for $v \in \mathcal{M}^\perp$, since by Theorem 3.2.6 $\pi(x)v \in \mathcal{M}^\perp$. Hence $\pi(x)P = P\pi(x)$. □

The picture is completed by Schur's lemma, which is one of the fundamental theorems in the subject.

Theorem 3.2.11 (Schur's Lemma)**.**

(a) A unitary representation π of G is irreducible if and only if $\mathcal{C}(\pi)$ contains only scalar multiples of the identity.

(b) Suppose π_1 and π_2 are irreducible unitary representations of G. If π_1 and π_2 are equivalent then $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional. Otherwise $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.

Proof. (a) If π is reducible, then $\mathcal{C}(\pi)$ contains nontrivial projections.

Now let π be irreducible and $T \in \mathcal{C}(\pi)$. First we assume that T is normal. Assume $\mathcal{H}$ is nontrivial. Let $\lambda \in \sigma(T)$. Then we can find $f \in C(\sigma(T))$ such that $f \neq 0$ and f vanishes outside an open neighborhood of λ. Let $\Phi : C(\sigma(T)) \to A_T$ be the isometry of the spectral theorem. Then $W = \overline{\Phi(f)H}$ is invariant under π. This is so since $\Phi(f)$ is a limit of polynomials in T, T^* and I, each of which commutes with $\pi(g)$ for every $g \in G$. In other words if $\Phi(f) = \lim_{n\to\infty} p_n(T)$, where $p_n(T)$ is a polynomial in T, T^* and I, we have

$$\pi(g)\Phi(f)H = \pi(g) \lim_{n\to\infty} p_n(T)H = \lim_{n\to\infty} p_n(T)\pi(g)H = \Phi(f)H,$$

so $W = \overline{\Phi(f)H}$ is also invariant. Since π is irreducible and f is nontrivial, we have $W = H$ (otherwise we would have $W = \{0\}$, and so $\Phi(f) = 0$).

Now suppose that $\sigma(T)$ is not a singleton. Then there exists some $\mu \in \sigma(T)$ distinct from λ, so we can pick two nonzero functions $f, h \in C(\sigma(T))$ such that their supports are disjoint. But then

$$\{0\} = \Phi(h)\Phi(f)H$$

and $W \neq H$, since otherwise we would have $\Phi(h)H = \Phi(h)W = \{0\}$ so $\Phi(h) = 0$ which is a contradiction. Hence $\sigma(T)$ contains at most one point, so $T = \lambda I$.

Now let $T \in \mathcal{C}(\pi)$. Then $T = A + iB$, where $A = \frac{1}{2}(T+T^*)$ and $B = \frac{1}{2i}(T-T^*)$ are self-adjoint and therefore normal. Furthermore $A, B \in \mathcal{C}(\pi)$ so $A = cI$ and $B = dI$. Therefore $T = (c + id)I$ proving that $\mathcal{C}(\pi) \in \mathbb{C}I$ when π is irreducible.

(b) If $T \in \mathcal{C}(\pi_1, \pi_2)$ then $T^* \in \mathcal{C}(\pi_2, \pi_1)$ because

$$T^*\pi_2(x) = [\pi_2(x^{-1})T]^* = [T\pi_1(x^{-1})]^* = \pi_1(x)T^*.$$

It follows that $T^*T \in \mathcal{C}(\pi_1)$ and $TT^* \in \mathcal{C}(\pi_2)$, so $T^*T = cI$ and $TT^* = dI$ for some $c, d \in \mathbb{R}$. In fact $c = d$, since if $c \neq 0$ then $c^2I = T^*TT^*T = dT^*T = cdI$ so $cI = dI$. Similarly if $d \neq 0$ then $d^2I = TT^*TT^* = cTT^* = cdI$ so $cI = dI$. If $T^*T = 0$, then $\|Tu\|^2 = \langle T^*Tu, u\rangle = 0$ for all $u \in \mathcal{H}_{\pi_1}$. Hence, either $T = 0$ or $c^{-1/2}T$ is unitary. This shows precisely that $\mathcal{C}(\pi_1, \pi_2) = \{0\}$ when π_1 and π_2 are inequivalent, and that $\mathcal{C}(\pi_1, \pi_2)$ consists of scalar multiples of unitary operators. If $T_1, T_2 \in \mathcal{C}(\pi_1, \pi_2)$ are unitary then $T_2^{-1}T_1 = T_2^*T_1 \in \mathcal{C}(\pi_1)$, so $T_2^{-1}T_1 = cI$ and $T_1 = cT_2$, so $\dim \mathcal{C}(\pi_1, \pi_2) = 1$. □

When studying reducibility it is often more convenient to work with operators, as we shall see.

As an immediate corollary we get a description of the irreducible representations of abelian groups.

Corollary 3.2.12. If G is abelian, then every irreducible representation of G is one-dimensional.

Proof. If π is a representation of G, the operators $\pi(x)$ all commute with one another and so belong to $\mathcal{C}(\pi)$. If π is irreducible, we have $\pi(x) = c_x I$ for each $x \in G$. But then every one-dimensional subspace of $\mathcal{H}_\pi$ is invariant, so $\dim \mathcal{H}_\pi = 1$. □

The representation theory of locally compact abelian groups is well understood. The irreducible representations form a group $\widehat{G}$ called the *dual group* of G. For more on this topic see [5, Chapter 4].

Irreducible unitary representations of a locally compact group are the basic building blocks of harmonic analysis associated to that group, just like prime numbers are the building blocks of integers. However the relationship between an arbitrary unitary representation of a group and the irreducible unitary representations of that group is not quite as straightforward as the one between an integer and its factorization to prime numbers. For starters it is not obvious that a given group has any irreducible representations other than the trivial one-dimensional representation $\pi_0(x) = 1$. But in fact there are enough irreducible unitary representations to separate points of G. This is the Gelfand-Raikov theorem, which we shall prove at the end of this chapter. Hence the basic questions of representation theory of G are the following.

(i) Describe all the irreducible unitary representations of G, up to equivalence.

(ii) Determine how arbitrary unitary representations of G can be built from irreducible ones.

(iii) Given a specific unitary representation of G such as the regular representation, show concretely how to build it out of irreducible ones.

The answer to (i) naturally depends strongly on the particular group. The irreducible representations have been determined for many groups, however we do not discuss any of these examples in this text. See for instance [5] or [9].

As to question (ii), one might wish that every unitary representation would be a direct sum of irreducible subrepresentations. When the group is compact this is true as we shall see, but it is not true generally. Consider the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$, $[\pi_L(x)f](t) = f(t-x)$. This representation has no irreducible subrepresentations. If there was one it would be one-dimensional by Corollary 3.2.12, so the invariant subspace would be of the form $\{cf : c \in \mathbb{C}\}$ for some $f \in L^2(\mathbb{R}) \setminus \{0\}$. But then we would have $f(t-x) = [\pi_L(x)f](t) = c_x f(t)$ for some $c_x \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, so $|f|$ is a constant function. This is impossible for $f \in L^2(G)$ unless $f = 0$. For cases such as these one needs a *direct integral* of irreducible representations. Direct sum is a special case of this. We will not go into this direct integrals in this text, see for instance [5, Section 7.4.].

The Peter-Weyl theorem answers question (iii) for the left regular representation of compact groups.

If G is a locally compact group, then $L^1(G)$ is a Banach *-algebra under the convolution product and the involution $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$.

Definition 3.2.13. Let A be a Banach *-algebra and $\mathcal{H}$ a Hilbert space. A mapping $\phi : A \to L(\mathcal{H})$ is a *nondegenerate *-representation* of A on $\mathcal{H}$ if it is *-homomorphism and $\phi(A)\mathcal{H} = \{\phi(a)v : a \in A, v \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

The nondegeneracy condition can be easily seen to be equivalent with the condition that for every $v \in \mathcal{H} \setminus \{0\}$ there exists $a \in A$ such that $\phi(a)v \neq 0$. Indeed if there exists $v \in \mathcal{H} \setminus \{0\}$ such that $\phi(a)v = 0$ for every $a \in A$, then $\langle v, \phi(a)u\rangle = \langle \phi(a^*)v, u\rangle = 0$ for every $a \in A$ and $u \in \mathcal{H}$, so $v \in (\phi(A)\mathcal{H})^\perp$. Hence $\phi(A)\mathcal{H}$ is not dense in $\mathcal{H}$. On the other hand if $v \in (\phi(A)\mathcal{H})^\perp$, then $0 = \langle v, \phi(a)u\rangle = \langle \phi(a^*)v, u\rangle$ for all $a \in A$ and $u \in \mathcal{H}$. Hence $\phi(a)v = 0$ for every $a \in A$, so by assumption $v = 0$. Therefore $\phi(A)\mathcal{H}$ is dense in $\mathcal{H}$.

Note that *-representations of Banach *-algebras are not assumed to be continuous in any topology. They are in fact automatically continuous by the following lemma.

Lemma 3.2.14. *Let A be a Banach *-algebra and B a C*-algebra. If $\phi : A \to B$ is *-homomorphism, then $\|\phi\| \leq 1$.*

Proof. If A is not unital we can adjoin an identity to it. Now $\phi(e_A)$ is an identity of $\phi(A)$, so we may assume that B is unital and $\phi(e_A) = e_B$.

For every $x \in A$ we have $\sigma(\phi(x)) \subset \sigma(x)$, so

$$\|\phi(x)\|^2 = \|\phi(x^*x)\| = r(\phi(x^*x)) \leq r(x^*x) \leq \|x^*x\| \leq \|x\|^2.$$

Here the first two equalities hold for C*-algebras (recall Lemma 1.2.11). □

Any unitary representation π of G determines a representation of $L^1(G)$, still denoted by π, in the following way. If $f \in L^1(G)$, we define a bounded operator $\pi(f)$ on $\mathcal{H}_\pi$ by

$$\pi(f) = \int f(x)\pi(x)dx.$$

We will explain how this integral should be interpreted. For any $u \in \mathcal{H}_\pi$, we define $\pi(f)u$ by specifying its inner product with an arbitrary $v \in \mathcal{H}_\pi$, which is given by

$$\langle \pi(f)u, v\rangle = \int f(x)\langle \pi(x)u, v\rangle dx.$$

Since $\langle \pi(\cdot)u, v\rangle$ is a bounded continuous function on G, the integral on the right is an ordinary integral of a function in $L^1(G)$. It is clear from the above formula that $\langle \pi(f)u, v\rangle$ depends linearly on u and antilinearly on v and that $|\langle \pi(f)u, v\rangle| \leq \|f\|_1\|u\|\|v\|$, so $\pi(f)$ really is a bounded operator on $\mathcal{H}_\pi$ with norm $\|\pi(f)\| \leq \|f\|_1$.

Example 3.2.15. Let π_L be the left regular representation of G. Then $\pi_L(f)$ is given by convolution with f on the left, since if $f \in L^1(G)$ and $g, h \in L^2(G)$, then

$$\begin{aligned}\langle \pi_L(f)g, h\rangle &= \int f(x)\left(\int [\pi_L(x)g](y)\overline{h(y)}dy\right)dx \\ = &\int\left(\int f(x)g(x^{-1}y)dx\right)\overline{h(y)}dy = \int (f*g)(y)\overline{h(y)}dy = \langle f*g, h\rangle.\end{aligned}$$

Theorem 3.2.16. *Let π be a unitary representation of G. The map $f \mapsto \pi(f)$ is nondegenerate *-representation of $L^1(G)$ on $\mathcal{H}_\pi$. Moreover, for $x \in G$ and $f \in L^1(G)$,*

$$\pi(x)\pi(f) = \pi(L_x f), \quad \pi(f)\pi(x) = \Delta(x^{-1})\pi(R_{x^{-1}}f).$$

Proof. The correspondence $f \mapsto \pi(f)$ is clearly linear. Now for $u, v \in \mathcal{H}_\pi$ we have

$$\begin{aligned}\langle \pi(f*g)u, v\rangle &= \int (f*g)(x)\langle \pi(x)u, v\rangle dx \\ = &\int\int f(y)g(y^{-1}x)\langle \pi(x)u, v\rangle dxdy = \int\int f(y)g(x)\langle \pi(yx)u, v\rangle dxdy \\ = &\int\int f(y)g(x)\langle \pi(y)\pi(x)u, v\rangle dxdy = \int f(y)\int g(x)\langle \pi(x)u, \pi(y)^*v\rangle dxdy \\ = &\int f(y)\langle \pi(g)u, \pi(y)^*v\rangle dy = \int f(y)\langle \pi(y)\pi(g)u, v\rangle dy = \langle \pi(f)\pi(g)u, v\rangle,\end{aligned}$$

$$\begin{aligned}\langle \pi(f^*)u, v\rangle &= \int \Delta(x^{-1})\overline{f(x^{-1})}\langle \pi(x)u, v\rangle dx = \int \overline{f(x)}\langle \pi(x^{-1})u, v\rangle dx \\ &= \int \langle u, f(x)\pi(x)v\rangle dx = \langle u, \pi(f)v\rangle = \langle \pi(f)^*u, v\rangle,\end{aligned}$$

$$\begin{aligned}
\langle \pi(x)\pi(f)u, v\rangle &= \langle \pi(f)u, \pi(x)^* v\rangle = \int f(y)\langle \pi(xy)u, v\rangle dy \\
&= \int f(x^{-1}y)\langle \pi(y)u, v\rangle dy = \langle \pi(L_x f)u, v\rangle, \\
\langle \pi(f)\pi(x)u, v\rangle &= \int f(y)\langle \pi(y)\pi(x)u, v\rangle dy = \int f(y)\langle \pi(yx)u, v\rangle dy \\
&= \Delta(x^{-1})\int f(yx^{-1})\langle \pi(y)u, v\rangle dy = \Delta(x^{-1})\langle \pi(R_{x^{-1}} f)u, v\rangle.
\end{aligned}$$

This shows that π is a *-homomorphism. To see that π is nondegenerate, suppose $u \in \mathcal{H}_\pi \setminus \{0\}$. Pick a compact neighborhood V of e such that $\|\pi(x)u - u\| < \|u\|$ for $x \in V$, and set $f = |V|^{-1}\chi_V$. Then

$$\|\pi(f)u - u\| = \frac{1}{|V|}\sup_{\|v\|=1}\left|\int_V \langle \pi(x)u - u, v\rangle dx\right| < \|u\|$$

and in particular $\pi(f)u \neq 0$. □

Conversely nondegenerate *-representation of $L^1(G)$ defines a unitary representation of G.

Theorem 3.2.17. *Suppose π is a nondegenerate *-representation of $L^1(G)$ on a Hilbert space $\mathcal{H}$. Then π arises from a unique unitary representation of G on $\mathcal{H}$ in the way we described above.*

We will not prove this claim, as we don't need this theorem. The proof can be found in [5, Theorem 3.11.]. The idea is that if $\{\psi_U\}$ is an approximate identity in G, then $\pi(x)$ should be the limit of $\pi(L_x\psi_U)$.

3.3 The Gelfand-Raikov Theorem

It is not obvious where one should look for nontrivial irreducible unitary representations for a group G. In this section we shall describe a method of turning the group algebra $L^1(G)$ into Hilbert spaces on which the group G acts unitarily. In fact every cyclic unitary representation, and in particular every irreducible unitary representation, arises in this way up to unitary equivalence.

Definition 3.3.1. A *function of positive type* on a locally compact group G is a function $\phi \in L^\infty(G)$ that defines a positive linear functional on the Banach *-algebra $L^1(G)$, that is

$$\int (f^* * f)\phi \geq 0 \text{ for all } f \in L^1(G).$$

We have

$$\begin{aligned}\int (f^* * f)\phi &= \int\int \Delta(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\phi(x)dydx \\ &= \int\int \overline{f(y)}f(yx)\phi(x)dydx = \int\int f(x)\overline{f(y)}\phi(y^{-1}x)dxdy.\end{aligned}$$

It turns out that every function of positive type agrees locally almost everywhere with a continuous function, as we shall see. Let

$$\mathcal{P} = \mathcal{P}(G) = \text{ the set of all continuous functions of positive type on } G.$$

Theorem 3.3.2. *If ϕ is of positive type, then so is $\overline{\phi}$.*

Proof. For any $f \in L^1(G)$, we have

$$\int (f^* * f)\overline{\phi} = \int\int \overline{f(y)}f(yx)\overline{\phi(x)}dydx = \overline{\int [(\overline{f})^* * \overline{f}]\phi} \geq 0.$$

□

There is a beautiful connection between functions of positive type and unitary representations. The following theorem provides the first clue of this.

Theorem 3.3.3. *If π is a unitary representation of G and $u \in \mathcal{H}_\pi$, let $\phi(x) = \langle \pi(x)u, u\rangle$. Then $\phi \in \mathcal{P}$.*

Proof. Since representations are assumed to be strongly continuous, we have $|\langle \pi(x)u, u\rangle - \langle \pi(y)u, u\rangle| \leq \|\pi(x)u - \pi(y)u\|\|u\| \to 0$ as $y \to x$, so ϕ is continuous. Also $\phi(y^{-1}x) = \langle \pi(y^{-1})\pi(x)u, u\rangle = \langle \pi(x)u, \pi(y)u\rangle$, so if $f \in L^1(G)$,

$$\begin{aligned}&\int\int f(x)\overline{f(y)}\phi(y^{-1}x)dxdy = \int\int \langle f(x)\pi(x)u, f(y)\pi(y)u\rangle dxdy \\ &= \int (f^* * f)(x)\langle \pi(x)u, u\rangle dx = \langle \pi(f^* * f)u, u\rangle \\ &= \langle \pi(f)u, \pi(f)u\rangle = \|\pi(f)u\|^2 \geq 0.\end{aligned}$$

□

Corollary 3.3.4. If $f \in L^2(G)$, let $\widetilde{f}(x) = \overline{f(x^{-1})}$. Then $f * \widetilde{f} \in \mathcal{P}$.

Proof. Let π_L be the left regular representation. Then

$$\langle \pi_L(x)f, f\rangle = \int f(x^{-1}y)\overline{f(y)}dy = \int \overline{\widetilde{f}(y^{-1}x)}\,\overline{f(y)}dy = \overline{f * \widetilde{f}(x)}.$$

Hence $f * \widetilde{f} \in \mathcal{P}$. □

We shall show that every nonzero function of positive type arises from a unitary representation. If $\phi \neq 0$ is of positive type, it defines a positive semi-definite Hermitian form on $L^1(G)$ by

$$\langle f, g\rangle_\phi = \int (g^* * f)\phi = \int\int f(x)\overline{g(y)}\phi(y^{-1}x)dxdy,$$

which clearly satisfies

$$|\langle f, g\rangle_\phi| \leq \|\phi\|_\infty \|f\|_1 \|g\|_1. \tag{3.2}$$

Let $\mathcal{N} = \{f \in L^1(G) : \langle f, f\rangle_\phi = 0\}$. By the Schwarz inequality $f \in \mathcal{N}$ if and only if $\langle f, g\rangle_\phi = 0$ for all $g \in L^1(G)$. The form $\langle\cdot,\cdot\rangle_\phi$ therefore induces an inner product on the quotient space $L^1(G)/\mathcal{N}$, still denoted by $\langle\cdot,\cdot\rangle_\phi$. We denote the Hilbert space completion of $L^1(G)/\mathcal{N}$ by $\mathcal{H}_\phi$, and we denote the image of $f \in L^1(G)$ in $L^1(G)/\mathcal{N} \subset \mathcal{H}_\phi$ by $\widetilde{f}$. By the inequality (3.2)

$$\|\widetilde{f}\|_{\mathcal{H}_\phi} \leq \|\phi\|_\infty^{1/2} \|f\|_1.$$

Now, if $f, g \in L^1(G)$ and $x \in G$,

$$\begin{aligned}\langle L_x f, L_x g\rangle_\phi &= \int\int f(x^{-1}y)\overline{g(x^{-1}y)}\phi(z^{-1}y)dydz \\ &= \int\int f(y)\overline{g(y)}\phi((xz)^{-1}(xy))dydz = \langle f, g\rangle_\phi.\end{aligned}$$

In particular, $L_x(\mathcal{N}) \subset \mathcal{N}$, so the operators L_x yield a unitary representation π_ϕ of G on $\mathcal{H}_\phi$ that is determined by

$$\pi_\phi(x)\widetilde{f} = (L_x f)^\sim \quad (f \in L^1(G)).$$

It is easy to verify that the corresponding representation of $L^1(G)$ on $\mathcal{H}_\phi$ is given by $\pi_\phi(f)g^\sim = (f * g)^\sim$.

Theorem 3.3.5. *Given a function ϕ of positive type on G, let $\mathcal{H}_\phi$ the Hilbert space determined as above by the Hermitian form and let π_ϕ be the unitary representation of G on $\mathcal{H}_\phi$. There is a cyclic vector ϵ for π_ϕ such that $\pi_\phi(f)\epsilon = \widetilde{f}$ for all $f \in L^1(G)$ and $\phi(x) = \langle \pi_\phi(x)\epsilon, \epsilon\rangle$ locally almost everywhere.*

Proof. Let $\{\psi_U\}$ be an approximate identity. Now $\{\psi_U^*\}$ is a left approximate identity, that is $\psi_U^* * f \to f$ for all $f \in L^1(G)$. Therefore for any $f \in L^1(G)$, $\langle \widetilde{f}, \widetilde{\psi}_U\rangle_\phi = \int(\psi_U^* * f)\phi \to \int f\phi$. Also $\|\widetilde{\psi}_U\|_{\mathcal{H}_\phi} \leq \|\phi\|_\infty^{1/2}\|\psi_U\|_1 = \|\phi\|_\infty^{1/2}$. It follows that the functional $f \mapsto \lim_U \langle \widetilde{f}, \widetilde{\psi}_U\rangle_\phi$ is bounded on $L^1(G)/\mathcal{N}$, so it extends to a bounded functional on the completion $\mathcal{H}_\phi$. Therefore $\lim\langle v, \widetilde{\psi}_U\rangle_\phi$ exists for all

$v \in \mathcal{H}_\phi$, and hence that $\widetilde{\psi}_U$ converges weakly in $\mathcal{H}_\phi$ to an element ϵ such that $\langle \widetilde{f}, \epsilon \rangle = \int f\phi$ for all $f \in L^1(G)$.

If $f, g \in L^1(G)$ and $y \in G$, we have

$$\begin{aligned} \langle \widetilde{g}, \pi_\phi(y)\epsilon \rangle_\phi &= \langle \pi_\phi(y^{-1})\widetilde{g}, \epsilon \rangle_\phi = \langle (L_{y^{-1}}g)^\sim, \epsilon \rangle_\phi \\ &= \int g(yx)\phi(x)dx = \int g(x)\phi(y^{-1}x)dx, \end{aligned}$$

and hence

$$\langle \widetilde{g}, \widetilde{f} \rangle_\phi = \int \langle \widetilde{g}, \pi_\phi(y)\epsilon \rangle_\phi \overline{f(y)}dy = \langle \widetilde{g}, \pi_\phi(f)\epsilon \rangle_\phi.$$

It follows that $\widetilde{f} = \pi_\phi(f)\epsilon$ for all $f \in L^1(G)$. It also follows that if $\langle \widetilde{g}, \pi_\phi(y)\epsilon \rangle = 0$ for all $y \in G$, then by the above $\langle \widetilde{g}, \widetilde{f} \rangle_\phi = 0$ for all $f \in L^1(G)$, so the linear span $\{\pi_\phi(y)\epsilon : y \in G\}$ is dense in $\mathcal{H}_\phi$ and ϵ is a cyclic vector. Moreover

$$\begin{aligned} \int \langle \epsilon, \pi_\phi(y)\epsilon \rangle_\phi \overline{f(y)}dy &= \lim \int \langle \widetilde{\psi}_U, \pi_\phi(y)\epsilon \rangle_\phi \overline{f(y)}dy = \lim \langle \widetilde{\psi}_U, \pi_\phi(f)\epsilon \rangle_\phi \\ &= \lim \langle \widetilde{\psi}_U, \widetilde{f} \rangle_\phi = \langle \epsilon, \widetilde{f} \rangle_\phi = \overline{\langle \widetilde{f}, \epsilon \rangle_\phi} = \int \overline{\phi(y) f(y)}dy \end{aligned}$$

for every $f \in L^1(G)$, and hence

$$\langle \pi_\phi(y)\epsilon, \epsilon \rangle_\phi = \overline{\langle \epsilon, \pi_\phi(y)\epsilon \rangle_\phi} = \phi(y) \text{ locally almost everywhere.}$$

□

Corollary 3.3.6. Every function of positive type agrees locally almost everywhere with a continuous function.

Corollary 3.3.7. If $\phi \in \mathcal{P}$ then $\|\phi\|_\infty = \phi(e)$ and $\phi(x^{-1}) = \overline{\phi(x)}$.

Proof. We have $\phi(x) = \langle \pi(x)u, u \rangle$ for some π and u, so $|\phi(x)| = |\langle \pi(x)u, u \rangle| \leq \|u\|^2 = \phi(e)$ and $\phi(x^{-1}) = \langle \pi(x^{-1})u, u \rangle = \langle u, \pi(x)u \rangle = \overline{\phi(x)}$. □

Theorems 3.3.3 and 3.3.5 establish a correspondence between cyclic representations and functions of positive type. Note that in Theorem 3.3.3 we didn't assume π was cyclic, however the expression $\langle \pi(\cdot)u, u \rangle$ only depends on the subrepresentation of π on the cyclic subspace generated by u. Moreover representations with the same associated function of positive type are equivalent.

Theorem 3.3.8. *Suppose π and ρ are cyclic representations of G with cyclic vectors u and v. If $\langle \pi(x)u, u \rangle = \langle \rho(x)v, v \rangle$ for all $x \in G$, then π and ρ are unitarily equivalent. More precisely there exists a unitary $T \in \mathcal{C}(\pi, \rho)$ such that $Tu = v$.*

Proof. Define $T[\pi(x)u] = \rho(x)v$. This extends to a well-defined isometry from the span of $\{\pi(x)u : x \in G\}$ to the span of $\{\rho(x)v : x \in G\}$. To check that T is well-defined, let $\sum_{i=1}^n \alpha_i \pi(x_i)u = 0$. Now

$$\begin{aligned}\left\|\sum_{i=1}^n \alpha_i \rho(x_i)v\right\|^2 &= \sum_{i,j=1}^n \alpha_i\overline{\alpha_j}\langle \rho(x_j^{-1}x_i)v, v\rangle = \sum_{i,j=1}^n \alpha_i\overline{\alpha_j}\langle \pi(x_j^{-1}x_i)u, u\rangle \\ &= \left\|\sum_{i=1}^n \alpha_i \pi(x_i)u\right\|^2 = 0,\end{aligned}$$

so $\sum_{i=1}^n \alpha_i\rho(x_i)v = T[\sum_{i=1}^n \alpha_i\pi(x_i)u] = 0$ so T is well-defined. By the above it is also an isometry. By continuity it extends to a unitary map from $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$. Since $\rho(y)T[\pi(x)u] = \rho(yx)v = T[\pi(y)\pi(x)u]$ we have $\rho(y)T = T\pi(y)$, so $T \in \mathcal{C}(\pi,\rho)$. Also $Tu = T[\pi(e)u] = \rho(e)v = v$. $\square$

Corollary 3.3.9. If π is a cyclic representation of G with cyclic vector u and $\phi(x) = \langle\pi(x)u, u\rangle$, then π is unitarily equivalent to the representation π_ϕ.

Proof. If u is a cyclic vector of π, then by Theorem 3.3.5 $\phi(x) = \langle\pi(x)u,u\rangle = \langle\pi_\phi(x)\epsilon,\epsilon\rangle$, so we can apply the above theorem. $\square$

The set $\mathcal{P}$ of continuous functions of positive type is a convex cone. We single out two subsets of $\mathcal{P}$ for special attention. Let

$$\mathcal{P}_1 = \{\phi \in \mathcal{P} : \|\phi\|_\infty = 1\} = \{\phi\in\mathcal{P} : \phi(e) = 1\},$$

$$\mathcal{P}_0 = \{\phi \in \mathcal{P} : \|\phi\|_\infty \le 1\} = \{\phi\in\mathcal{P} : 0 \le \phi(e) \le 1\}.$$

These are bounded convex sets, and denote

$$\mathcal{E}(\mathcal{P}_j) = \text{ the set of extreme points of } \mathcal{P}_j, \quad (j = 0, 1).$$

The extreme points of $\mathcal{P}_1$ are of particular interest because of the following theorem.

Theorem 3.3.10. *If $\phi \in \mathcal{P}_1$, then $\phi \in \mathcal{E}(\mathcal{P}_1)$ if and only if the representation π_ϕ is irreducible.*

Proof. Suppose π_ϕ is reducible, say $\mathcal{H}_\phi = \mathcal{M}\oplus\mathcal{M}^\perp$ where $\mathcal{M}$ is nontrivial and invariant under π_ϕ. Let $\epsilon \in \mathcal{H}_\phi$ be a cyclic vector for π_ϕ. Since ϵ is cyclic, it cannot belong to $\mathcal{M}$ or $\mathcal{M}^\perp$, so $\epsilon = u + v$ with $u\in\mathcal{M}$, $v\in\mathcal{M}^\perp$ and $u \ne 0 \ne v$. But then

$$\phi(x) = \langle\pi_\phi(x)\epsilon,\epsilon\rangle_\phi = \langle\pi_\phi(x)u,u\rangle_\phi + \langle\pi_\phi(x)v,v\rangle_\phi = c_1\psi_1(x) + c_2\psi_2(x)$$

where $\psi_1, \psi_2 \in \mathcal{P}_1$, $c_1 = \|u\|^2$, $c_2 = \|v\|^2$, and $c_1 + c_2 = \phi(e) = 1$. It remains to show that $\psi_1 \neq \psi_2$.

Suppose towards contradiction that $\langle \pi_\phi(\cdot)v, v\rangle_\phi = c\langle \pi_\phi(\cdot)u, u\rangle_\phi$ for some constant c, which must necessarily be positive. Choose $\delta > 0$ such that

$$\delta < \frac{c\|u\|^2}{\|v\| + c\|u\|}.$$

It follows that $\delta\|v\| < c\|u\|^2 - \delta c\|u\|$.

Since ϵ is cyclic in $\mathcal{H}_\phi$, there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $x_1, \ldots x_n \in G$ such that

$$\left\|\sum_{k=1}^{n} \alpha_k \pi(x_k)\epsilon - u\right\| < \delta.$$

By the above we have

$$\begin{aligned}
\left|\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)u, u\rangle - \langle u, u\rangle\right| &= \left|\langle \sum_{k=1}^{n} \alpha_k \pi(x_k)\epsilon - u, u\rangle\right| \\
&\leq \left\|\sum_{k=1}^{n} \alpha_k \pi(x_k)\epsilon - u\right\| \|u\| < \delta\|u\|,
\end{aligned}$$

so $\|u\|^2 - \delta\|u\| < |\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)u, u\rangle|$. On the other hand we have

$$\begin{aligned}
\left|\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)v, v\rangle\right| &= \left|\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)\epsilon, v\rangle - \langle u, v\rangle\right| \\
&\leq \left\|\sum_{k=1}^{n} \alpha_k \pi(x_k)\epsilon - u\right\| \|v\| < \delta\|v\|.
\end{aligned}$$

Combining the above inequalities we have

$$\left|\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)v, v\rangle\right| < \delta\|v\| < c\|u\|^2 - \delta c\|u\| < c\left|\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)u, u\rangle\right|,$$

so $\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)v, v\rangle \neq c\sum_{k=1}^{n} \alpha_k \langle \pi(x_k)u, u\rangle$, which implies that for some k $\alpha_k \langle \pi(x_k)v, v\rangle \neq c\alpha_k \langle \pi(x_k)u, u\rangle$ and hence $\psi_1 \neq \psi_2$. This shows that ϕ is not extreme.

Conversely, suppose π_ϕ is irreducible, but that $\phi = \psi + \psi'$ with $\psi, \psi' \in \mathcal{P}$. Then for any $f, g \in L^1(G)$, we have

$$\langle f, f\rangle_\psi = \langle f, f\rangle_\phi - \langle f, f\rangle_{\psi'} \leq \langle f, f\rangle_\phi$$

and hence

$$|\langle f,g\rangle_\psi| \le \langle f,f\rangle_\psi^{1/2}\langle g,g\rangle_\psi^{1/2} \le \langle f,f\rangle_\phi^{1/2}\langle g,g\rangle_\phi^{1/2}.$$

Thus the map $(f,g)\mapsto\langle f,g\rangle_\psi$ induces a bounded Hermitian form on $\mathcal{H}_\phi$, so by Theorem 3.1.4 there is a bounded self-adjoint operator T on $\mathcal{H}_\phi$ such that $\langle f,f\rangle_\psi = \langle T\widetilde{f},\widetilde{g}\rangle_\phi$ for all $f,g\in L^1(G)$. Now if $x\in G$ and $f,g\in L^1(G)$ we have

$$\begin{aligned}\langle T\pi_\phi(x)\widetilde{f},\widetilde{g}\rangle_\phi &= \langle T(L_xf)^\sim,\widetilde{g}\rangle_\phi = \langle L_xf,g\rangle_\psi = \langle f,L_{x^{-1}}g\rangle_\psi\\ &= \langle T\widetilde{f},(L_{x^{-1}}g)^\sim\rangle_\phi = \langle T\widetilde{f},\pi_\phi(x^{-1})\widetilde{g}\rangle_\phi = \langle\pi_\phi(x)T\widetilde{f},\widetilde{g}\rangle_\phi.\end{aligned}$$

Therefore, $T\in\mathcal{C}(\pi_\phi)$, so by Schur's lemma, $T=cI$ and $\langle f,g\rangle_\psi = c\langle f,g\rangle_\phi$ for all f,g. Letting g be an approximate identity we get $\int f\psi = c\int f\phi$ for every $f\in L^1(G)$. This implies $\psi=c\phi$ and hence $\psi'=(1-c)\phi$, so ϕ is extreme. □

Recall the following theorems from functional analysis.

Theorem 3.3.11 (Alaoglu's Theorem)**.** *The norm closed unit ball of the dual of a normed space is weak* compact.*

Theorem 3.3.12 (The Krein-Milman Theorem)**.** *If C is a compact convex subset of a real or complex locally convex Hausdorff space X, then C is the closed convex hull of its extreme points.*

The proof of Alaouglu's theorem can be found in [12, p. 229, Theorem 2.6.18.] and the proof of the Krein-Milman theorem in [12, p. 265, Theorem 2.10.6.].

The condition $\int(f^* * f)\phi\ge 0$ is clearly preserved under weak* limits, so $\mathcal{P}_0$ is a weak* closed subset of the closed unit ball in $L^\infty(G)$. By Alaoglu's theorem $\mathcal{P}_0$ is compact, and then by Krein-Milman theorem it is the closed convex hull of its extreme points. However $\mathcal{P}_1$ is in general not weak* closed, although if G is discrete, then the point mass δ_e at e is in $L^1(G)$, and $\int\delta_e\phi=\phi(e)$ implies that $\mathcal{P}_1$ is weak* closed. In spite of this the conclusion of the Krein-Milman holds for $\mathcal{P}_1$ too.

Lemma 3.3.13. $\mathcal{E}(\mathcal{P}_0)=\mathcal{E}(\mathcal{P}_1)\cup\{0\}$.

Proof. Suppose $\phi_1,\phi_2\in\mathcal{P}_0$, $c_1,c_2>0$ and $c_1+c_2=1$. If $c_1\phi_1+c_2\phi_2=0$, then $c_1\phi_1(e)+c_2\phi_2(e)=0$, which implies that $\phi_1(e)=\phi_2(e)=0$ and hence $\phi_1=\phi_2=0$. Thus $0\in\mathcal{E}(\mathcal{P}_0)$.

Now suppose $\phi\in\mathcal{E}(\mathcal{P}_1)$ and $c_1\phi_1+c_2\phi_2=\phi$. Then $c_1\phi_1(e)+c_2\phi_2(e)=\phi(e)=1$. This implies that $\phi_1(e)=\phi_2(e)=1$, since otherwise $c_1\phi_1+c_2\phi_2<c_1+c_2=1$ which is a contradiction. Therefore $\phi_1,\phi_2\in\mathcal{P}_1$. Since $\phi\in\mathcal{E}(\mathcal{P}_1)$ we have $\phi_1=\phi_2$, so $\phi\in\mathcal{E}(\mathcal{P}_0)$.

Finally, no element of $\mathcal{E}(\mathcal{P}_0) \setminus (\mathcal{E}(\mathcal{P}_1) \cup \{0\})$ is extreme in $\mathcal{E}(\mathcal{P}_1)$, since if $\phi \in \mathcal{E}(\mathcal{P}_0) \setminus (\mathcal{E}(\mathcal{P}_1) \cup \{0\})$ then $0 < \phi(e) < 1$, so ϕ is interior to the line segment joining 0 and $\phi/\phi(e)$. □

Theorem 3.3.14. *The convex hull of $\mathcal{E}(\mathcal{P}_1)$ is weak* dense in $\mathcal{P}_1$.*

Proof. Suppose $\phi \in \mathcal{P}_1$. Now $\phi \in \mathcal{P}_1 \subset \mathcal{P}_0 = \overline{co}\mathcal{E}(\mathcal{P}_0) = \overline{co}(\mathcal{E}(\mathcal{P}_1) \cup \{0\})$, so ϕ is the weak* limit of a net of functions ϕ_α of the form $c_1\psi_1 + \cdots + c_n\psi_n + c_{n+1}0 = \sum_{j=1}^n c_j\psi_j$, where $\psi_1, \ldots, \psi_n \in \mathcal{E}(\mathcal{P}_1)$, $c_1, \ldots, c_{n+1} \geq 0$ and $\sum_{j=1}^{n+1} c_j = 1$.

Now $\|\lim \phi_\alpha\|_\infty = 1$ and $\|\phi_\alpha\|_\infty \leq 1$, so $\lim \|\phi_\alpha\|_\infty \leq 1$. In fact we have $\lim \|\phi_\alpha\| = 1$. Towards contradiction assume that $c = \lim \|\phi_\alpha\| < 1$. Now for some α_0 we have $|\|\phi_\alpha\| - c| < \frac{1-c}{2}$ whenever $\alpha \geq \alpha_0$. It follows that $\|\phi_\alpha\|_\infty < \frac{1+c}{2} < 1$ whenever $\alpha \geq \alpha_0$. Now since the set $\{f \in L^\infty(G) : \|f\|_\infty \leq \frac{1+c}{2}\}$ is weak* closed, we have $\|\lim \phi_\alpha\|_\infty \leq \frac{1+c}{2} < 1$, which is a contradiction.

Now if we set $\phi'_\alpha = \phi_\alpha/\phi_\alpha(e)$, we have

$$\phi'_\alpha = \frac{1}{\phi_\alpha(e)} \sum_{j=1}^n c_j\psi_j, \quad \frac{1}{\phi_\alpha(e)} \sum_{j=1}^n c_j = \frac{\phi_\alpha(e)}{\phi_\alpha(e)} = 1.$$

Thus ϕ'_α is in the convex hull of $\mathcal{E}(\mathcal{P}_1)$ and $\phi = \lim \phi'_\alpha$. □

Next we show that the weak* topology that $\mathcal{P}_1$ inherits from $L^\infty(G)$ coincides with the topology of uniform convergence on compact subsets of G.

Definition 3.3.15. On $C^b(G)$ the *topology of compact convergence* on G is the topology of uniform convergence on compact subsets of G. A neighborhood base at the function ϕ_0 consists of sets of the form

$$N(\phi_0; \varepsilon, K) = \{\phi : |\phi(x) - \phi_0(x)| < \varepsilon \text{ for } x \in K\},$$

where $\varepsilon > 0$ and $K \subset G$ is compact.

The proof of the aforementioned remarkable claim relies on the following lemma.

Lemma 3.3.16. *Suppose X is a Banach space and B is a norm-bounded subset of X^*. On B, the weak* topology coincides with the topology of compact convergence on X.*

Proof. The weak* topology is the topology of pointwise convergence on X, so it is weaker than the topology of compact convergence. On the other hand, if $\lambda_0 \in B, \varepsilon > 0$ and $K \subset X$ is compact, let $C = \sup\{\|\lambda\| : \lambda \in B\}$ and $\delta = \varepsilon/3C$.

Then there exists $\xi_1, \ldots, \xi_n \in K$ such that the balls $B(\xi_j, \delta)$ cover K. If $\lambda \in B$ and $\xi \in K$ then $\|\xi - \xi_j\| < \delta$ for some j, so that

$$\begin{aligned}|\lambda(\xi) - \lambda_0(\xi)| &< |\lambda(\xi - \xi_j)| + |(\lambda - \lambda_0)(\xi_j)| + |\lambda_0(\xi_j - \xi)| \\ &\leq \|\lambda\|\|\xi - \xi_j\| + |(\lambda - \lambda_0)(\xi_j)| + \|\lambda_0\|\|\xi_j - \xi\| \\ &< \frac{2\varepsilon}{3} + |(\lambda - \lambda_0)(\xi_j)|\end{aligned}$$

so the weak* neighborhood $\bigcap_{j=1}^n \{\lambda : |(\lambda - \lambda_0)(\xi_j)| < \varepsilon/3\}$ of λ_0 is contained in the neighborhood $N(\lambda_0; \varepsilon, K)$ for the topology of compact convergence. □

As a corollary we obtain the following lemma.

Lemma 3.3.17. *Suppose $\phi_0 \in \mathcal{P}_1$ and $f \in L^1(G)$. For every $\varepsilon > 0$ and every compact $K \subset G$ there is a weak* neighborhood Φ of ϕ_0 in $\mathcal{P}_1$ such that $|f * \phi(x) - f * \phi_0(x)| < \varepsilon$ for all $\phi \in \Phi$ and $x \in K$.*

Proof. By Corollary 3.3.7 we have $f*\phi(x) = \int f(xy)\phi(y^{-1})dy = \int (L_{x^{-1}}f)\overline{\phi}$. Since $x \mapsto L_{x^{-1}}f$ is continuous from G to $L^1(G)$, $\{L_{x^{-1}}f : x \in K\}$ is compact in $L^1(G)$, and we can apply Lemma 3.3.16. □

Lemma 3.3.18. *If $\phi \in \mathcal{P}_1$, $|\phi(x) - \phi(y)|^2 \leq 2 - 2\mathrm{Re}\phi(yx^{-1})$.*

Proof. We have $\langle \pi(x)u, u\rangle$ for some unitary representation π and some unit vector $u \in \mathcal{H}_\pi$, so

$$\begin{aligned}|\phi(x) - \phi(y)|^2 &= |\langle [\pi(x) - \pi(y)]u, u\rangle|^2 = |\langle u, [\pi(x^{-1}) - \pi(y^{-1})]u\rangle|^2 \\ &\leq \|\pi(x^{-1})u - \pi(y^{-1})u\|^2 = 2 - 2\mathrm{Re}\langle \pi(x^{-1})u, \pi(y^{-1})u\rangle \\ &= 2 - 2\mathrm{Re}\langle \pi(yx^{-1})u, u\rangle = 2 - 2\mathrm{Re}\phi(yx^{-1}).\end{aligned}$$

□

Theorem 3.3.19. *On $\mathcal{P}_1$, the weak* topology coincides with the topology of compact convergence on G.*

Proof. If $f \in L^1(G)$ and $\varepsilon > 0$, there is a compact $K \subset G$ such that $\int_{G\setminus K} |f| < \frac{1}{4}\varepsilon$. If $\phi, \phi_0 \in \mathcal{P}_1$ and $|\phi - \phi_0| < \varepsilon/2\|f\|_1$ on K then

$$\begin{aligned}\left|\int (f\phi - f\phi_0)\right| &\leq \int_K |f||\phi - \phi_0| + \int_{G\setminus K} |f||\phi - \phi_0| \\ &< \frac{1}{2}\varepsilon + \int_{G\setminus K} |f|(\|\phi\| + \|\phi_0\|) < \frac{1}{2}\varepsilon + 2\frac{1}{4}\varepsilon = \varepsilon,\end{aligned}$$

so compact convergence on G implies weak* convergence.

Conversely suppose $\phi_0 \in \mathcal{P}_1$, $\varepsilon > 0$ and $K \subset G$ is compact. We wish to find a weak* neighborhood Φ of ϕ_0 in $\mathcal{P}_1$ such that $|\phi - \phi_0| < \varepsilon$ on K when $\phi \in \Phi$. First, if $\eta > 0$ there is a compact neighborhood V of e such that $|\phi_0(x) - \phi_0(e)| = |\phi_0(e) - 1| < \eta$ for all $x \in V$. Let

$$\Phi_1 = \left\{ \phi \in \mathcal{P} : \left| \int_V (\phi - \phi_0) \right| < \eta |V| \right\}.$$

Now Φ_1 is a weak* neighborhood of ϕ_0 since $\chi_V \in L^1$. If $\phi \in \Phi_1$, then

$$\left| \int_V (1 - \phi) \right| \leq \left| \int_V (1 - \phi_0) \right| + \left| \int_V (\phi_0 - \phi) \right| < 2\eta |V|. \tag{3.3}$$

Also, if $\phi \in \Phi_1$ and $x \in G$, we have

$$|\chi_V * \phi(x) - |V|\phi(x)| = \left| \int_V [\phi(y^{-1}x) - \phi(x)] dy \right| \leq \int_V |\phi(y^{-1}x) - \phi(x)| dy.$$

By Lemma 3.3.18, Schwarz inequality and inequality (3.3), the right side of the above inequality is bounded by

$$\begin{aligned} \int_V [2 - 2\mathrm{Re}\phi(y)]^{1/2} dy &\leq \left(\int_V [2 - 2\mathrm{Re}\phi(y)] dy \right)^{1/2} |V|^{1/2} \\ &\leq 2^{1/2} \left| \int_V (1 - \phi) \right|^{1/2} |V|^{1/2} < 2|V|\sqrt{\eta}. \end{aligned}$$

By Lemma 3.3.17, there exists a weak* neighborhood Φ_2 of ϕ_0 in $\mathcal{P}_1$ such that $|\chi_V * \phi(x) - \chi_V * \phi_0(x)| < \eta |V|$ for $\phi \in \Phi_2$ and $x \in K$. Hence, if $\phi \in \Phi_1 \cap \Phi_2$ and $x \in K$, $|\phi(x) - \phi_0(x)|$ is bounded by

$$\begin{aligned} & \tfrac{1}{|V|} \big[||V|\phi(x) - \chi_V * \phi(x)| + |\chi_V * (\phi - \phi_0)(x)| + \\ & \quad |\chi_V * \phi_0(x) - |V|\phi_0(x)| \big] \\ \leq \; & \frac{1}{|V|} (2|V|\sqrt{\eta} + |V|\eta + 2|V|\sqrt{\eta}) = \eta + 4\sqrt{\eta} \end{aligned}$$

Therefore, if we choose η so that $\eta + 4\sqrt{\eta} < \varepsilon$ and take $\Phi = \Phi_1 \cap \Phi_2$, we are done. □

One more simple approximation result is needed for the proof of the Gelfand-Raikov theorem.

Theorem 3.3.20. *The linear span of $C_c(G) \cap \mathcal{P}$ includes all functions of the form $f * g$ with $f, g \in C_c(G)$. It is dense in $C_c(G)$ in the uniform norm, and dense in $L^p(G)$ $(1 \leq p < \infty)$ in the $L^p(G)$ norm.*

Proof. By the Corollary 3.3.4 the set $\overline{C_c(G) \cap \mathcal{P}}$ includes all functions of the form $f * \widetilde{f}$ with $f \in C_c(G)$, where $\widetilde{f}(x) = \overline{f(x^{-1})}$. By polarization, its linear span also includes all functions of the form $f * \widetilde{h}$ and hence all functions of the form $f * g$ with $f, g \in C_c(G)$. Now $\{f * g : f, g \in C_c(G)\}$ is dense in $C_c(G)$ in the uniform norm and L^p norm because g can be taken to be the approximate identity ψ_U, and $C_c(G)$ is itself dense in $L^p(G)$. □

Now we can prove the main result of this section.

Theorem 3.3.21 (The Gelfand-Raikov Theorem)**.** *If G is any locally compact group, the irreducible unitary representations of G separate points on G. That is, if $x, y \in G$ with $x \neq y$, there is an irreducible representation π such that $\pi(x) \neq \pi(y)$.*

Proof. If $x \neq y$ there exists $f \in C_c(G)$ such that $f(x) \neq f(y)$, and by Theorem 3.3.20 f can be taken to be a linear combination of functions of positive type. By Theorems 3.3.14 and 3.3.19, there is a linear combination g of extreme points of $\mathcal{P}_1$ that approximates f on the compact set $\{x, y\}$ closely enough so that $g(x) \neq g(y)$. Hence there must be an extreme point ϕ of $\mathcal{P}_1$ such that $\phi(x) \neq \phi(y)$. The associated representation π_ϕ is irreducible by Theorem 3.3.10 and it satisfies

$$\langle \pi_\phi(x)\epsilon, \epsilon \rangle = \phi(x) \neq \phi(y) = \langle \pi_\phi(y)\epsilon, \epsilon \rangle$$

so $\pi_\phi(x) \neq \pi_\phi(y)$. □

When the group G is neither abelian nor compact the irreducible representations may be infinite-dimensional, and often the finite-dimensional representations do not separate points of G. An example of such group a group is $SL(2, \mathbb{R})$, for the proof see for instance [18, p. 113, Corollary 3.].

The construction of unitary representations of a group from functions of positive type is in fact very similiar to the Gelfand-Naimark-Segal construction in the theory of C*-algebras. Indeed in the language of C*-algebras a *state* is a positive linear functional of norm 1, and the GNS construction states that for every state ϕ of a C*-algebra A, there exists a cyclic *-representation π_ϕ of A with cyclic vector ξ such that $\rho = \langle \pi_\phi(\cdot)\xi, \xi \rangle$. Moreover irreducible *-representations of A correspond to *pure states*, which are the extreme points in the *state space*. The proof can be found for instance in [3, p. 31, Theorem 7.7.]. The GNS construction is at the heart of the proof of the noncommutative Gelfand-Naimark theorem.

An examination of the ideas of this chapter reveals the importance of locally compactness in representation theory. Haar measure allows us to construct the first good unitary representations, the regular representations, and perhaps even more importantly to consider the group algebra, which had an essential role in the construction of cyclic and irreducible representations.

Outside of locally compact groups no comparable representation theory is known. Every topological Hausdorff group does have isometric Banach representations, namely the action given by left (right) translation by group elements on the left (right) uniformly continuous functions $LUC(G)$ ($RUC(G)$). This result is known as Teleman's theorem. However Banach representations are far less geometric and useful than representations on Hilbert spaces. Almost none of the ideas and results of this chapter work for Banach representations. For more on Teleman's theorem see [14].

Chapter 4

Compact Groups

In classical harmonic analysis a function on a compact interval $[-\pi, \pi]$ (or the unit circle $\mathbb{T}$) is represented or approximated by sums of trigonometric functions (or complex exponentials). These simpler functions provide an orthonormal basis for the space $L^2([-\pi, \pi])$ (or $L^2(\mathbb{T})$). In this chapter we present some of the basic theory of representations of compact groups. The results of this chapter are summarized in the Peter-Weyl theorem, which among other things gives an orthonormal basis of functions for $L^2(G)$ for a compact group G.

Throughout this chapter G is a compact group with a normalized Haar measure $|G| = 1$, which is both left and right invariant.

4.1 Representations of Compact Groups

We begin by proving some basic results about unitary representations on compact groups. The following lemma is important.

Lemma 4.1.1. *Suppose π is a unitary representation of the compact group G. Fix a unit vector $u \in \mathcal{H}_\pi$, and define the operator T on $\mathcal{H}_\pi$ by*

$$Tv = \int \langle v, \pi(x)u \rangle \pi(x) u dx.$$

Then T is positive, nonzero and compact, and $T \in \mathcal{C}(\pi)$.

Proof. For any $v \in \mathcal{H}_\pi$ we have

$$\langle Tv, v \rangle = \int \langle v, \pi(x)u \rangle \langle \pi(x)u, v \rangle dx = \int |\langle v, \pi(x)u \rangle|^2 dx \geq 0,$$

so T is positive. Moreover, if we take $u = v$, $|\langle u, \pi(x)u \rangle|^2$ is strictly positive on a neighborhood of e, so $\langle Tu, u \rangle > 0$ and hence $T \neq 0$.

Since G is compact, $x \mapsto \pi(x)u$ is uniformly continuous. Hence, given $\varepsilon > 0$, we can find a partition of G into disjoint sets $E_1, \ldots, E_n$ and points $x_j \in E_j$ such that $\|\pi(x)u - \pi(x_j)u\| < \frac{1}{2}\varepsilon$ for $x \in E_j$. Indeed there is a neighborhood V of e such that $\|\pi(x)u - u\| < \frac{1}{2}\varepsilon$ for every $x \in V$. Now let U be a symmetric neighborhood of e such that $UU \subset V$. The translates of U cover G, so by compactness there exists a finite subcover $\{g_jU\}_{j=1}^n$. Then let $E_1 = g_1U$ and $E_j = g_jU \setminus \bigcup_{k=1}^{j-1} g_kU$ whenever $1 < k \leq n$. The finite subcover can be chosen so that every E_j is nonempty, so we may pick $x_j \in E_j$ for every j. Now if $x \in E_j$, then $x_j^{-1}x \in Ug_j^{-1}g_jU \subset V$. Hence $\|\pi(x)u - \pi(x_j)u\| = \|\pi(x_j^{-1}x)u - u\| < \frac{1}{2}\varepsilon$ for every $x \in E_j$. Now we have

$$\begin{aligned} & \|\langle v, \pi(x)u\rangle\pi(x)u - \langle v, \pi(x_j)u\rangle\pi(x_j)u\| \\ \leq \ & \|\langle v, [\pi(x) - \pi(x_j)]u\rangle\pi(x)u\| + \|\langle v, \pi(x_j)u\rangle[\pi(x) - \pi(x_j)]u\| \\ < \ & \varepsilon\|v\| \end{aligned}$$

for $x \in E_j$, so if we set

$$T_\varepsilon v = \sum_{j=1}^n |E_j|\langle v, \pi(x_j)u\rangle\pi(x_j)u$$

we have

$$\|Tv - T_\varepsilon v\| \leq \sum_j \int_{E_j} \|\langle v, \pi(x)u\rangle\pi(x)u - \langle v, \pi(x_j)u\rangle\pi(x_j)u\|dx < \varepsilon\|v\|$$

for all v. But the range of T_ε is the linear span of $\{\pi(x_j)u\}_1^n$, so T_ε has finite rank. Therefore T is compact, being the norm limit of operators of finite rank.

Finally $T \in \mathcal{C}(\pi)$ because

$$\begin{aligned} \pi(y)Tv &= \int \langle v, \pi(x)u\rangle\pi(yx)udx = \int \langle v, \pi(y^{-1}x)u\rangle\pi(x)udx \\ &= \int \langle \pi(y)v, \pi(x)u\rangle\pi(x)udx = T\pi(y)v. \end{aligned}$$

□

Theorem 4.1.2. *If G is compact, then every irreducible representation of G is finite-dimensional, and every unitary representation of G is a direct sum of irreducible representations.*

Proof. Suppose π is irreducible, and let T be as in Lemma 4.1.1. By Schur's lemma, $T = cI$ with $c \neq 0$. So the identity operator $\mathcal{H}_\pi$ is compact, and hence $\dim \mathcal{H}_\pi < \infty$.

Now let π be an arbitrary unitary representation of G. Since T is compact, nonzero, and self-adjoint, it has a nonzero eigenvalue λ whose eigenspace $\mathcal{E}_\lambda$ is necessarily finite-dimensional, since the restriction of T to $\mathcal{E}_\lambda$ is λI and compact. Since $T \in \mathcal{C}(\pi)$, $\mathcal{E}_\lambda$ is invariant because for any $x \in G$ and $u \in \mathcal{E}_\lambda$ we have $T\pi(x)u = \pi(x)Tu = \lambda\pi(x)u$, so indeed $\pi(x)u \in \mathcal{E}_\lambda$. Hence π has a finite-dimensional subrepresentation. But every finite-dimensional representation is a direct sum of irreducible representations. To see this, if a finite-dimensional representation $(\pi, \mathcal{H})$ has an invariant subspace $\mathcal{M}$ with $0 < \dim \mathcal{M} < \dim \mathcal{H}$, then π is the direct sum of two subrepresentations with dimension smaller than $\dim \mathcal{H}$. Since $\dim \mathcal{H}$ is finite, this process of decomposing eventually ends.

Now by Zorn's lemma there is a maximal family $\{\mathcal{M}_\alpha\}$ of mutually orthogonal irreducible invariant subspaces for π. If $\mathcal{N}$ is the orthogonal complement $\bigoplus \mathcal{M}_\alpha$, then $\mathcal{N}$ is invariant, and by the above argument $\pi^{\mathcal{N}}$ has an irreducible invariant subspace, contradicting maximality unless $\mathcal{N}$. Thus $\mathcal{H}_\pi = \bigoplus \mathcal{M}_\alpha$. □

We denote $\widehat{G}$ the set of unitary equivalence classes of irreducible unitary representations of G. This is indeed a valid set since by the above theorem the representations are all finite-dimensional. We denote the equivalence class of π by $[\pi]$. Writing "$[\pi] \in \widehat{G}$" will be a convenient shorthand for the statement "π is an irreducible unitary representation of G".

It is worth noting that if ρ is a possibly nonunitary representation of the compact group G on a Hilbert space $\mathcal{H}$, then there is an inner product on $\mathcal{H}$ with respect to which ρ is unitary. To see this, if $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$ then define a new inner product by

$$\langle u, v \rangle_\rho = \int \langle \rho(x)u, \rho(x)v \rangle dx.$$

Then $\langle \cdot, \cdot \rangle_\rho$ is a ρ-invariant inner product, for

$$\langle \rho(y)u, \rho(y)v \rangle_\rho = \int \langle \rho(xy)u, \rho(xy)v \rangle dx = \int \langle \rho(x)u, \rho(x)v \rangle dx = \langle u, v \rangle_\rho.$$

Moreover by Theorem 3.1.4 there exists a positive $P \in L(\mathcal{H})$ such that $\langle u, v \rangle_\rho = \langle Pu, v \rangle$. By the spectral theorem P has the unique square root $\sqrt{P} = S$. Now $x \mapsto S\rho(x)S^{-1}$ defines a unitary representation of G on $\mathcal{H}$ since if $u, v \in \mathcal{H}$ and $x \in G$, then

$$\begin{aligned} & \langle S\rho(x)S^{-1}u, S\rho(x)S^{-1}v \rangle = \langle P\rho(x)S^{-1}u, \rho(x)S^{-1}v \rangle \\ = \; & \langle \rho(x)S^{-1}u, \rho(x)S^{-1}v \rangle_\rho = \langle S^{-1}u, S^{-1}v \rangle_\rho = \langle PS^{-1}u, S^{-1}v \rangle = \langle u, v \rangle. \end{aligned}$$

A close study of the above argument shows that the claim can be generalized. A locally compact group is called *amenable* if the space of bounded functions

$L^\infty(G)$ admits an invariant mean, that is there exists a functional $m \in L^\infty(G)^*$ with $m(1) = \|m\| = 1$ and $m(L_x f) = m(f)$ for every $x \in G$ and $f \in L^\infty(G)$. All compact groups and abelian groups are amenable. More on amenable groups can be found on [13], [15] and [1, Chapter G].

A representation $\rho : G \to L(\mathcal{H})$ is a *uniformly bounded* representation if $\sup_{x\in G} \|\rho(x)\| < \infty$. In the example above a strongly continuous representation of a compact group G is uniformly bounded since $\rho(G)u$ is compact and hence bounded for every $u \in \mathcal{H}$, so by the Banach-Steinhaus theorem $\sup_{x\in G} \|\rho(x)\| < \infty$. Now $\langle u, v\rangle_\rho = m(\varphi_{u,v})$ defines a ρ-invariant inner product, where $\varphi_{u,v}(x) = \langle \rho(x)u, \rho(x)v\rangle$ is a bounded continuous function. If we denote $|\rho| = \sup_{x\in G} \|\rho(x)\|$, then the inequalities $|\rho|^{-1}\|u\| \leq \|u\|_\rho \leq |\rho|\|u\|$ hold, so again we find a positive invertible S such that $S\rho(x)S^{-1}$ is unitary for every $x \in G$. A group is called *unitarizable* if for every uniformly bounded representation $(\pi, \mathcal{H})$ we can find $S \in L(\mathcal{H})$ such that $x \mapsto S\pi(x)S^{-1}$ is a unitary representation. By what we just showed amenable groups are unitarizable.

Naturally one may ask if the converse holds, that is is every unitarizable group amenable. This was conjectured by Dixmier in 1950, and it is still open. Some partial results have been obtained, see for instance [16].

4.2 The Peter-Weyl Theorem

We shall define a non-abelian analog of the trigonometric functions and complex exponentials of classical harmonic analysis.

Definition 4.2.1. If π is any unitary representation of G, the functions

$$\phi_{u,v}(x) = \langle \pi(x)u, v\rangle \quad (u, v \in \mathcal{H}_\pi)$$

are called *matrix elements* or *matrix coefficients* of π. If u and v are members of an orthonormal basis $\{e_j\}$ for $\mathcal{H}_\pi$, $\phi_{u,v}$ is one of the entries of the matrix for $\pi(x)$ with respect to that basis, namely

$$\pi_{ij}(x) = \phi_{e_j,e_i}(x) = \langle \pi(x)e_j, e_i\rangle.$$

We denote the linear span of the matrix coefficients of π by $\mathcal{E}_\pi$.

The space $\mathcal{E}_\pi$ is a subspace of $C(G)$ and hence also of $L^p(G)$ for all p.

Theorem 4.2.2. *The space $\mathcal{E}_\pi$ depends only on the unitary equivalence class of π. It is invariant under left and right translations. If $\dim \mathcal{H}_\pi = n < \infty$ then $\dim \mathcal{E}_\pi \leq n^2$.*

Proof. If T is a unitary equivalence of π and π', so that $\pi'(x) = T\pi(x)T^{-1}$, then $\langle \pi(x)u, v\rangle = \langle \pi'(x)Tu, Tv\rangle$. Now

$$\phi_{u,v}(y^{-1}x) = \langle \pi(y^{-1}x)u, v\rangle = \langle \pi(x)u, \pi(y)v\rangle = \phi_{u,\pi(y)v}(x),$$

and likewise $\phi_{u,v}(xy) = \phi_{\pi(y)u,v}(x)$. Finally if $\dim \mathcal{H}_\pi = n$, then $\mathcal{E}_\pi$ is clearly spanned by the n^2 functions π_{ij}. □

Theorem 4.2.3. *If $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ then $\mathcal{E}_\pi = \sum_{j=1}^n \mathcal{E}_{\pi_j}$.*

Proof. Clearly $\mathcal{E}_{\pi_j} \subset \mathcal{E}_\pi$ for all j. On the other hand if $u = \sum u_j$ and $v = \sum v_j$ with $u_j, v_j \in \mathcal{H}_{\pi_j}$, then $\langle \pi(x)u_j, v_i\rangle = 0$ for every $i \neq j$ and hence $\phi_{u,v} = \sum \phi_{u_j,v_j} \in \sum \mathcal{E}_{\pi_j}$. □

Note that in the above theorem the sum $\sum_{j=1}^n \mathcal{E}_{\pi_j}$ need not be direct.

The matrix coefficients of irreducible representations can be used to make an orthonormal basis for $L^2(G)$. Let $d_\pi = \dim \mathcal{H}_\pi$, and denote the trace of a matrix A by TrA.

Theorem 4.2.4 (The Schur Orthogonality Relations)**.** *Let π and π' be irreducible unitary representations of G, and consider $\mathcal{E}_\pi$ and $\mathcal{E}_{\pi'}$ as subspaces of $L^2(G)$.*

(a) If $[\pi] \neq [\pi']$ then $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$.

(b) If $\{e_j\}$ is any orthonormal basis for $\mathcal{H}_\pi$ then $\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \ldots, d_\pi\}$ is an orthonormal basis for $\mathcal{E}_\pi$.

Proof. If A is any linear map from $\mathcal{H}_\pi$ to $\mathcal{H}_{\pi'}$, let

$$\widetilde{A} = \int \pi'(x^{-1})A\pi(x)dx.$$

Then

$$\widetilde{A}\pi(y) = \int \pi'(x^{-1})A\pi(xy)dx = \int \pi'(yx^{-1})A\pi(x)dx = \pi'(y)\widetilde{A},$$

so $A \in \mathcal{C}(\pi, \pi')$. Given $v \in \mathcal{H}_\pi$ and $v' \in \mathcal{H}_{\pi'}$, let us define A by $Au = \langle u, v\rangle v'$. Then for any $u \in \mathcal{H}_\pi$ and $u' \in \mathcal{H}_{\pi'}$,

$$\begin{aligned}
\langle \widetilde{A}u, u'\rangle &= \int \langle A\pi(x)u, \pi'(x)u'\rangle dx \\
&= \int \langle \pi(x)u, v\rangle \langle v', \pi'(x)u'\rangle dx \\
&= \int \phi_{u,v}(x)\overline{\phi_{u',v'}(x)}dx.
\end{aligned}$$

We now apply Schur's lemma. If $[\pi] \neq [\pi']$ then $\widetilde{A} = 0$, so $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$. This proves (a). If $\pi = \pi'$ then $\widetilde{A} = cI$, so if we take $U = e_i, u' = e_{i'}, v = e_j$ and $v' = e_{j'}$ we get

$$\int \pi_{ij}(x)\overline{\pi_{i'j'}(x)} = c\langle e_i, e_{i'}\rangle = c\delta_{ii'}.$$

But

$$cd_\pi = \mathrm{Tr}\widetilde{A} = \int \mathrm{Tr}[\pi(x^{-1})A\pi(x)]dx = \mathrm{Tr}A,$$

and since $Au = \langle u, e_j\rangle e_{j'}$ we have $\mathrm{Tr}A = \sum\langle Ae_k, e_k\rangle = \sum\langle e_k, e_j\rangle\langle e_{j'}, e_k\rangle = \delta_{jj'}$. Hence

$$\int \pi_{ij}(x)\overline{\pi_{i'j'}(x)} = \frac{1}{d_\pi}\delta_{ii'}\delta_{jj'}$$

so $\{\sqrt{d_\pi}\pi_{ij}\}$ is an orthonormal set. Since $\dim \mathcal{E}_\pi \leq d_\pi^2$, it is a basis. □

We observed in Theorem 4.2.2 that $\mathcal{E}_\pi$ is invariant under the left and right translations L and R. Note that since G is compact we have $\pi_R(x) = R_x$. For the next theorem we shall simplify our notation by letting $L = \pi_L$ be the left regular representation and $R = \pi_R$ be the right regular representation. One may then ask what are the irreducible subrepresentations of L and R on $\mathcal{E}_\pi$.

Theorem 4.2.5. *Suppose π is irreducible. For $i = 1, \ldots, d_\pi$ let $\mathcal{R}_i$ be the linear span of $\pi_{i1}, \ldots, \pi_{id_\pi}$ (the ith row of the matrix (π_{ij})) and let $\mathcal{C}_i$ be the linear span of $\pi_{1i}, \ldots, \pi_{d_\pi i}$ (the ith column). Then $\mathcal{R}_i$ (respectively $\mathcal{C}_i$) is invariant under the right (left) regular representation, and $R^{\mathcal{R}_i}$ ($L^{\mathcal{C}_i}$) is equivalent to π ($\overline{\pi}$). The equivalence is given by*

$$\sum c_j e_j \mapsto \sum c_j \pi_{ij} \quad (\sum c_j e_j \mapsto \sum c_j \pi_{ij}).$$

Proof. In terms of the basis $\{e_j\}$ for $\mathcal{H}_\pi$, π is given by

$$\pi(x)\left(\sum_{j=1}^{d_\pi} c_j e_j\right) = \sum_{k,j=1}^{d_\pi} \pi_{kj}(x)c_j e_k.$$

Moreover $\pi(yx) = \pi(y)\pi(x)$, so $\pi_{ij}(yx) = \sum_k \pi_{ik}(y)\pi_{kj}(x)$. In other words, $R_x\pi_{ij} = \sum_k \pi_{kj}(x)\pi_{ik}$, so

$$R_x\left(\sum_{j=1}^{d_\pi} c_j\pi_{ij}\right) = \sum_{j,k=1}^{d_\pi} \pi_{kj}(x)c_j\pi_{ik}.$$

Comparing the two above lines we see that π is equivalent to $R^{\mathcal{R}_i}$. In the same way, for left translations we see that

$$L_x\left(\sum_{j=1}^{d_\pi} c_j\pi_{ji}\right) = \sum_{j,k=1}^{d_\pi} \pi_{jk}(x)c_j\pi_{ki},$$

and since π is unitary, we have $\pi_{jk}(x^{-1}) = \overline{\pi}_{kj}(x)$. □

Now let

$$\mathcal{E} = \text{the linear span of } \bigcup_{[\pi]\in\widehat{G}} \mathcal{E}_\pi.$$

So $\mathcal{E}$ consists of *finite* linear combinations of matrix coefficients of irreducible representations. By Theorem 4.2.3, $\mathcal{E}$ is also the the linear span of matrix coefficients of finite-dimensional representations of G. The space $\mathcal{E}$ could be considered as the space of "trigonometric functions" on G.

Theorem 4.2.6. *$\mathcal{E}$ is an algebra.*

Proof. It is sufficient to show that if $[\pi], [\rho] \in \widehat{G}$ and π_{ij}, ρ_{kl} are matrix coefficients, then $\pi_{ij}\rho_{kl}$ is a matrix coefficients of some finite-dimensional representation of G. We shall construct another representation $\pi \otimes \rho$ of G using the *Kronecker product* for matrices. That is if $A = [a_{ij}]$ is an $n \times m$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $np \times mq$ block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.$$

It is easy to verify that $(A \otimes B)(C \otimes D) = AC \otimes BD$ if one can form the matrices AC and BD, and hence $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ if A^{-1} and B^{-1} exist, and $(A\otimes B)^* = A^*\otimes B^*$. Now define the new representation by $(\pi\otimes\rho)(x) = \pi(x)\otimes\rho(x)$ on $\mathbb{C}^{nm}$, where $n = \dim \pi$ and $m = \dim \rho$. By the above mentioned properties of the Kronecker product this is a unitary representation of G. Moreover it is quite clear from the resulting matrix

$$(\pi \otimes \rho)(x) = \begin{pmatrix} \pi_{11}(x)\rho(x) & \cdots & \pi_{1n}(x)\rho(x) \\ \vdots & \ddots & \vdots \\ \pi_{n1}(x)\rho(x) & \cdots & \pi_{nn}(x)\rho(x) \end{pmatrix}$$

that $\pi_{ij}(x)\rho_{kl}(x)$ appears as a matrix coefficient. Indeed the desired coefficient is $\langle (\pi \otimes \rho)(x)e_{(j-1)m+l}, e_{(i-1)m+k}\rangle$. Now by Theorem 4.1.2, $\pi \otimes \rho$ is a direct sum of irreducible representations, so by Theorem 4.2.3, we have $\pi_{ij}\rho_{kl} \in \mathcal{E}$. □

Remark 4.2.7. Equivalently we could have defined $\pi \otimes \rho$ in the above proof by the action $(\pi \otimes \rho)(x)T = \pi(x)T\overline{\rho}(x^{-1})$, where T is a $n \times m$ matrix.

We are almost done with proving the Peter-Weyl theorem.

Theorem 4.2.8. *$\mathcal{E}$ is dense in $C(G)$ in the uniform norm, and dense in $L^p(G)$ in the L^p norm for $p < \infty$.*

Proof. It is enough to show that $\mathcal{E}$ is dense in $C(G)$ since $C(G)$ is dense in $L^p(G)$. But $\mathcal{E}$ is an algebra that separates points by the Gelfand-Raikov theorem, is closed under conjugation since every representation has a contragredient and contains constant functions because of the trivial representation of G on $\mathbb{C}$. Therefore by Stone-Weierstrass $\mathcal{E}$ is dense in $C(G)$. □

The original proof by Hermann Weyl and Fritz Peter (1927) is in fact older than either Gelfand-Raikov theorem (1943) or Stone-Weierstrass (1937). The first proof can be found in [5, Theorem 5.11.]. The proof is based on studying convolution operators $T_\psi f = \psi * f$ on $L^2(G)$, where ψ is a continuous symmetric function. The operator is proven to be compact using Arzela-Ascoli theorem, so by the spectral theorem for compact operators $L^2(G)$ can be seen as direct sum of finite-dimensional eigenspaces of T_ψ. Moreover each eigenspace is invariant under right translations, and it will follow that the eigenspaces are contained in $\mathcal{E}$. Hence $\mathcal{E} \cap \text{Range}(T_\psi)$ will be uniformly dense in $\text{Range}(T_\psi)$, and taking the union of ranges of T_ψ as ψ runs through an approximate identity is dense in $C(G)$, proving the theorem.

Combining Theorem 4.2.8 with the Schur orthogonality relations, we see that $L^2(G)$ is the orthogonal direct sum of the spaces $\mathcal{E}_\pi$ as $[\pi]$ ranges over $\widehat{G}$, and that we obtain an orthonormal basis for $L^2(G)$ by fixing an element π of each irreducible equivalence class $[\pi]$ and taking the matrix coefficients corresponding to an orthonormal basis of $\mathcal{H}_\pi$. In the statement of the Peter-Weyl theorem we assume that one representation has been picked from each equivalence class.

The main theorem is a summary of the results of this chapter.

Theorem 4.2.9 (The Peter-Weyl Theorem)**.** *Let G be a compact group. Then $\mathcal{E}$ is uniformly dense in $C(G), L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$, and*

$$\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \ldots, d_\pi, [\pi] \in \widehat{G}\}$$

is an orthonormal basis for $L^2(G)$. Each $[\pi] \in \widehat{G}$ occurs in the right and left regular representations of G with multiplicity d_π. More precisely, for each $i = 1, \ldots, d_\pi$ the subspace of $\mathcal{E}_\pi$ (respectively $\mathcal{E}_{\overline{\pi}}$) spanned by the ith row (ith column) of the matrix (π_{ij}) $((\overline{\pi}_{ij}))$ is invariant under the right (left) regular representation, and the latter representation is equivalent to π.

As an application of the ideas of this thesis we obtain a characterization of compact groups.

Corollary 4.2.10. Every compact group is a product of closed subgroups of unitary matrices.

Proof. Consider the mapping

$$F : G \to \prod_{[\pi] \in \widehat{G}} \pi(G) \subset \prod_{[\pi] \in \widehat{G}} U(\mathcal{H}_\pi), \quad g \mapsto (\pi(g))_{\pi \in \widehat{G}},$$

where the image of F has the product topology. By Gelfand-Raikov theorem this is injective. Note that strong operator topology on $U(\mathcal{H}_\pi)$ coincides with the norm topology since $\mathcal{H}_\pi$ is finite-dimensional. So the function F is continuous and a homomorphism since $\pi(xy) = \pi(x)\pi(y)$ for every $[\pi] \in \widehat{G}$. Hence G is topologically isomorphic to its image $F(G)$. □

Using the Peter-Weyl theorem a Fourier transform can be defined for functions on a compact group. Moreover the transform has some of the same properties as the classical Fourier transform, such as taking convolutions of functions to pointwise products. More on Fourier analysis on compact groups can be found in [5, p. 133-138].

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