

**Self-similar sets of Hausdorff measure zero and
positive packing measure**

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Introduction

Fractals are a rather new concept in mathematics. The word *fractal* was first used by Benoit Mandelbrot in 1975. However, ideas relating to fractals have appeared as early as in the 17th century. In the 19th century, they became mathematically more rigorous and in the 20th century they grew to be a major field of study. The word fractal originates from the Latin word *fractus*, meaning broken or fractured. This word illustrates well the difference between fractals and classical geometric objects. If we examine the boundary of an object of classical geometry by zooming in on the picture, at some point we see a solid interior and a clear exterior separated by a simple curve. If we zoom in more, the curve will also begin to resemble a line. With fractals, this is not the case. Fractals contain fine details at every level of this kind of zoom – no matter how far we go, the picture will not simplify. Fractals are shredded figures with arbitrarily small shreds. There is no formal general definition for what a fractal is.

If we would like to study the geometric properties of a fractal, we have to assume some properties for it. The most classic examples are self-similar sets, which include the Cantor set and Sierpinski gasket. Self-similar sets are a kind of fractal which can be divided into a finite amount of pieces, each of which is similar to the whole set. Being similar here means that they are the same sets up to translation, rotation, scaling and reflection. Self-similar sets are usually defined by a family of functions executing these operations.

Some of the tools that are used for studying classical geometry are inadequate when studying fractals. These common ideas have to be refined so that we can describe the 'length', 'area', 'volume' or 'dimension' of a fractal set. One useful tool is the Hausdorff measure and Hausdorff dimension introduced by Felix Hausdorff in 1918. The Hausdorff dimension is a generalization of the common notion of dimension and it does not have to be an integer. One can then measure the set in its own dimension with the corresponding Hausdorff measure, which is a generalization of a length, area or volume. This definition is convenient, since it works for every set. However, it is only one definition. Claude Tricot introduced the packing measure and packing dimension in 1982. These two dimensions are different from each other but work in the same way.

When self-similar sets are studied, they are usually assumed to satisfy some separation conditions, as the open set condition or strong separation condition. These conditions limit how much the individual similar parts may overlap each other. Without any kind of separation

condition, the self-similar set can be very complex. However, it is known that the Hausdorff and packing dimensions agree on self-similar sets even without separation conditions. The behavior of the corresponding Hausdorff and packing measures remain quite mysterious. Another question concerning self-similar sets, is what happens to the measure when we take a projection of a self-similar set? Marstrand's theorem [5, Theorem 6.1] guarantees that, in a projection the Hausdorff dimension is maintained for almost every direction, but how does the Hausdorff measure change in the process? The purpose of this thesis is to address these problems. It is based on an article by Peres, Simon and Solomyak [11]. We restrict our focus to self-similar sets which are defined by functions which only translate and scale, thus rotations and reflections are not allowed.

We will see that there are self-similar sets that have zero Hausdorff measure but positive and finite packing measure in their dimension. This behavior is common for such self-similar sets on the line, which are projections of a certain type of planar self-similar sets. This kind of self-similar set on a line does not necessarily satisfy any separation conditions and, of course, the interesting cases are exactly those projections where overlapping occurs. By projecting certain types of planar self-similar sets, we achieve whole families of self-similar sets on a line with the above property. We also see that if there is overlapping in this kind of projection, the Hausdorff measure drops to zero in almost every case. However, the corresponding packing measure is positive and finite. This suggests that the packing measure is a more suitable measure for sets of this kind. The result concerning packing measure is shown for a general Borel set. We need to make the assumption about the dimension of that Borel set with respect to the Hausdorff measure, since packing measure may drop for almost all projections, see [9] and [6]. Thus Hausdorff measure is still needed and cannot be completely omitted.

Chapter 1 contains preliminary information needed in the thesis. In Chapter 2 we introduce one-parameter families of iterated function systems and present some properties about them. These families are the abstract counterparts of the projections of a planar self-similar set. The results concerning the Hausdorff measure and packing measure are in the Chapters 3 and 4, respectively. Finally in Chapter 5 we present two examples.

Chapter 1

Preliminaries

In this chapter we introduce definitions and general concepts which are used throughout the paper. Many of the proofs are omitted but they can be found in most books concerning the subject at hand. For theory concerning fractals, dimensions and other geometric measure theory see [5] or [10]. For general measure theory see [12].

1.1 Basic notations

The following notations are in the form for metric spaces (X, d) , even though we mostly work in \mathbb{R} or \mathbb{R}^2 with the Euclidean metric.

The closed ball with centre $x \in X$ and radius $r > 0$ is denoted by

$$B(x, r) = \{y \in X : d(x, y) \leq r\}.$$

The diameter of a set $A \subset X$ is denoted by

$$d(A) = \sup\{d(x, y) : x, y \in A\}.$$

In case of an interval $A \subset \mathbb{R}$ we may write $d(A) = |A|$. If A is a non-empty subset of X , then for the distance from $x \in X$ to A we set

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

For $\varepsilon > 0$ the ε -neighbourhood of $A \subset X$ is the set

$$A(\varepsilon) = \{x \in X : d(x, A) \leq \varepsilon\}.$$

For a set $A \subset X$, \bar{A} is the closure, $\text{Conv}(A)$ is the convex hull and $\#A$ is the cardinality of A .

1.2 The space of continuously differentiable functions

The functions of this class will be used frequently.

Definition 1.1. We say that a function $f: I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$ is compact interval, belongs to the set $C^1(I)$ if and only if it has a continuous derivative on I .

Sets C^k are defined similarly for all $k = 0, 1, \dots$, but we only deal with the case $k = 1$. One can define different kinds of metrics and norms for these sets to get function spaces.

Theorem 1.2. Let $I \subset \mathbb{R}$ be a compact interval. Then

$$\|f\|_{C^1(I)} = \sup_{x \in I} |f(x)| + \sup_{x \in I} |f'(x)|$$

is a norm on $C^1(I)$. Moreover, this norm makes $C^1(I)$ a Banach space. This norm is called the C^1 norm.

Note that $\sup_{x \in A} |\cdot|$ is usually called the *uniform norm*, *supremum norm* or *infinity norm* and denoted by $\|\cdot\|_\infty$. One can easily show that this is a norm for bounded functions. The proof follows from the fact that the uniform norm combined with continuous functions on a compact metric space form a Banach space. This norm also seems meaningful intuitively, since it considers both the function and its derivative.

1.3 General measure theory

We assume that the reader has some knowledge of basic measure theory and is familiar with the Lebesgue measure \mathcal{L} .

Definition 1.3. Let X be a set. A function $\mu: \{A : A \subset X\} \rightarrow [0, \infty]$ is called a *measure* if

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$, when $A \subset B \subset X$
3. $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$, when $A_1, A_2, \dots \subset X$.

So it is a non-negative, *monotonic* and (countably) *subadditive* set function, which vanishes on the empty set. In measure theory the above kind of function is usually defined to be only an *outer measure*, while a measure is also countably additive (see condition 3. in the next theorem) and defined on some σ -algebra of X . Our definition seemingly differs from this, and the justification is presented below.

Definition 1.4. A set $A \subset X$ is μ *measurable* if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$$

for every $E \subset X$.

Here we collect the basic properties of measurable sets.

Theorem 1.5. *Let μ be an outer measure on X and let \mathcal{M} be the set of all μ measurable sets.*

1. \mathcal{M} is a σ -algebra, that is,
 - (a) $\emptyset \in \mathcal{M}$,
 - (b) if $A \in \mathcal{M}$, then $X \setminus A \in \mathcal{M}$,
 - (c) if $A_1, A_2, \dots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.
2. If $\mu(A) = 0$, then $A \in \mathcal{M}$.
3. If $A_1, A_2, \dots \in \mathcal{M}$ are disjoint, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.
4. If $A_1, A_2, \dots \in \mathcal{M}$, then
 - (i) $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$ provided $A_1 \subset A_2 \subset \dots$,
 - (ii) $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$ provided $A_1 \supset A_2 \supset \dots$ and $\mu(A_1) < \infty$.

It is a convenience rather than a restriction to define measures as in Definition 1.3. If ν is a countably additive non-negative set function on a σ -algebra Σ of X , then it can be extended to a measure, as in Definition 1.3, by

$$\nu^*(A) = \inf\{\nu(B) : A \subset B \in \Sigma\}.$$

On the other hand, a measure μ gives a countably additive set function when we restrict it to the σ -algebra of μ measurable sets.

Definition 1.6. Let X be a topological space. The smallest σ -algebra containing all open sets of X is called the Borel family. A member of this family is a Borel set.

This kind of smallest σ -algebra always exists [12, Theorem 1.10]. Note that if we speak about some σ -algebra or Borel sets on X , it implies that we have some topology on X .

Definition 1.7. Let μ be a measure on X .

1. μ is *locally finite* if for every $x \in X$ there is $r > 0$ such that $\mu(B(x, r)) < \infty$.
2. μ is a *Borel measure* if all Borel sets are μ measurable.
3. μ is *Borel regular* if it is a Borel measure and if for every $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.
4. μ is a *Radon measure* if it is a Borel measure and
 - (a) $\mu(K) < \infty$ for compact sets $K \subset X$,
 - (b) $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact}\}$ for open sets $V \subset X$,
 - (c) $\mu(A) = \inf\{\mu(V) : A \subset V, V \text{ is open}\}$ for $A \subset X$.

Definition 1.8. The *support* of a Borel measure μ , $\text{spt } \mu$, on a separable metric space X is the smallest closed set $S \subset X$ such that $\mu(X \setminus S) = 0$.

Definition 1.9. Let $f: X \rightarrow Y$ and μ be a measure on X . The *image measure* of μ under f is defined by

$$f_*\mu(A) = \mu(f^{-1}(A))$$

for $A \subset Y$.

Clearly $f_*\mu$ is a measure on Y . Also, it can be shown that if μ is a Borel measure and f a Borel function, then $f_*\mu$ is a Borel measure. Moreover, $\text{spt } f_*\mu = f(\text{spt } \mu)$.

Definition 1.10. Let μ and ν be measures on metric spaces X and Y , with collection of measurable sets \mathcal{M}_μ and \mathcal{M}_ν respectively. For every $A \subset X \times Y$ we set

$$(\mu \times \nu)(C) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : C \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{M}_\mu \text{ and } B_i \in \mathcal{M}_\nu \right\}.$$

This is the *product measure* $\mu \times \nu$ on $X \times Y$.

In the above definition we treat $0 \cdot \infty = \infty \cdot 0 = 0$. It can be shown that if both μ and ν are Borel, Borel regular or Radon measures, then $\mu \times \nu$ has the same property.

Next, we give two useful theorems about integration, Fubini's theorem and integration with respect to an image measure.

Theorem 1.11. Let X and Y be separable metric spaces and μ and ν locally finite Borel measures on X and Y , respectively. If f is a non-negative Borel function on $X \times Y$, then

$$\int f \, d(\mu \times \nu) = \iint f(x, y) \, d\mu(x) \, d\nu(y) = \iint f(x, y) \, d\nu(y) \, d\mu(x).$$

One can also use this in cases where there are seemingly no iterated integrals or product measures.

Corollary 1.12. Let μ be a Borel measure and f a non-negative Borel function on a separable metric space X . Then

$$\int f \, d\mu = \int_0^{\infty} \mu(\{x \in X : f(x) \geq t\}) \, dt,$$

where the latter integral is taken with respect to the Lebesgue measure.

Theorem 1.13. Suppose that $f: X \rightarrow Y$ is a Borel mapping, μ is a Borel measure on X , and g is a non-negative Borel function on Y . Then

$$\int g \, df_*\mu = \int (g \circ f) \, d\mu.$$

Finally we present a useful theorem concerning the Lebesgue measure.

Theorem 1.14. Let \mathcal{L} be the Lebesgue measure on \mathbb{R} . If $A \subset \mathbb{R}$ is measurable, then the limit

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}(A \cap B(x, r))}{\mathcal{L}(B(x, r))}$$

equals 1 for Lebesgue almost every $x \in A$ and equals 0 for Lebesgue almost every $x \in \mathbb{R} \setminus A$.

The above theorem is known as the *Lebesgue density point theorem*. A point $x \in A$ for which this limit is 1 is called a *Lebesgue density point*.

1.4 Measures and dimensions

In this section we introduce Hausdorff and packing measures, which are general tools in fractal geometry. They both give a dimension for the set examined, which coincides with the common notion of dimension in cases of simple objects of classical geometry. However, these definitions of dimensions are very general. They assign a dimension for every subset of the space, moreover, this dimension can be non-integral. We begin with a bit simpler generalization of dimension, called the box-counting dimension.

Box-counting dimension

Definition 1.15. Let $A \subset \mathbb{R}^n$ be bounded. The *lower* and *upper box-counting dimensions* of A are

$$\underline{\dim}_{\text{B}} A = \liminf_{\delta \downarrow 0} \frac{\log N_{\delta}(A)}{-\log \delta}$$

$$\overline{\dim}_{\text{B}} A = \limsup_{\delta \downarrow 0} \frac{\log N_{\delta}(A)}{-\log \delta}.$$

If these agree, then the *box-counting dimension* of A is

$$\dim_{\text{B}} A = \lim_{\delta \downarrow 0} \frac{\log N_{\delta}(A)}{-\log \delta}.$$

Here $N_{\delta}(A)$ is any of the following:

- (a) the smallest number of sets of diameter at most δ that cover A ,
- (b) the smallest number of closed balls of radius δ that cover A ,
- (c) the smallest number of cubes of side length δ that cover A ,
- (d) the number of δ -mesh cubes that intersect A ,
- (e) the largest number of disjoint balls of radius δ with centres in A .

The name relates closest to the definition (d). Suppose that A lies in a evenly spaced mesh of cubes. Then one has to count the amount of n -cubes required to cover A , while the mesh is made finer and finer. This dimension is easier to calculate than more refined definitions of dimension which we introduce later. However, it has some unwanted properties. For example, a set has the same lower and upper box-counting dimension as its closure has. Then, by taking A to be the rationals on the unit interval we see that $\dim_{\text{B}} A = 1$. Thus a countable set can have non-zero box-counting dimension. Also, since the box-counting dimension of a singleton is zero, we see that $\dim_{\text{B}} \bigcup_{i=1}^{\infty} A_i = \sup_i \dim_{\text{B}} A_i$ does not hold generally. This dimension can still be useful. It can be used to approximate the more complicated dimensions we meet later on.

Hausdorff measure and dimension

Definition 1.16. Let X be a separable metric space and $0 \leq s < \infty$. Define

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^{\infty} d(E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) \leq \delta \right\}$$

and

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A)$$

for every $A \subset X$. We call \mathcal{H}^s the s -dimensional *Hausdorff measure*.

Remark 1.17. Here we interpret $0^0 = 1$ and $d(\emptyset)^s = 0$ for every s . The limit exists and agrees with the supremum since $\mathcal{H}_\delta^s(A)$ is non-increasing with respect to δ . Note that this measure is gained through Carathéodory's construction, so it is a Borel measure. Also, Hausdorff measure does not change if we consider only coverings with open sets E_i . By this, it is also Borel regular. See [10, Theorem 4.4.].

Next we present some properties of the Hausdorff measure.

Theorem 1.18. Let $A \subset X$ and $0 \leq s < t < \infty$.

- (1) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.
- (2) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$.

Proof. To prove (1), let $A \subset \bigcup_{i=1}^{\infty} E_i$ with $d(E_i) \leq \delta$ and $\sum_{i=1}^{\infty} d(E_i)^s \leq \mathcal{H}_\delta^s(A) + 1$. Since

$$\mathcal{H}_\delta^t(A) \leq \sum_{i=1}^{\infty} d(E_i)^t \leq \delta^{t-s} \sum_{i=1}^{\infty} d(E_i)^s \leq \delta^{t-s} (\mathcal{H}_\delta^s(A) + 1),$$

we get (1) as $\delta \rightarrow 0$. Statement (2) is clearly equivalent to (1). \square

The previous theorem enables us to make the following definition.

Definition 1.19. The *Hausdorff dimension* of a set $A \subset X$ is

$$\begin{aligned} \dim_{\text{H}} A &= \sup \{s : \mathcal{H}^s(A) > 0\} = \sup \{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf \{t : \mathcal{H}^t(A) < \infty\} = \inf \{t : \mathcal{H}^t(A) = 0\}. \end{aligned}$$

This dimension has the properties one would expect from a dimension. It is monotonic and countably stable, that is, it satisfies

$$\dim_{\text{H}} A \leq \dim_{\text{H}} B \quad \text{for } A \subset B \subset X$$

and

$$\dim_{\text{H}} \bigcup_{i=1}^{\infty} A_i = \sup_i \dim_{\text{H}} A_i \quad \text{for } A_i \subset X, i = 1, 2, \dots$$

It is also convenient that $\dim_{\mathbb{H}} A = s$ is defined for every set A , though we do not know anything about the measure $\mathcal{H}^s(A)$ in general. It can be zero, finite or infinite. It is worth noting that integral dimensional Hausdorff measures play a special role. They can be thought of as a generalization of the Lebesgue measure of the same dimension. \mathcal{H}^0 is readily seen to be a counting measure. For a rectifiable curve $\Gamma \subset \mathbb{R}^n$, $\mathcal{H}^1(\Gamma)$ is the length of Γ . More generally, it can be shown that if M is a sufficiently regular m -dimensional surface in \mathbb{R}^n , $1 \leq m \leq n$, then the restriction $\mathcal{H}^m|_M$ gives a constant multiple of the surface measure on M . See [10, Subsection 4.3].

Packing measure and dimension

Definition 1.20. Let $0 \leq s < \infty$. For $A \subset \mathbb{R}^n$ put

$$P_\delta^s(A) = \sup \sum_{i \in I} d(B_i)^s,$$

where the supremum is taken over all countable collections of disjoint balls of diameter smaller than δ with centres in A . Then we define the *packing pre-measure* of A by

$$P^s(A) = \lim_{\delta \downarrow 0} P_\delta^s(A) = \inf_{\delta > 0} P_\delta^s(A).$$

Remark 1.21. Again the limit exists and agrees with the infimum as $P^s(A)$ is non-decreasing with respect to δ . Clearly P^s is monotonic and $P^s(\emptyset) = 0$, but it is not subadditive, so it really is not a measure. But we get a measure out of it with the following definition.

Definition 1.22. Let $A \subset \mathbb{R}^n$ and $0 \leq s < \infty$. The s -dimensional *packing measure* of A is

$$\mathcal{P}^s(A) = \inf \left\{ \sum_{i=1}^{\infty} P^s(A_i) : A = \bigcup_{i=1}^{\infty} A_i \right\}$$

Theorem 1.23. \mathcal{P}^s is a Borel regular measure on \mathbb{R}^n .

Proof. See [10, Subsection 5.10] □

Theorem 1.24. Let $A \subset \mathbb{R}^n$ and $0 \leq s < t < \infty$.

(1) If $\mathcal{P}^s(A) < \infty$, then $\mathcal{P}^t(A) = 0$.

(2) If $\mathcal{P}^t(A) > 0$, then $\mathcal{P}^s(A) = \infty$.

Proof. See [10, Theorem 5.11]. □

As this is the case, we can define a dimension the same way we did with the Hausdorff measures.

Definition 1.25. The *packing dimension* of a set $A \subset \mathbb{R}^n$ is

$$\begin{aligned} \dim_{\mathcal{P}} A &= \sup\{s : \mathcal{P}^s(A) > 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\} \\ &= \inf\{t : \mathcal{P}^t(A) < \infty\} = \inf\{t : \mathcal{P}^t(A) = 0\}. \end{aligned}$$

As we now see, the packing measure and dimension behaves much like Hausdorff measure and dimension. This dimension is monotonic and countably stable. Also, the measure \mathcal{P}^s behaves as \mathcal{H}^s with integral dimensions. But how do these measures and dimensions relate to each other?

Theorem 1.26. *For every $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) \leq \mathcal{P}^s(A)$.*

Proof. See [10, Theorem 5.12]. □

There is an alternative definition for the packing dimension, which does not use measures. This is called the *modified upper box-counting dimension*, and is presented on the following theorem.

Theorem 1.27. *Let $A \subset \mathbb{R}^n$. Then $\dim_{\mathcal{P}} A = \inf\{\sup_i \overline{\dim}_{\mathcal{B}} E_i : A \subset \bigcup_{i=1}^{\infty} E_i\}$.*

Proof. See [10, Theorem 5.11]. □

The *modified lower box-counting dimension* is defined in similar way, but this does not concern us.

After these facts the content of the following theorem is not surprising.

Theorem 1.28. *Let $A \subset \mathbb{R}^n$ be bounded. Then*

$$\dim_{\mathcal{H}} A \leq \dim_{\mathcal{P}} A \leq \overline{\dim}_{\mathcal{B}} A.$$

Proof. The claim follows immediately from Theorem 1.26 and Theorem 1.27. □

It is harder to see that all of the above inequalities can be strict. For such examples see [14].

Theorem 1.29. *Let $c > 0$, $0 \leq \alpha \leq 1$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $|f(x) - f(y)| \leq c|x - y|^\alpha$ for all $x, y \in \mathbb{R}^n$. Then*

$$\dim f(A) \leq \frac{1}{\alpha} \dim A,$$

where \dim can be any of the above definitions for dimension.

Proof. First we show this for the box-counting dimensions. Let $(E_i)_{i=1}^{\infty}$ be a δ -cover of A . Then $(f(E_i))_{i=1}^{\infty}$ is a $c\delta^\alpha$ -cover of $f(A)$, since $d(f(E_i)) = \sup_{x,y \in E_i} |f(x) - f(y)| \leq c \sup_{x,y \in E_i} |x - y|^\alpha = c\delta^\alpha$. Therefore $N_\delta(A) \geq N_{c\delta^\alpha} f(A)$. Now

$$\begin{aligned} \overline{\dim}_{\mathcal{B}} f(A) &= \limsup_{\delta \rightarrow 0} \frac{\log N_{c\delta^\alpha}(f(A))}{-\log c\delta^\alpha} \leq \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(A)}{-\log c\delta^\alpha} = \limsup_{\delta \rightarrow 0} \frac{1}{\alpha} \frac{\log N_\delta(A)}{-\log c - \log \delta} \\ &= \frac{1}{\alpha} \overline{\dim}_{\mathcal{B}}(A). \end{aligned}$$

The same is true also for the lower box-counting dimension.

For Hausdorff dimension, let $s \geq 0$, $(E_i)_{i=1}^{\infty}$ be a δ -cover of A . Again, $(f(E_i))_{i=1}^{\infty}$ is a $c\delta^\alpha$ -cover of $f(A)$. Thus $\sum_{i=1}^{\infty} d(f(E_i))^{s/\alpha} \leq c^{s/\alpha} \sum_{i=1}^{\infty} d(E_i)^s$ so $\mathcal{H}_{c\delta^\alpha}^{s/\alpha}(f(A)) \leq c^{s/\alpha} \mathcal{H}_\delta^s(A)$. Letting $\delta \rightarrow 0$ gives $\mathcal{H}^{s/\alpha}(f(A)) \leq c^{s/\alpha} \mathcal{H}^s(A)$. If $s > \dim_{\mathcal{H}} A$, then $\mathcal{H}^{s/\alpha}(f(A)) \leq c^{s/\alpha} \mathcal{H}^s(A) = 0$. This implies that $\dim_{\mathcal{H}} f(A) \leq \frac{s}{\alpha}$ for all $s > \dim_{\mathcal{H}} A$ so $\dim_{\mathcal{H}} f(A) \leq \frac{1}{\alpha} \dim_{\mathcal{H}} A$.

For the packing dimension, let $A \subset \bigcup_{i=1}^{\infty} E_i$, where E_i is bounded for every i . Using monotonicity and countable stability of packing measure, Theorem 1.28 and the first part of this proof we get

$$\dim_{\text{P}} f(A) \leq \dim_{\text{P}} \bigcup_{i=1}^{\infty} f(E_i) = \sup_i \dim_{\text{P}} f(E_i) \leq \sup_i \overline{\dim}_{\text{B}} f(E_i) \leq \frac{1}{\alpha} \sup_i \overline{\dim}_{\text{B}} E_i.$$

Taking infimum over all coverings with bounded sets we get

$$\dim_{\text{P}} f(A) \leq \frac{1}{\alpha} \inf \left\{ \sup_i \overline{\dim}_{\text{B}} E_i : A \subset \bigcup_{i=1}^{\infty} E_i \right\},$$

which yields the claim with Theorem 1.27. □

A function satisfying the above condition is called an α -Hölder map. Finally we give a powerful tool called the *Frostman's lemma*.

Theorem 1.30. *Let $A \subset \mathbb{R}^n$ be a Borel set. Then $\mathcal{H}^s(A) > 0$ if and only if there exists a Radon measure μ such that, $0 < \mu(\mathbb{R}^n) < \infty$, $\text{spt } \mu$ is compact and contained in A and $\mu(B(x, r)) \leq r^s$ for $x \in \mathbb{R}^n$ and $r > 0$.*

Proof. See [10, Theorem 8.8]. □

Such a measure μ is sometimes called a *Frostman measure*.

1.5 The space of sequences

Now we make ourselves familiar with spaces of infinite words. Suppose that $\mathcal{A} = \{1, \dots, m\}$, a finite set which contains some elements. Then, for any interger $n \geq 1$, \mathcal{A}^n is the set of every sequence of length n with members from \mathcal{A} , that is

$$\mathcal{A}^n = \{i_1 i_2 \dots i_n : i_j \in \mathcal{A} \text{ for every } 1 \leq j \leq n\}.$$

Further, the set $\mathcal{A}^{\mathbb{N}}$ is the set of every sequence of infinite length with members from \mathcal{A} , that is

$$\mathcal{A}^{\mathbb{N}} = \{i_1 i_2 \dots : i_j \in \mathcal{A} \text{ for every } j \in \mathbb{N}\}.$$

We now define some notations concerning sequences. Denote the length of the sequence ω by $|\omega|$. For example, if $u \in \mathcal{A}^n$, then $|u| = n$ and if $\omega \in \mathcal{A}^{\mathbb{N}}$, then $|\omega| = \infty$. The length of the empty sequence is zero. Denote $\omega \wedge \tau$ the unique longest common prefix of sequences ω and τ . For example if $u = 112233$ and $\omega = 111\dots$, then $u \wedge \omega = 11$. The longest common prefix always exists since the empty sequence is always a prefix of any sequence. If $u = u_1 \dots u_n$ and ω is any

sequence, then $u\omega = u_1 \dots u_n \omega_1 \omega_2 \dots$. Note that this notion has meaning only when the first sequence is finite. For any sequence ω with $|\omega| \geq n$, we denote $\omega|n = \omega_1 \dots \omega_n$. Suppose we have some collection of numbers $\{r_1, \dots, r_m\}$ and some sequence ω with members from \mathcal{A} , then

$$r_\omega = \prod_{i=1}^{|\omega|} r_{\omega_i}.$$

We agree that for the empty sequence this is zero. Now we restrict our attention to the set $\mathcal{A}^{\mathbb{N}}$ and denote it by Ω . We equip it with the product topology, while $\{1, \dots, m\}$ carries the discrete topology. So, by definition, we obtain the topology by taking as a base the sets of the form

$$\prod_{i=1}^{\infty} A_i,$$

where A_i is open in \mathcal{A} for every i and $A_i \neq \mathcal{A}$ only for finitely many i . Note that as \mathcal{A} has discrete topology, every subset is open. However, Ω is not discrete as the product is infinite.

Theorem 1.31. *The set Ω is compact.*

Proof. This follows from Tychonoff's theorem. \square

Now we will introduce a metric to the set Ω .

Theorem 1.32. *Suppose that $\mathcal{A} = \{1, \dots, m\}$ and $0 < r_i < 1$ for each $i = 1, \dots, m$. Then $d(\omega, \tau) = r_{\omega \wedge \tau}$ is a metric on Ω .*

Proof. Let $\omega, \tau, \eta \in \Omega$. Clearly $d(\omega, \tau) = 0$ if and only if $\omega = \tau$ and $d(\omega, \tau) = d(\tau, \omega)$, so we only need to check that the triangle inequality holds. Now $|\omega \wedge \tau| \geq \min\{|\omega \wedge \eta|, |\tau \wedge \eta|\}$, so $r_{\omega \wedge \tau} \leq \max\{r_{\omega \wedge \eta}, r_{\tau \wedge \eta}\} \leq r_{\omega \wedge \eta} + r_{\tau \wedge \eta}$. \square

Theorem 1.33. *The topology induced by the metric $d(\omega, \tau) = r_{\omega \wedge \tau}$ is the same as the product topology on Ω .*

Proof. We prove this by showing that the topologies are equally strong. Let $U = \prod_{i=1}^{\infty} A_i$, $U \neq \emptyset$ such that $A_i \neq \mathcal{A}$ only for finitely many i . Let n be the smallest integer such that $A_i = \mathcal{A}$ for every $i \geq n$. Choose some $\tau \in U$ and $r > 0$ small enough to $r < r_{\tau|n}$. Now, if $\eta \in B(\tau, r)$, then $d(\tau, \eta) < r_{\tau|n}$. This implies that $|\tau \wedge \eta| \geq n$ so $\eta \in U$ and $B(\tau, r) \subset U$.

On the other hand let $\tau \in \Omega$ and $r > 0$. Let n be smallest integer such that $r_{\tau|n} < r$. Choose $U = \{\omega \in \Omega : \omega|n = \tau|n\}$. Clearly $U \subset B(\tau, r)$ and U is of the form required. \square

Remark 1.34. A sequence $(\omega^n)_{n=1}^{\infty}$ in Ω converges to ω if and only if $|\omega^n \wedge \omega| \rightarrow \infty$ as $n \rightarrow \infty$.

Also note that Ω is uncountable but separable. Let $W = \bigcup_{i=1}^{\infty} \mathcal{A}^i$, the countable set of all finite sequences with members from \mathcal{A} . Then $\{u111\dots : u \in W\}$ is countable dense set in Ω .

Lemma 1.35. *Suppose that $\mathcal{A} = \{1, \dots, m\}$ and $0 < r_i < 1$ for each $i = 1, \dots, m$. Then, with the metric $d(\omega, \tau) = r_{\omega \wedge \tau}$, $\dim_{\text{H}} \Omega = s$ and $\mathcal{H}^s(\Omega) > 0$, where $\sum_{i=1}^m r_i^s = 1$.*

Proof. As Ω is compact and we want to calculate the infimum of $\sum_{i=1}^{\infty} d(E_i)^s$ over all open coverings $(E_i)_{i=1}^{\infty} \supset \Omega$, it is enough to consider finite coverings by open sets. By definition, an open set $U \subset \Omega$ is of the form $\prod_{i=1}^{\infty} A_i$, where $A_i = \mathcal{A}$ when $i \geq k$, $k \in \mathbb{N}$. Therefore, the set $\{d(\omega, \tau) : \omega, \tau \in U\}$ has only finite amount of elements. Thus, if $U \subset \Omega$ is some open set, then $d(U) = \max_{\omega, \tau \in U} d(\omega, \tau) = r_{\omega_0 \wedge \tau_0}$ for some $\omega_0, \tau_0 \in U$. Denote $S_u := \{\omega \in \Omega : |\omega \wedge u| \geq |u|\}$, where $u \in \bigcup_{i=1}^{\infty} \mathcal{A}^i$. We have $d(S_{\omega_0 \wedge \tau_0}) = r_{\omega_0 \wedge \tau_0}$ and $U \subset S_{\omega_0 \wedge \tau_0}$, so we may also assume that the cover consists of sets of the form S_u . Finally, if S and S' are covering sets of this kind and $S \cap S' \neq \emptyset$, then $S \subset S'$ or $S' \subset S$, so we may also assume that these sets are disjoint. This kind of cover is of the form $\{S_u\}_{u \in I}$, where I is a finite set such that for every $\omega \in \Omega$ there exists a unique $u \in I$ which is a prefix of ω . Then we claim that

$$\sum_{u \in I} d(S_u)^s = \sum_{u \in I} r_u^s = 1, \quad (1.1)$$

whence $\mathcal{H}^s(\Omega) = 1$ and $\dim_{\text{H}} \Omega = s$. To see why (1.1) holds, start with the set $Q_1 = \{1, \dots, m\}$. If $Q_1 = I$ then we are done. If it is not, then construct Q_2 as follows. For every $u \in Q_1$ and $u \notin I$, add the sequences $u1, \dots, um$ to the set Q_2 . For every $u \in Q_1$ and $u \in I$, add $u \in Q_2$. At every step $k \in \mathbb{N}$ we have

$$\sum_{u \in Q_k} r_u^s = 1$$

by the definition of s . After a finite amount of repetitions, namely when $k = \max_{u \in I} |u|$, we have $Q_k = I$. \square

1.6 Self-similar sets

It is hard to study fractals in general. To get a grip of them we have to restrict ourselves to certain type of fractals. Self-similar sets are roughly those kind of fractals which consists of arbitrarily small geometrically similar copies of itself. We will now present some theory concerning self-similar sets. For a more detailed introduction see [5] or [8].

Definition 1.36. A function $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a *similitude* if there exists $0 < r < 1$ such that

$$|S(x) - S(y)| = r|x - y|$$

for $x, y \in \mathbb{R}^n$.

So a similitude can rotate, reflect and zoom out a set with respect to the *similarity ratio* r . To state the description of a self-similar set more rigorously, we want it to be a compact set $\mathcal{K} \subset \mathbb{R}^n$, which satisfies

$$\mathcal{K} = \bigcup_{i=1}^m S_i(\mathcal{K}) \quad (1.2)$$

for some similitudes $\mathcal{S} = \{S_1, \dots, S_m\}$. We may write $S_i(\mathcal{K}) = \mathcal{K}_i$ for simplicity. Some definitions also require this union to be 'nearly disjoint', that is,

$$\mathcal{H}^s(\mathcal{K}_i \cap \mathcal{K}_j) = 0 \quad (1.3)$$

for $i \neq j$ and $\dim_{\mathbb{H}} \mathcal{K} = s$, but we omit this condition. We have a nice definition, but how can we find these sets? To this end some work has to be done.

Definition 1.37. A collection $\{f_1, \dots, f_m\}$ of *contractions* on \mathbb{R}^n , that is, functions $f_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$ which satisfy $|f_i(x) - f_i(y)| < c_i|x - y|$, $0 < c_i < 1$ for every $x, y \in \mathbb{R}^n$ is called an *iterated function system* (i.f.s.).

Clearly similitudes are contractions.

Definition 1.38. Let \mathcal{C} be the set of all non-empty compact subsets of \mathbb{R}^n . For all $A, B \in \mathcal{C}$ set

$$h(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

which is called the *Hausdorff metric*.

Theorem 1.39. *The set \mathcal{C} equipped with the Hausdorff metric forms a complete metric space.*

It is easy to show that this is a metric space. It requires a bit more work to show that the space is complete. For detailed proof see [2]. Next, observe that for any i.f.s. $\{f_1, \dots, f_m\}$, especially with similitudes, the function $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}$, $\mathcal{F}(A) = \bigcup_{i=1}^m f_i(A)$ is a contraction on \mathcal{C} . Then we use the Banach fixed point theorem to conclude that there exists a unique fixed point, the *attractor* of the i.f.s, a compact set on \mathbb{R}^n for which (1.2) holds. The set can be very complex if arbitrary overlaps are permitted in the union. Therefore we usually have some separation conditions for self-similar sets. In the case of the union (1.2) being disjoint, we say that \mathcal{S} satisfies the *strong separation condition*. This condition really is strong, it doesn't allow any overlap. To allow some, but not too much overlap, we have the following condition.

Definition 1.40. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be an i.f.s. of similitudes. \mathcal{S} then satisfies the *open set condition* if there is a non-empty open set O such that

$$\bigcup_{i=1}^m S_i(O) \subset O \quad \text{and} \quad S_i(O) \cap S_j(O) = \emptyset$$

for $i \neq j$.

Strong separation condition clearly implies open set condition, but the inverse is not true. If a self-similar set \mathcal{K} is generated by i.f.s. which satisfies, say, open set condition, then for simplicity we may just say that \mathcal{K} satisfies the open set condition.

Theorem 1.41. *If $\mathcal{S} = \{S_1, \dots, S_m\}$ satisfies the open set condition, then for the self-similar set \mathcal{K} we have $0 < \mathcal{H}^s(\mathcal{K}) \leq \mathcal{P}^s(\mathcal{K}) < \infty$, where $s = \dim_{\mathbb{H}}\mathcal{K} = \dim_{\mathbb{P}}\mathcal{K} = \dim_{\mathbb{B}}\mathcal{K}$. Moreover, s is the unique number for which*

$$\sum_{i=1}^m r_i^s = 1, \quad (1.4)$$

where r_i is the similarity ratio of S_i .

The number s in the above theorem is called the *similarity dimension* of \mathcal{S} . The value s is easily determined if the i.f.s. is *homogenous*, that is, $r_i = r$ for each i . In this case $s = \log m / \log r^{-1}$. Observe that a self-similar set on \mathbb{R} generated by similitudes $S_i(x) = rx + a_i$, $i = 1, \dots, m$ is the set

$$\mathcal{K} = \left\{ \sum_{n=0}^{\infty} r^n c_n : c_n \in \{a_1, \dots, a_m\} \right\}.$$

This is the case since

$$S_i \left(\left\{ \sum_{n=0}^{\infty} r^n c_n : c_n \in \{a_1, \dots, a_m\} \right\} \right) = \left\{ a_i + \sum_{n=1}^{\infty} r^n c_n : c_n \in \{a_1, \dots, a_m\} \right\}$$

and thus it is the unique set satisfying condition (1.2). Now the set

$$\mathcal{K} - \mathcal{K} = \left\{ \sum_{n=0}^{\infty} r^n c_n : c_n \in \Gamma \right\},$$

where $\Gamma = \{a_i - a_j : i, j \leq m\}$, is also self-similar. The set $\mathcal{K} - \mathcal{K}$ is generated by a homogenous i.f.s. which contains $\#\{a_i - a_j : i, j \leq m\}$ similitudes.

Self-similar sets satisfying the open set condition are quite well understood, but with arbitrary overlaps permitted they are still quite mysterious. It is known that even for a general self-similar set the Hausdorff, packing and box-counting dimensions agree [4], but much less is known about the Hausdorff and packing measures of these sets. This is a problem which motivates this thesis.

1.7 Orthogonal projections of fractal sets

This section consists of some known results concerning projections of fractals and some related questions. Theorem 1.43 is known as Marstrand's theorem and can be generalized to higher dimensions naturally [5, Theorem 6.1]. The projection of a planar self-similar set results in a new self-similar set on a line, which may have significant overlaps. This connection, however, allows us to make deductions about the resulting set based on the properties of the original set.

Definition 1.42. Denote by L_θ the line $y \cos \theta = x \sin \theta$. Let $\text{proj}_\theta : \mathbb{R}^2 \rightarrow L_\theta$, $\theta \in [0, \pi)$ be the orthogonal projection on L_θ , that is, $\text{proj}_\theta(x, y) = ((x, y) \cdot (\cos \theta, \sin \theta))(\cos \theta, \sin \theta)$, where \cdot is the inner product on \mathbb{R}^2 .

Theorem 1.43. *Let $A \subset \mathbb{R}^2$ be a Borel set. Then*

$$\dim_{\text{H}}(\text{proj}_{\theta}A) = \min\{\dim_{\text{H}}A, 1\}$$

for almost all $\theta \in [0, \pi)$.

So the dimension is maintained in almost every case, unless the set projected has a dimension greater than that of the line. This raises a question what happens to the measure of the set? In case of a self-similar set \mathcal{K} generated by i.f.s satisfying the open set condition we know that $0 < \mathcal{H}^s(\mathcal{K}) < \infty$ and $\dim_{\text{H}}(\text{proj}_{\theta}\mathcal{K}) = s$ with the similarity dimension s and a typical θ , but how does $\mathcal{H}^s(\text{proj}_{\theta}\mathcal{K})$ behave?

From now on we make a restriction to consider only iterated function systems that consists of similitudes of the form $S(x) = rx + a$, thus only a translation is allowed in addition to the compulsory contraction. So the self-similar sets we work with hereafter are usually generated by an i.f.s. $\{S_i\}_{i=1}^m$, where

$$S_i(x) = r_i x + a_i \tag{1.5}$$

for some $r_i \in (0, 1)$ and $a_i \in \mathbb{R}^n$, where n is one or two.

Let $\tilde{\mathcal{K}} \subset \mathbb{R}^2$ be a self-similar set generated by i.f.s. $\{\tilde{S}_i\}_{i=1}^m$, where $\tilde{S}_i: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\tilde{S}_i(x) = r_i x + c_i$, $0 < r_i < 1$ and $c_i = (a_i, b_i) \in \mathbb{R}^2$ for $i = 1, \dots, m$. Then $\text{proj}_{\theta} \circ \tilde{S}_i(x, y) = r_i \cdot \text{proj}_{\theta}(x, y) + \text{proj}_{\theta}(a_i, b_i)$ is a similitude when restricted to the set L_{θ} , so the set $\{\text{proj}_{\theta} \circ \tilde{S}_i\}_{i=1}^m$ is an i.f.s. on L_{θ} . For $A \subset \mathbb{R}^2$, let $P_{\theta}(A) = \{L_{\theta+\frac{\pi}{2}} + (x, y) : (x, y) \in A\}$. Then $\text{proj}_{\theta}A = L_{\theta} \cap P_{\theta}(A)$. Sets $\tilde{S}_i(P_{\theta}(x, y))$ and $P_{\theta}(\tilde{S}_i(x, y))$ are both lines perpendicular to L_{θ} and they share the point $\tilde{S}_i(x, y)$. Hence $\tilde{S}_i(P_{\theta}(x, y)) = P_{\theta}(\tilde{S}_i(x, y))$, and $\tilde{S}_i(P_{\theta}(A)) = P_{\theta}(\tilde{S}_i(A))$. Therefore

$$\begin{aligned} \text{proj}_{\theta}\tilde{\mathcal{K}} &= L_{\theta} \cap P_{\theta}(\tilde{\mathcal{K}}) = L_{\theta} \cap \bigcup_{i=1}^m P_{\theta}(\tilde{S}_i(\tilde{\mathcal{K}})) = \bigcup_{i=1}^m L_{\theta} \cap P_{\theta}(\tilde{S}_i(\tilde{\mathcal{K}})) = \bigcup_{i=1}^m L_{\theta} \cap \tilde{S}_i(P_{\theta}(\tilde{\mathcal{K}})) \\ &= \bigcup_{i=1}^m L_{\theta} \cap \tilde{S}_i(P_{\theta}(\text{proj}_{\theta}\tilde{\mathcal{K}})) = \bigcup_{i=1}^m L_{\theta} \cap P_{\theta}(\tilde{S}_i(\text{proj}_{\theta}\tilde{\mathcal{K}})) = \bigcup_{i=1}^m \text{proj}_{\theta}(\tilde{S}_i(\text{proj}_{\theta}\tilde{\mathcal{K}})). \end{aligned}$$

Thus $\text{proj}_{\theta}\tilde{\mathcal{K}}$ is the self-similar set generated by the i.f.s. $\{\text{proj}_{\theta} \circ \tilde{S}_i\}_{i=1}^m$. To examine $\text{proj}_{\theta}\tilde{\mathcal{K}}$ we identify L_{θ} with \mathbb{R} in natural way by rotating the set to zero angle. Now we have a family of iterated function systems $\{\text{proj}_{\theta} \circ \tilde{S}_i\}_{i=1}^m$ which generates a self-similar set on \mathbb{R} for every $\theta \in [0, \pi)$. There is also a different way to represent this family, one which we use more often. For the parameter we do not take the angle θ , but $\lambda = \tan \theta$, with the similitudes $S_i^{\lambda}: \mathbb{R} \rightarrow \mathbb{R}$, $S_i^{\lambda}(x) = r_i x + (a_i + b_i \lambda)$. The attractor of $\{S_i^{\lambda}\}_{i=1}^m$ is not technically the same as $\text{proj}_{\theta}\tilde{\mathcal{K}}$ but affine equivalent with it. It can be normalized by scaling as $\text{proj}_{\theta}\tilde{\mathcal{K}} = \frac{1}{\sqrt{1+\lambda^2}}\mathcal{K}^{\lambda}$. Now we reach every case by $\lambda \in \mathbb{R}$ as we would with θ , with the exception of a projection on the y -axis, which would be the case ' $\lambda = \infty$ '. However, one point is not a problem, especially since the results are of the type 'almost every'.

Example 1.44. The s -dimensional Sierpinski gasket \mathcal{G}^r of 'unit size' is generated by similitudes $\tilde{S}_1(x, y) = r(x, y)$, $\tilde{S}_2(x, y) = r(x, y) + (1-r, 0)$ and $\tilde{S}_3(x, y) = r(x, y) + (0, 1-r)$, where $0 < r \leq \frac{1}{2}$ and $s = \log 3 / \log r^{-1}$. By above, the orthogonal projections of \mathcal{G}^r on L_θ is up to scaling the set generated by similitudes $S_1^\lambda(x) = rx$, $S_2^\lambda(x) = rx + 1 - r$ and $S_3^\lambda(x) = rx + (1-r)\lambda$, where $\lambda = \tan \theta$. Moreover, this set is

$$\left\{ \sum_{n=0}^{\infty} r^n c_n : c_n \in C \right\}, \quad (1.6)$$

where $C = \{0, 1-r, (1-r)\lambda\}$. As affine transformations won't matter much when studying the properties of a self-similar set we can adopt a simpler representation for this set. Dividing by $1-r$ we can assume that $C = \{0, 1, \lambda\}$. Further, observe that it is enough to consider $\lambda \geq 2$, as we can scale C so that the distance between the two points closest to each other is one and then translate and reflect accordingly, unless λ is 0 or 1. Thus essentially every projection of \mathcal{G}^r is achieved with the set (1.6) by assuming for example that $C = \{0, 1, \lambda\}$ and $\lambda \geq 2$.

Chapter 2

Self-similar sets on a line

2.1 One-parameter families of iterated function systems

The goal is to study orthogonal projections of a planar self-similar sets onto lines of different angles. We restrict our attention only to self-similar sets generated by similitudes of the form (1.5), that is, similitudes which do not rotate or reflect. As discussed above, the projection of this kind of self-similar set is also self-similar set on the line, which depends on the original set by one parameter. The parameter can be taken as the direction of the line the set is being projected onto. The problem is that the resulting set need not satisfy any separation conditions. We now consider general one-parameter families of iterated function systems $\{S_1^\lambda, \dots, S_m^\lambda\}_{\lambda \in J}$ where

$$S_i^\lambda(x) = r_i(\lambda)x + a_i(\lambda) \quad (2.1)$$

for $x \in \mathbb{R}$ and $J \subset \mathbb{R}$ is a compact interval. For our purposes we can assume that $a_i(\lambda)$ and $r_i(\lambda)$ belong to $C^1(J)$, the space of continuously differentiable functions. Because the similitudes are contractive, we also have

$$0 < \beta \leq r_i(\lambda) \leq \rho < 1 \quad (2.2)$$

for all $i \leq m$ and $\lambda \in J$. Let \mathcal{K}^λ be the self-similar set corresponding to λ and $s(\lambda)$ the similarity dimension of the i.f.s. Denote $\mathcal{A} = \{1, \dots, m\}$ and $\Omega = \mathcal{A}^{\mathbb{N}}$, the space of all infinite words from \mathcal{A} . For $u \in \mathcal{A}^n$ we write $S_u^\lambda = S_{u_1}^\lambda \circ \dots \circ S_{u_n}^\lambda$.

Definition 2.1. The map $\Pi(\lambda, \cdot): \Omega \rightarrow \mathcal{K}^\lambda$ defined by

$$\Pi(\lambda, \omega) = \lim_{n \rightarrow \infty} S_{\omega_1 \dots \omega_n}^\lambda(0) = \sum_{n=1}^{\infty} r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda) \quad (2.3)$$

will be called the *natural projection map*.

Next we present some properties of the natural projection map.

Lemma 2.2. *With the above definitions*

- (i) $\Pi(\cdot, \omega) \in C^1(J)$ for all $\omega \in \Omega$.
- (ii) With Ω equipped with the product topology, the map $\Omega \rightarrow C^1(J): \omega \mapsto \Pi(\cdot, \omega)$ is continuous.

Proof. (i) Let $\omega \in \Omega$. From equation (2.2) we have

$$\sup\{r_i(\lambda) : \lambda \in J, i = 1, \dots, m\} \leq \rho \quad (2.4)$$

and since J is compact and a_i is continuous for all $i \in \mathbb{N}$, we can choose

$$a := \max_i \|a_i\|_{C(J)}.$$

Then

$$\begin{aligned} \sup_{\lambda \in J} \left| \Pi(\lambda, \omega) - \sum_{n=1}^N r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda) \right| &= \sup_{\lambda \in J} \left| \sum_{n=N+1}^{\infty} r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda) \right| \\ &\leq \sup_{\lambda \in J} \sum_{n=N+1}^{\infty} |r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda)| \leq \sum_{n=N+1}^{\infty} \rho^{n-1} a \\ &= a \rho^N \sum_{n=0}^{\infty} \rho^n = \frac{a \rho^N}{1 - \rho} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

This shows that $S_{\omega_1 \dots \omega_N}^\lambda(0)$, as function of λ , converges to $\Pi(\lambda, \omega)$ (even uniformly) as $N \rightarrow \infty$. For $\Pi(\cdot, \omega)$ to be of class $C^1(J)$, we show that $\frac{d}{d\lambda} S_{\omega_1 \dots \omega_N}^\lambda(0)$, as a function of λ , converges uniformly. First choose

$$M_1 := \max_i \|r'_i\|_{C(J)} \quad \text{and} \quad M_2 = \max_i \|a'_i\|_{C(J)}.$$

Note that these exist since the derivatives are continuous and J is compact. By calculating

$$\begin{aligned} \left| \frac{d}{d\lambda} (r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda)) \right| &= |r'_{\omega_1}(\lambda) \cdot r_{\omega_2}(\lambda) \cdots r_{\omega_{n-1}}(\lambda) \cdot a_{\omega_n}(\lambda) \\ &\quad + r_{\omega_1}(\lambda) \cdot r'_{\omega_2}(\lambda) \cdots r_{\omega_{n-1}}(\lambda) \cdot a_{\omega_n}(\lambda) \\ &\quad \vdots \\ &\quad + r_{\omega_1}(\lambda) \cdot r_{\omega_2}(\lambda) \cdots r_{\omega_{n-1}}(\lambda) \cdot a'_{\omega_n}(\lambda)| \end{aligned}$$

and using above approximations we get

$$\begin{aligned} \left| \frac{d}{d\lambda} (r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda)) \right| &\leq (n-1)M_1 a \rho^{n-2} + \rho^{n-1} M_2 \leq (n-1)M_1 a \rho^{n-2} + \rho^{n-2} M_2 \\ &= ((n-1)M_1 a + M_2) \rho^{n-2} \leq c(n-1) \rho^{n-2}, \end{aligned}$$

where $c > 0$ and $n > 1$. Now

$$\begin{aligned} \sup_{\lambda \in J} \left| \sum_{n=N}^{\infty} \frac{d}{d\lambda} (r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda)) \right| &\leq \sum_{n=N}^{\infty} \sup_{\lambda \in J} \left| \frac{d}{d\lambda} (r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda)) \right| \\ &\leq c \sum_{n=N}^{\infty} (n-1) \rho^{n-2} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

since $\rho < 1$. Hence $\Pi(\cdot, \omega) \in C^1(J)$.

- (ii) Let $\omega \in \Omega$ and a , M_1 and M_2 be as in (i). Let $\omega^N \rightarrow \omega$ as $N \rightarrow \infty$ and let $k_N = |\omega \wedge \omega^N|$. Note that by Remark 1.34 $k_N \rightarrow \infty$ as $N \rightarrow \infty$. Now

$$\begin{aligned} \sup_{\lambda \in J} |\Pi(\lambda, \omega) - \Pi(\lambda, \omega^N)| &= \sup_{\lambda \in J} \left| \sum_{n=k_N}^{\infty} r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda) - \sum_{n=k_N}^{\infty} r_{\omega_1^N \dots \omega_{n-1}^N}(\lambda) a_{\omega_n^N}(\lambda) \right| \\ &\leq 2 \sum_{n=k_N}^{\infty} \rho^{n-1} a = 2a\rho^{k_N} \frac{1}{1-\rho} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Also, as in (i),

$$\begin{aligned} \sup_{\lambda \in J} |\Pi'(\lambda, \omega) - \Pi'(\lambda, \omega^N)| &= \sup_{\lambda \in J} \left| \sum_{n=k_N}^{\infty} \frac{d}{d\lambda} (r_{\omega_1 \dots \omega_{n-1}}(\lambda) a_{\omega_n}(\lambda)) - \sum_{n=k_N}^{\infty} \frac{d}{d\lambda} (r_{\omega_1^N \dots \omega_{n-1}^N}(\lambda) a_{\omega_n^N}(\lambda)) \right| \\ &\leq 2 \sum_{n=k_N}^{\infty} (n-1) M_1 \rho^{n-2} a + \rho^{n-1} M_2 \\ &= 2M_1 a \sum_{n=k_N}^{\infty} (n-1) \rho^{n-2} + 2M_2 \sum_{n=k_N}^{\infty} \rho^{n-1} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

□

Denote $f_{\omega, \tau}(\lambda) = \Pi(\lambda, \omega) - \Pi(\lambda, \tau)$.

Remark 2.3. Lemma 2.2 implies the following facts.

- (i) The set $\{f_{\omega, \tau} : \omega, \tau \in Y\}$ is compact in $C^1(J)$ as long as $Y \subset \Omega$ is compact.
- (ii) The function $(\omega, \tau, \lambda) \mapsto f_{\omega, \tau}(\lambda) : \Omega \times \Omega \times J \rightarrow \mathbb{R}$ is continuous with compact domain, so it also is uniformly continuous function.

Next we give two useful definitions.

Definition 2.4. The *transversality condition* holds on J if for any $\omega, \tau \in \Omega$

$$\text{if there is } \lambda \in J \text{ such that } f_{\omega, \tau}(\lambda) = f'_{\omega, \tau}(\lambda) = 0 \text{ then } f_{\omega, \tau} \equiv 0.$$

Definition 2.5. The set of *intersection parameters* is

$$\mathcal{IP} = \{\lambda \in J : f_{\omega, \tau}(\lambda) = 0 \text{ for some } \omega, \tau \in \Omega \text{ but } f_{\omega, \tau} \not\equiv 0\}.$$

We are to study projections of a planar self-similar set \mathcal{K} . We want to assume that \mathcal{K} satisfies strong separation condition and that it is not on a line. The analogues of these assumptions in this general setting are

$$\text{for any } \lambda \in J, \mathcal{K}^\lambda \text{ is not a singleton} \quad (2.5)$$

and

$$f_{\omega, \tau} \equiv 0 \text{ implies } \omega = \tau. \quad (2.6)$$

If \mathcal{K} is on a line, then the projection of that set onto a line of a certain direction is a singleton, hence the condition (2.5). If $x, y \in \mathcal{K}$ and $|\text{proj}_\theta(x) - \text{proj}_\theta(y)| = 0$ for every θ , then $x = y$. If \mathcal{K} satisfies strong separation condition, then the point x has a unique representation with a sequence, hence the condition 2.6.

Next we give some consequences of the latter condition (2.6).

Lemma 2.6. *Suppose that (2.6) holds. Then*

- (i) $\mathcal{IP} = \{\lambda \in J : \mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset \text{ for some } i \neq j\}$,
 - (ii) \mathcal{IP} is compact,
 - (iii) If $\lambda \notin \mathcal{IP}$, then $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) > 0$,
 - (iv) Suppose that transversality holds on J . There exists $\delta > 0$ such that for all $\omega, \tau \in \Omega$ with $\omega_1 \neq \tau_1$,
- $$\text{if } |f_{\omega, \tau}(\lambda)| < \delta, \quad \text{then } |f'_{\omega, \tau}(\lambda)| > \delta. \quad (2.7)$$

Proof.

- (i) Let $\lambda \in \mathcal{IP}$. Then there are $\omega \neq \tau$ such that $f_{\omega, \tau}(\lambda) = 0$. Choose $\alpha \in \mathcal{A}^n$ such that $\alpha\omega' = \omega$, $\alpha\tau' = \tau$ and $\omega'_1 \neq \tau'_1$. (If $\omega_1 \neq \tau_1$, then α is not needed.) Now

$$0 = f_{\omega, \tau}(\lambda) = \Pi(\lambda, \alpha\omega') - \Pi(\lambda, \alpha\tau')$$

which is equivalent to $S_\alpha(\Pi(\lambda, \omega')) = S_\alpha(\Pi(\lambda, \tau'))$. Because S_α is a bijection, we have $\Pi(\lambda, \omega') = \Pi(\lambda, \tau')$. This clearly states that $\mathcal{K}_{\omega'_1}^\lambda \cap \mathcal{K}_{\tau'_1}^\lambda \neq \emptyset$, hence $\mathcal{IP} \subset \{\lambda \in J : \mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset \text{ for some } i \neq j\}$.

Suppose now that $\mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset$ and $i \neq j$. Then there exists $\omega, \tau \in \Omega$ such that $\omega_1 = j \neq i = \tau_1$ and $\Pi(\lambda, \omega) = \Pi(\lambda, \tau)$ so $f_{\omega, \tau}(\lambda) = 0$. By (2.6) and $\omega \neq \tau$ we have $f_{\omega, \tau} \neq 0$. So $\{\lambda \in J : \mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda \neq \emptyset \text{ for some } i \neq j\} \subset \mathcal{IP}$.

- (ii) Let $\lambda_0 \in \overline{\mathcal{IP}}$. This, with (i), means that there is a sequence $\lambda_n \xrightarrow{n \rightarrow \infty} \lambda_0$ such that for every $n \in \mathbb{N}$ there are $\omega^n, \tau^n \in \Omega$ with $f_{\omega^n, \tau^n}(\lambda_n) = 0$ and $\omega_1^n \neq \tau_1^n$. Using compactness, we can form two subsequences ω^{n_i} and τ^{n_i} which converges to ω^0 and τ^0 respectively. Note also that $\omega_1^0 \neq \tau_1^0$. By Lemma 2.2 we have

$$f_{\omega^0, \tau^0}(\lambda_0) = \lim_{i \rightarrow \infty} f_{\omega^{n_i}, \tau^{n_i}}(\lambda_{n_i}) = 0,$$

but as in (i), $f_{\omega^0, \tau^0} \neq 0$ since $\omega_1^0 \neq \tau_1^0$. This means that $\lambda_0 \in \mathcal{IP}$, so as a closed subset of a bounded set, the set is compact.

- (iii) If $\lambda \notin \mathcal{IP}$, then by (i) we have $\mathcal{K}_i^\lambda \cap \mathcal{K}_j^\lambda = \emptyset$ for every $i \neq j$. So $\{S_i^\lambda\}_{i=1}^m$ meets the strong separation condition and the claim follows by Theorem 1.41.

(iv) If (2.7) does not hold, we have $\omega^n, \tau^n \in \Omega$, $\omega_1^n \neq \tau_1^n$ and $\lambda_n \in J$ such that

$$|f_{\omega^n, \tau^n}(\lambda_n)| < \frac{1}{n} \quad \text{and} \quad |f'_{\omega^n, \tau^n}(\lambda_n)| \leq \frac{1}{n}$$

for every $n \in \mathbb{N}$. As in (ii), by choosing suitable subsequences we can get an index sequence $(n_i)_{i=1}^\infty$ such that $\omega^{n_i} \rightarrow \omega^0$, $\tau^{n_i} \rightarrow \tau^0$, $\omega_1^0 \neq \tau_1^0$ and also $\lambda_{n_i} \rightarrow \lambda_0$ as $i \rightarrow \infty$, since J is compact too. When $i \rightarrow \infty$ we get

$$f_{\omega^0, \tau^0}(\lambda_0) = 0 \quad \text{and} \quad f'_{\omega^0, \tau^0}(\lambda_0) = 0$$

by Lemma 2.2. This is a contradiction with Definition 2.4, since again by (2.6) we have $f_{\omega, \tau} \not\equiv 0$.

□

2.2 Families of projections

Let $\mathcal{K} = \bigcup_{i=1}^m (r_i \mathcal{K} + b_i) \subset \mathbb{R}^2$ be a self-similar set. Recall Definition 1.42. As mentioned in Section 1.7, $\text{proj}_\theta \mathcal{K}$ is also a self-similar set on L_θ . This set is the attractor of $\{S_1^\theta, \dots, S_m^\theta\}$, where $S_i^\theta(x) = r_i x + \text{proj}_\theta b_i$. We now have a one-parameter family of i.f.s.

Lemma 2.7. *The one-parameter family of i.f.s. $\{S_1^\theta, \dots, S_m^\theta\}_{\theta \in [0, \pi)}$ satisfies the transversality condition Definition 2.4.*

Proof. Note first that if $(x, y) \in \mathbb{R}^2$, then, with \cdot being the inner product, $(x, y) \cdot (\cos \theta, \sin \theta) = x \cos \theta + y \sin \theta$ is the projection of (x, y) on the line L_θ . Now

$$\begin{aligned} \frac{d}{d\theta}(x \cos \theta + y \sin \theta) &= -x \sin \theta + y \cos \theta = x \cos(\theta + \frac{\pi}{2}) + y \sin(\theta + \frac{\pi}{2}) \\ &= (x, y) \cdot (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) \end{aligned}$$

so the derivative is the projection of (x, y) on the line perpendicular to L_θ . Let $\tilde{\Pi}: \Omega \rightarrow \mathcal{K}$ be the natural projection map. Then

$$\Pi(\theta, \omega) = \text{proj}_\theta \circ \tilde{\Pi}(\omega) = \tilde{\Pi}(\omega) \cdot (\cos \theta, \sin \theta),$$

so simply by Pythagoras' theorem we have

$$|\Pi(\theta, \omega) - \Pi(\theta, \tau)|^2 + \left| \frac{d}{d\theta}(\Pi(\theta, \omega) - \Pi(\theta, \tau)) \right|^2 = |\tilde{\Pi}(\omega) - \tilde{\Pi}(\tau)|^2.$$

Now we see that $f_{\omega, \tau}(\lambda) = 0$ and $f'_{\omega, \tau}(\lambda) = 0$ if and only if $\tilde{\Pi}(\omega) = \tilde{\Pi}(\tau)$, hence the claim. □

Remark 2.8. For a family of projections $\theta \in \mathcal{IP}$ if and only if $\text{proj}_\theta: \mathcal{K} \rightarrow \mathbb{R}$ is not an injection.

Chapter 3

Zero Hausdorff measure

This chapter concentrates on the Hausdorff measures of the projections of planar self-similar sets. From this perspective the main result is Theorem 3.6. This follows easily from more general result Theorem 3.5, for which we need the following preliminary theorems.

Let $\mathcal{A} = \{1, \dots, m\}$ as in Chapter 2 and recall the notations from Section 1.5 regarding sequences in \mathcal{A} . Denote $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}^n$. Also, for $S: \mathbb{R}^d \rightarrow \mathbb{R}^d$ let $\|S\| = \sup\{|Sx| : |x| \leq 1\}$.

Theorem 3.1 (Bandt and Graf). *Let \mathcal{K} be the self-similar set generated by an i.f.s. $\{S_i\}_{i \in \mathcal{A}}$ of the form (2.1), and let s be the similarity dimension. Then $\mathcal{H}^s(\mathcal{K}) = 0$ if and only if for any $\varepsilon > 0$ there exists $u, v \in \mathcal{A}^*$, $u \neq v$ such that*

$$\|S_u^{-1} \circ S_v - \text{Id}\| < \varepsilon.$$

Proof. See [1]. □

The similitudes we consider are of the form $S_i^\lambda(x) = r_i(\lambda)x + a_i(\lambda)$, so

$$\begin{aligned} ((S_u^\lambda)^{-1} \circ S_v^\lambda - \text{Id})(x) &= \left(\frac{r_v(\lambda)}{r_u(\lambda)} - 1\right)x + \frac{S_v^\lambda(0) - S_u^\lambda(0)}{r_u(\lambda)} \\ &= \left(\frac{r_v(\lambda)}{r_u(\lambda)} - 1\right)(x - x_0(\lambda)) + \frac{S_v^\lambda(x_0(\lambda)) - S_u^\lambda(x_0(\lambda))}{r_u(\lambda)}. \end{aligned}$$

Here $x_0(\lambda)$ is arbitrary and is added for conveniency. We choose $x_0(\lambda) = \Pi(\lambda, \bar{1})$, where $\bar{1} = 111\dots \in \Omega$, so we have $S_v^\lambda(x_0(\lambda)) = \Pi(\lambda, v\bar{1})$. With the upcoming proof in mind, we make the following definition in the spirit of the previous theorem.

Definition 3.2. Let \mathcal{V}_ε be the set of $\lambda \in J$ such that there exists $u, v \in \mathcal{A}^*$, $u \neq v$ such that

$$e^{-\varepsilon} < \frac{r_u(\lambda)}{r_v(\lambda)} < e^\varepsilon \tag{3.1}$$

and

$$|f_{v\bar{1}, u\bar{1}}(\lambda)| = |\Pi(\lambda, v\bar{1}) - \Pi(\lambda, u\bar{1})| < \varepsilon r_u(\lambda). \tag{3.2}$$

Lemma 3.3. *For any $\gamma > 0$, $\beta \in (0, 1)$, and $\lambda_0 \in J$ there exists $N \in \mathbb{N}$ with the following property: For any $s, t \in \mathcal{A}^*$, with $r_s(\lambda_0)/r_t(\lambda_0) \in [\beta, \beta^{-1}]$, there exists $u, v \in \mathcal{A}^*$ such that s, t are their respective prefixes, $|u| - |s| \leq N$, $|v| - |t| \leq N$, and*

$$e^{-\gamma} < \frac{r_u(\lambda_0)}{r_v(\lambda_0)} < e^\gamma \quad (3.3)$$

Proof. Let $\gamma > 0$, $\beta \in (0, 1)$ and $\lambda_0 \in J$ be fixed. For any finite sequence $s = s_1, s_2, \dots, s_n \in \mathcal{A}^*$ we denote $\sum_{i \in s} f(i) = \sum_{i=s_1}^{s_n} f(i)$. Suppose that $s, t \in \mathcal{A}^*$ such that $r_s(\lambda_0)/r_t(\lambda_0) \in [\beta, \beta^{-1}]$. We want to find $p, q \in \mathcal{A}^*$ such that

$$\frac{r_s(\lambda_0)}{r_t(\lambda_0)} \cdot \frac{r_p(\lambda_0)}{r_q(\lambda_0)} \in (e^{-\gamma}, e^\gamma) \quad (3.4)$$

and $|p|, |q| \leq N$, where $N \in \mathbb{N}$ is a uniform bound, that is, it depends only on γ, β and λ_0 . In other words

$$\sum_{i \in s} \log r_i(\lambda_0) - \sum_{i \in t} \log r_i(\lambda_0) \in [\log \beta, -\log \beta], \quad (3.5)$$

and we want to find $p, q \in \mathcal{A}^*$ as above such that

$$\sum_{i \in s} \log r_i(\lambda_0) - \sum_{i \in t} \log r_i(\lambda_0) + \sum_{i \in p} \log r_i(\lambda_0) - \sum_{i \in q} \log r_i(\lambda_0) \in (-\gamma, \gamma).$$

Let $a_i = \log r_i(\lambda_0)$. Suppose first that $\frac{a_i}{a_j} \in \mathbb{Q}$ for all $i, j \in \mathcal{A}$. Therefore for any $i \in \mathcal{A}$ we can write $a_i = q_i a_1$, where $q_i \in \mathbb{Q}$. The set of all possible values in (3.5) is

$$L := \left\{ \sum_{i \in s} a_i - \sum_{i \in t} a_i : s, t \in \mathcal{A}^* \right\} \cap [\log \beta, -\log \beta]. \quad (3.6)$$

Scale this set with the lowest common denominator of the rationals $\{q_i\}_{i=1}^m$ and with the inverse of a_1 to achieve the set

$$\left\{ \sum_{i \in s} k_i - \sum_{i \in t} k_i : s, t \in \mathcal{A}^* \right\} \cap [-c, c],$$

where $k_i \in \mathbb{N}$ for every $i \in \mathcal{A}$ and $c > 0$. This set contains only integers from a bounded interval. Therefore L is finite. For every point $x \in L$ choose some $s_x, t_x \in \mathcal{A}^*$ such that $x = \sum_{i \in s_x} a_i - \sum_{i \in t_x} a_i$. Since L is finite, there is $N \in \mathbb{N}$ such that $\max\{|s_x|, |t_x|\} \leq N$ for every $x \in L$. Thus for any $r_s(\lambda_0)/r_t(\lambda_0) \in [\beta, \beta^{-1}]$ we can choose $p = t_x$ and $q = s_x$. Then

$$\sum_{i \in s} a_i + \sum_{i \in t_x} a_i - \sum_{i \in t} a_i - \sum_{i \in s_x} a_i = 0,$$

which means that

$$\log \frac{r_{st_x}(\lambda_0)}{r_{ts_x}(\lambda_0)} = \log \left(\frac{r_s(\lambda_0)}{r_t(\lambda_0)} \cdot \frac{r_{t_x}(\lambda_0)}{r_{s_x}(\lambda_0)} \right) = 0$$

and (3.4) is satisfied. Next we deal with the opposite case. Suppose that there are some $i, j \in \mathcal{A}$ such that $\frac{a_i}{a_j} \notin \mathbb{Q}$. Then the set

$$\{x : x = na_i \pmod{a_j} \text{ for some } n \in \mathbb{N}\}$$

is dense in $[0, a_j]$. For this see for example [3, page 4]. By this we can choose $k, l \in \mathbb{N}$ such that $ka_i - la_j = \delta < 2\gamma$. Let $p = iii \dots i$ and $q = jjj \dots j$, with $|p| = k$ and $|q| = l$. Then $\delta = \sum_{i \in p} a_i - \sum_{i \in q} a_i$. Now by choosing a suitable $b \in \mathbb{Z}$, $|b| \leq \frac{2 \log \beta}{\delta}$ we have

$$\sum_{i \in s} a_i - \sum_{i \in t} a_i + b\delta \in (-\gamma, \gamma).$$

Equivalently

$$\frac{r_s(\lambda_0)}{r_t(\lambda_0)} \cdot \left(\frac{r_p(\lambda_0)}{r_q(\lambda_0)} \right)^b \in (e^{-\gamma}, e^\gamma).$$

Thus for any $r_s(\lambda_0)/r_t(\lambda_0) \in [\beta, \beta^{-1}]$ we can extend s and t by not more than $\frac{2 \log \beta}{\delta} \max\{k, l\}$ elements to satisfy (3.3). \square

Lemma 3.4. *For any $u \in \mathcal{A}^*$ and $\lambda_1, \lambda_2 \in J$,*

$$\frac{r_u(\lambda_2)}{r_u(\lambda_1)} \leq e^{\frac{L}{\beta} |u| |\lambda_2 - \lambda_1|}$$

where β is from (2.2) and $L := \max_{i \in \mathcal{A}} \|r'_i\|_{C(J)}$.

Proof. By applying the mean value theorem we get

$$\begin{aligned} \log \frac{r_i(\lambda_2)}{r_i(\lambda_1)} &= \log r_i(\lambda_2) - \log r_i(\lambda_1) \leq |\log r_i(\lambda_2) - \log r_i(\lambda_1)| \\ &\leq \frac{1}{\beta} |r_i(\lambda_2) - r_i(\lambda_1)| \leq \frac{L}{\beta} |\lambda_2 - \lambda_1| \end{aligned}$$

for any $i = 1, \dots, m$ and the claim follows. \square

Denote $C_1 = \frac{1}{1-\rho} \max_{i \in \mathcal{A}} \|a_i\|_{C(J)}$, where ρ is as in (2.2). Observe that

$$|f_{\omega, \tau}(\lambda)| \leq |\Pi(\lambda, \omega)| + |\Pi(\lambda, \tau)| \leq 2C_1,$$

that is, two points of \mathcal{K}^λ cannot be farther away from each other than this. However, this estimate can be easily tightened by every common step they share in their characterization. In other words,

$$|f_{\omega, \tau}(\lambda)| \leq 2C_1 r_{\omega \wedge \tau}(\lambda). \quad (3.7)$$

Recall that a set which is a countable intersections of opens sets is called a G_δ set.

Theorem 3.5. *Suppose that the one-parameter family of iterated function systems $\{S_1^\lambda, \dots, S_m^\lambda\}_{\lambda \in J}$ satisfies (2.1), (2.2) and Definition 2.4. Then*

- (i) $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0$ for Lebesgue almost every $\lambda \in \mathcal{IP}$.
- (ii) Suppose, in addition that (2.5) and (2.6) hold. Then \mathcal{IP} is a compact perfect set, and the set $\{\lambda \in \mathcal{IP} : \mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0\}$ is a dense G_δ set in \mathcal{IP} .

Note that (i) has content only when the set $\mathcal{I}P$ has positive Lebesgue measure. Checking this is discussed at the end of this chapter.

Proof. (i) Theorem 3.1 states that $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0$ if and only if $\lambda \in \mathcal{V}_\varepsilon$ for every $\varepsilon > 0$. To show that $\mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0$ for Lebesgue almost every $\lambda \in \mathcal{I}P$, we show that $\mathcal{L}(\mathcal{I}P \setminus \mathcal{V}_\varepsilon) = 0$ for every $\varepsilon > 0$. For this we check that $\mathcal{I}P \setminus \mathcal{V}_\varepsilon$ has no Lebesgue density points for any $\varepsilon > 0$. By (3.1) and (3.2) the set \mathcal{V}_ε is open as an inverse image of an open set for a continuous function. Therefore it is indeed Lebesgue measurable. Let $\varepsilon > 0$, $\lambda_0 \in \mathcal{I}P$ and ρ, β as in (2.2). By definition, there are such $\omega, \tau \in \Omega$ that $f_{\omega, \tau}(\lambda_0) = 0$ but $f_{\omega, \tau} \neq 0$. Let $k \in \mathbb{N}$ and choose minimal $n, p \in \mathbb{N}$ such that $r_{\omega|n}(\lambda_0) \leq \rho^k$ and $r_{\tau|p}(\lambda_0) \leq \rho^k$. Note that since n and p are minimal, we have $n, p \leq k$ and

$$\beta \rho^k \leq r_{\omega|n}(\lambda_0) \leq \rho^k \quad \text{and} \quad \beta \rho^k \leq r_{\tau|p}(\lambda_0) \leq \rho^k. \quad (3.8)$$

This shows that

$$\beta \leq \frac{r_{\omega|n}(\lambda_0)}{r_{\tau|p}(\lambda_0)} \leq \beta^{-1}.$$

Now we can use Lemma 3.3 to get $u, v \in \mathcal{A}^*$ with $|u| - n \leq N, |v| - p \leq N$, such that $u|n = \omega|n, v|p = \tau|p$ and

$$e^{-\varepsilon/3} \leq \frac{r_u(\lambda_0)}{r_v(\lambda_0)} \leq e^{\varepsilon/3}.$$

Lemma 3.4 states that if

$$|\lambda - \lambda_0| \leq \frac{\varepsilon \beta}{3L(k+N)}$$

then

$$\frac{r_u(\lambda)}{r_u(\lambda_0)} \leq e^{\frac{L}{\beta}|u| \frac{\varepsilon \beta}{3L(k+N)}} \leq e^{\varepsilon/3}.$$

Here the parameters λ and λ_0 are interchangeable, and the same argument holds for v , so we get an approximation for both values

$$\frac{r_u(\lambda)}{r_u(\lambda_0)}, \frac{r_v(\lambda)}{r_v(\lambda_0)} \in [e^{-\varepsilon/3}, e^{\varepsilon/3}]. \quad (3.9)$$

By this we see that also

$$e^{-\varepsilon} \leq \frac{e^{-\varepsilon/3}}{e^{\varepsilon/3}} \cdot \frac{r_u(\lambda_0)}{r_v(\lambda_0)} \leq \frac{r_u(\lambda)}{r_v(\lambda)} \leq \frac{e^{\varepsilon/3}}{e^{-\varepsilon/3}} \cdot \frac{r_u(\lambda_0)}{r_v(\lambda_0)} \leq e^\varepsilon. \quad (3.10)$$

Next we find a sub interval of J for which (3.2) holds. We have $f_{\omega, \tau}(\lambda_0) = 0$ which in other words means that $\Pi(\lambda_0, \omega) = \Pi(\lambda_0, \tau)$. Then, by (3.7),

$$\begin{aligned} |f_{v\bar{1}, u\bar{1}}(\lambda_0)| &\leq |\Pi(\lambda_0, v\bar{1}) - \Pi(\lambda_0, \tau)| + |\Pi(\lambda_0, \omega) - \Pi(\lambda_0, u\bar{1})| = |f_{v\bar{1}, \tau}(\lambda_0)| + |f_{\omega, u\bar{1}}(\lambda_0)| \\ &\leq 2C_1(r_{v\bar{1} \wedge \tau}(\lambda_0) + r_{u\bar{1} \wedge \omega}(\lambda_0)) \leq 2C_1(r_{\tau|p}(\lambda_0) + r_{\omega|n}(\lambda_0)) \leq 4C_1\rho^k \end{aligned} \quad (3.11)$$

Because $f_{\omega,\tau} \neq 0$ and $f_{\omega,\tau}(\lambda_0) = 0$ we have, by transversality Definition 2.4, $|f'_{\omega,\tau}(\lambda_0)| > \delta$ for some $\delta > 0$. By Remark 2.3(ii) there are $\gamma > 0$ and $k_0 \in \mathbb{N}$ such that

$$|f'_{v\bar{1},u\bar{1}}(\lambda)| > \delta \quad (3.12)$$

as long as $|\lambda - \lambda_0| < \gamma$ and $k \geq k_0$. Denote $F_k := [\lambda_0 - 4C_1\rho^k/\delta, \lambda_0 + 4C_1\rho^k/\delta]$. If $k \geq k_0$ and $4C_1\rho^k/\delta < \gamma$, then by (3.11) and (3.12) the function $f_{v\bar{1},u\bar{1}}$ has to reach 0 at some point of F_k . Thus, there exists $\lambda_1 \in F_k$ such that $f_{v\bar{1},u\bar{1}}(\lambda_1) = 0$. Suppose that k is so large that

$$\frac{4C_1\rho^k}{\delta} < \min\left\{\frac{\varepsilon\beta}{3L(k+N)}, \gamma\right\},$$

then (3.1) holds on F_k by (3.10). We have $\omega|n = u|n$ so by (3.8) we get

$$r_u(\lambda_0) = r_{u|n}(\lambda_0)r_{u_{n+1}\dots u_{|u|}}(\lambda_0) \geq \beta\rho^k\beta^{|u|-n} \geq \rho^k\beta^{N+1}.$$

Moreover, if we assume that ε is small enough so that $e^{\varepsilon/3} \leq 2$, then again by (3.9) we have

$$r_u(\lambda_0) \leq 2r_u(\lambda)$$

when $\lambda \in F_k$. Remark 2.3(i) makes it possible to take $C_2 > 0$ such that $\|f'_{\zeta,\xi}\|_{C(J)} \leq C_2$ for every $\zeta, \xi \in \Omega$. Because $f_{v\bar{1},u\bar{1}}(\lambda_1) = 0$ then $|f_{v\bar{1},u\bar{1}}(\lambda)| \leq C_2|\lambda - \lambda_1|$. Assume that ε is so small that

$$\frac{\beta^{N+1}\rho^k\varepsilon}{2C_2} \leq \frac{4C_1\rho^k}{\delta}.$$

Now, if

$$|\lambda - \lambda_1| < \frac{\beta^{N+1}\rho^k\varepsilon}{2C_2},$$

then $|f_{v\bar{1},u\bar{1}}(\lambda)| < (\varepsilon/2)\beta^{N+1}\rho^k \leq \varepsilon r_u(\lambda)$, so $\lambda \in \mathcal{V}_\varepsilon \cap F_k$. Finally

$$\frac{\mathcal{L}(F_k \cap \mathcal{V}_\varepsilon)}{\mathcal{L}(F_k)} \geq \frac{\beta^{N+1}\rho^k\varepsilon/(2C_2)}{8C_1\rho^k/\delta} = \frac{\delta\beta^{N+1}\varepsilon}{16C_1C_2}.$$

Now F_k is an interval with a middle point λ_0 . Because $\mathcal{L}(F_k) \rightarrow 0$ as $k \rightarrow \infty$, we see with the help of Theorem 1.14, that λ_0 cannot be a Lebesgue density point for the set $\mathcal{I}P \setminus \mathcal{V}_\varepsilon$. As λ_0 was arbitrary it follows that $\mathcal{I}P \setminus \mathcal{V}_\varepsilon$ has no Lebesgue density points.

- (ii) Here condition (2.6) was also assumed. So we can use Lemma 2.6, which states that $\mathcal{I}P$ is compact. This tells us that Baire category theorem holds in $\mathcal{I}P$. Let $\varepsilon > 0$ and let \mathcal{V}_ε be as above. As noted in the proof of (i) the set \mathcal{V}_ε is open. Thus $\mathcal{V}_{1/n} \cap \mathcal{I}P$ is open in $\mathcal{I}P$. In (i) we took arbitrary point from $\mathcal{I}P$ and found $\lambda_1 \in \mathcal{V}_\varepsilon \cap \mathcal{I}P$ from its neighbourhood. Thus the set $\mathcal{V}_\varepsilon \cap \mathcal{I}P$ is also dense. Therefore the dense G_δ property follows from the identity

$$\{\lambda \in \mathcal{I}P : \mathcal{H}^{s(\lambda)}(\mathcal{K}^\lambda) = 0\} = \bigcap_{n \in \mathbb{N}} (\mathcal{V}_{1/n} \cap \mathcal{I}P)$$

with Baire category theorem. Finally, to see that \mathcal{IP} is perfect, we check that it has no isolated points. To this end, let $\lambda_0 \in \mathcal{IP}$ and $\omega, \tau \in \Omega$ such that $f_{\omega, \tau}(\lambda_0) = 0$ but $f_{\omega, \tau} \neq 0$. For every $k \in \mathbb{N}$ we construct τ^k in the following way. For the beginning of the sequence τ^k we choose $\tau|k$. This guarantees that the points ω and τ^k will be close enough to each other. For the second infinite part look at the sequence $\omega' := \omega_{k+1}\omega_{k+2}\omega_{k+3} \dots$. Condition (2.5) ensures that \mathcal{K}^{λ_0} is not a singleton, so we can choose $\tau' \in \Omega$ such that $f_{\omega', \tau'}(\lambda_0) \neq 0$. Finally we add these parts together to get $\tau^k = \omega|k; \tau'$. Then $r_{\omega|k}, r_{\tau^k|k} \leq \rho^k$. Now $f_{\omega, \tau^k}(\lambda_0) \neq 0$, but $|f_{\omega, \tau^k}(\lambda_0)| \leq 4C\rho^k$ as in (3.11). Again, using transversality we can find $\delta > 0$ such that $|f'_{\omega, \tau}(\lambda_0)| > \delta$ and by Remark 2.3(ii) we can find $\gamma > 0$ and $k_0 \in \mathbb{N}$ such that $|f'_{\omega, \tau^k}(\lambda)| > \delta$ when $\lambda \in [\lambda_0 - \gamma, \lambda_0 + \gamma]$ and $k \geq k_0$. Suppose that $k \in \mathbb{N}$ is big enough that $4C\rho^k/\delta < \gamma$ and $k \geq k_0$. Then, as in (i), we see that there exists $\lambda_1 \in [\lambda_0 - 4C\rho^k/\delta, \lambda_0 + 4C\rho^k/\delta]$ such that $f_{\omega, \tau^k}(\lambda_1) = 0$. This means that $\lambda_1 \in \mathcal{IP}$, so λ_0 is not an isolated point. \square

Theorem 3.6. *Let $\mathcal{K} \subset \mathbb{R}^2$ be a self-similar set of dimension $s \in (0, 1)$ that is not on a line and is generated by i.f.s. $\{S_i\}_{i \leq m}$ of the form (1.5).*

- (i) $\mathcal{H}^s(\text{proj}_\theta \mathcal{K}) = 0$ for Lebesgue almost every $\theta \in \mathcal{IP}$.
- (ii) *If the i.f.s. $\{S_i\}_{i \leq m}$ satisfies the strong separation condition, then \mathcal{IP} is a compact perfect set, and the set $\{\theta \in \mathcal{IP} : \mathcal{H}^s(\text{proj}_\theta \mathcal{K}) = 0\}$ is a dense G_δ set in \mathcal{IP} .*

Proof. Condition (2.6) follows from the strong separation condition, so the claim follows immediately from Theorem 3.5 combined with Lemma 2.7 and Remark 2.8. \square

3.1 The Lebesgue measure of the set of intersection parameters

As mentioned before, Theorem 3.5(i) and Theorem 3.6(i) has content whenever \mathcal{IP} has positive Lebesgue measure.

Proposition 3.7. *Suppose that the one-parameter family of iterated function systems satisfy conditions (1.5), (2.2) and Definition 2.4. Then $\dim_{\mathbb{H}} \mathcal{IP} \leq 2s_{\max}$, where $s_{\max} = \sup_{\lambda \in J} s(\lambda)$. Thus, if $s_{\max} \in (0, \frac{1}{2})$ then $\mathcal{L}(\mathcal{IP}) = 0$.*

Proof. When $\delta > 0$, let \mathcal{IP}_δ be the set of all $\lambda \in J$ for which there exist $\omega, \tau \in \Omega$ such that

$$f_{\omega, \tau}(\lambda) = 0 \quad \text{and} \quad |f'_{\omega, \tau}(\lambda)| > \delta. \quad (3.13)$$

Transversality Definition 2.4 holds, so $\mathcal{IP} = \bigcup_{n \in \mathbb{N}} \mathcal{IP}_{1/n}$. Thus it is sufficient to show that $\dim_{\mathbb{H}} \mathcal{IP}_\delta \leq 2s_{\max}$ for any $\delta > 0$.

Let \mathcal{K} be the corresponding self-similar set. For $u \in \mathcal{A}^*$ let $I_u^\lambda := \text{Conv}(\mathcal{K}_u^\lambda)$. As scaling is not an issue, we can assume that $d(\mathcal{K}) = 1$. Then also $|I_u^\lambda| = r_u(\lambda) \leq d(S_u(\mathcal{K}))$. Let $\delta > 0$, $\lambda_0 \in \mathcal{I}P_\delta$ and $\omega, \tau \in \Omega$ satisfy (3.13). By Remark 2.3(ii) we may choose $k \in \mathbb{N}$ and $\gamma > 0$ such that

$$|f'_{\zeta, \xi}(\lambda)| \geq \frac{\delta}{2} \quad (3.14)$$

for $|\zeta \wedge \omega|, |\xi \wedge \tau| \geq k$ and $|\lambda_0 - \lambda| \leq \gamma$. This implies that the intervals I_u^λ and I_v^λ move relative to each other with a speed not less than $\delta/2$ as λ varies close to λ_0 and u, v are sufficiently long prefixes of ω, τ . Let $J = [\lambda_1, \lambda_2]$. Write $w \in \mathcal{C}_\varepsilon(\lambda_1)$ if $|I_w^\lambda| \leq \varepsilon$ and no prefix of w has this property. For $w \in \mathcal{C}_\varepsilon(\lambda_1)$, we can limit the size $|I_w^\lambda|$ uniformly for all $\lambda \in J$ by choosing $\varepsilon > 0$ small enough, since $|I_w^\lambda| \leq d(S_w(\mathcal{K}))$. Then, if $\varepsilon > 0$ is small enough, (3.14) says that the intervals I_u^λ and I_v^λ move past each other as $\lambda \in [\lambda_0 - \gamma, \lambda_0 + \gamma]$, where u, v are the unique prefixes of ω, τ from the set $\mathcal{C}_\varepsilon(\lambda_1)$. This means that when λ varies close to λ_0 , the sets I_u^λ and I_v^λ intersect each other on an interval of length at most $\frac{2}{\delta}(\max_{\lambda \in J} |I_u^\lambda| + \max_{\lambda \in J} |I_v^\lambda|)$, which is a subinterval of $[\lambda_0 - \gamma, \lambda_0 + \gamma]$. As J is a bounded interval, the number of these intervals where I_u^λ and I_v^λ move past each other, as u, v are fixed and λ is the variable, is at most $1 + \frac{|J|}{2\gamma}$. Choose $C = \max\{|J|, 2\}$. Then $\{\lambda \in J : I_u^\lambda \cap I_v^\lambda \neq \emptyset\}$ can be covered with a union of $1 + \frac{C}{2\gamma}$ sets of length $\frac{C}{\delta}(\max_{\lambda \in J} |I_u^\lambda| + \max_{\lambda \in J} |I_v^\lambda|)$.

Note that $|I_w^\lambda| \geq \beta\varepsilon$ for $w \in \mathcal{C}_\varepsilon(\lambda_1)$ and $\sum_{w \in \mathcal{C}_\varepsilon(\lambda_1)} r_w^{s(\lambda_1)} = 1$, which can be calculated as in the proof of Theorem 1.35. Therefore

$$1 = \sum_{w \in \mathcal{C}_\varepsilon(\lambda_1)} r_w^{s(\lambda_1)} \geq \#\mathcal{C}_\varepsilon(\lambda_1) \cdot (\beta\varepsilon)^{s(\lambda_1)}$$

giving

$$\beta^{-s(\lambda_1)} \varepsilon^{-s(\lambda_1)} \geq \#\mathcal{C}_\varepsilon(\lambda_1).$$

Thus the number of all possible pairs u, v from $\mathcal{C}_\varepsilon(\lambda_1)$, $u \neq v$, such that the sets I_u^λ and I_v^λ intersect for some $\lambda \in J$ is then at most $\beta^{-2s(\lambda_1)} \varepsilon^{-2s(\lambda_1)}$.

As implied at the beginning of the proof, for a fixed $w \in \mathcal{C}_\varepsilon(\lambda_1)$ the sets I_w^λ are roughly the same size for all $\lambda \in J$. More rigorously, if

$$t := \min \left\{ \frac{\log r_i(\lambda)}{\log r_i(\lambda_1)} : \lambda \in J, i = 1, \dots, m \right\},$$

then

$$\begin{aligned} |I_w^{\lambda_1}|^t &= (r_{w_1}(\lambda_1) \cdots r_{w_n}(\lambda_1))^t = e^{t \log(r_{w_1}(\lambda_1) \cdots r_{w_n}(\lambda_1))} = e^{t \log r_{w_1}(\lambda_1) + \cdots + t \log r_{w_n}(\lambda_1)} \\ &\geq e^{\log r_{w_1}(\lambda) + \cdots + \log r_{w_n}(\lambda)} = r_{w_1}(\lambda) \cdots r_{w_n}(\lambda) = |I_w^\lambda| \end{aligned}$$

for all $w \in \mathcal{A}^*$ and $\lambda \in J$, since

$$t \log r_j(\lambda_1) \geq \frac{\log r_j(\lambda)}{\log r_j(\lambda_1)} \log r_j(\lambda_1)$$

for every $j = 1, \dots, m$. Now $(\max_{\lambda \in J} |I_u^\lambda| + \max_{\lambda \in J} |I_v^\lambda|) \leq 2\varepsilon^t$.

In conclusion, if $\omega, \tau \in \Omega$ are such sequences that there is $\lambda \in J$ such that (3.13) holds, then by considering the prefixes $u, v \in \mathcal{C}_\varepsilon(\lambda_1)$ of ω, τ we see that every such λ belongs to a set which can be covered by $1 + \frac{C}{2^\gamma}$ sets of length $\frac{C}{\delta} 2\varepsilon^t$. In fact, this set covers all such λ for any $\omega', \tau' \in \Omega$ which have the same prefixes u and v from the set $\mathcal{C}_\varepsilon(\lambda_1)$ as ω and τ have. Thus the whole set \mathcal{IP}_δ can be covered with $(1 + \frac{C}{2^\gamma})\beta^{-2s(\lambda_1)}\varepsilon^{-2s(\lambda_1)}$ sets each of length $\frac{C}{\delta} 2\varepsilon^t$. This allows us to estimate the box-counting dimension of \mathcal{IP}_δ which in turn is an upper limit for Hausdorff dimension.

$$\overline{\dim}_{\mathbb{B}} \mathcal{IP}_\delta \leq \limsup_{\varepsilon \rightarrow 0} \frac{\log\left(\left(1 + \frac{C}{2^\gamma}\right)\beta^{-2s(\lambda_1)}\varepsilon^{-2s(\lambda_1)}\right)}{-\log \frac{C}{\delta} 2\varepsilon^t} = \frac{2s(\lambda_1)}{t} \leq \frac{2s_{\max}}{t}. \quad (3.15)$$

To get rid of t , divide J into small intervals and apply the above to each part of the division. Since Hausdorff dimension is countably stable, it is enough to consider the part with the biggest dimension. We can make this division as dense as we want, so by continuity of r_i , we can make t in (3.15) go as close to 1 as we want. Therefore $\dim_{\mathbb{H}} \mathcal{IP}_\delta \leq 2s_{\max}$. \square

In homogenous case there is easy way to check whether $\mathcal{L}(\mathcal{IP}) > 0$. This is presented in the following lemma.

Lemma 3.8. *Let $\{S_1^\lambda, \dots, S_m^\lambda\}_{\lambda \in J}$ be a one-parameter family of homogenous i.f.s. on \mathbb{R} satisfying (2.1) and (2.6). Suppose that for some $i \neq j$ there exists a subinterval $\tilde{J} \subset J$ such that*

$$\text{Conv}(\mathcal{K}_i^\lambda) \cap \text{Conv}(\mathcal{K}_j^\lambda) \neq \emptyset \quad \text{for every } \lambda \in \tilde{J} \quad (3.16)$$

and

$$\mathcal{K}^\lambda - \mathcal{K}^\lambda \quad \text{is an interval for every } \lambda \in \tilde{J}. \quad (3.17)$$

Then $\tilde{J} \subset \mathcal{IP}$.

Proof. For any $A \subset \mathbb{R}$ we have $A - A = \{t \in \mathbb{R} : A \cap (A + t) \neq \emptyset\}$. From condition (3.17) we have $\mathcal{K}^\lambda - \mathcal{K}^\lambda = \text{Conv}(\mathcal{K}^\lambda - \mathcal{K}^\lambda) = \text{Conv}(\mathcal{K}^\lambda) - \text{Conv}(\mathcal{K}^\lambda)$. The latter equality is true for any set. Also, \mathcal{K}_i^λ is a translate of $r(\lambda)\mathcal{K}^\lambda$ for every $i = 1, \dots, m$ and $\lambda \in J$ as the families are homogenous, in other words $\mathcal{K}_i^\lambda = r(\lambda)\mathcal{K}^\lambda + a_i$. Let $\lambda \in \tilde{J}$. Then $(\text{Conv}(r(\lambda)\mathcal{K}^\lambda) + a_i) \cap (\text{Conv}(r(\lambda)\mathcal{K}^\lambda) + a_j) \neq \emptyset$. This means that $a_j - a_i \in \{t \in \mathbb{R} : \text{Conv}(r(\lambda)\mathcal{K}^\lambda) \cap (\text{Conv}(r(\lambda)\mathcal{K}^\lambda) + t) \neq \emptyset\} = \text{Conv}(r(\lambda)\mathcal{K}^\lambda) - \text{Conv}(r(\lambda)\mathcal{K}^\lambda) = r(\lambda)\mathcal{K}^\lambda - r(\lambda)\mathcal{K}^\lambda = \{t \in \mathbb{R} : r(\lambda)\mathcal{K}^\lambda \cap r(\lambda)\mathcal{K}^\lambda + t \neq \emptyset\}$. Thus $(r(\lambda)\mathcal{K}^\lambda + a_i) \cap (r(\lambda)\mathcal{K}^\lambda + a_j) \neq \emptyset$ and $\lambda \in \mathcal{IP}$ by Lemma 2.6. \square

Remark 3.9. The conditions (3.16) and (3.17) can be verified by little calculations. Let $\{S_1, \dots, S_m\}_{i \leq m}$ be a homogenous i.f.s. on \mathbb{R} , with $S_i(x) = rx + a_i$, and \mathcal{K} be the corresponding self-similar set. Suppose that g is the minimal gap between two consecutive elements of $\{a_i\}_{i \leq m}$ and G is the maximal gap between two consecutive elements of $\Gamma := \{a_i - a_j\}_{i, j \leq m}$. We claim that $\text{Conv}(\mathcal{K}_i) \cap \text{Conv}(\mathcal{K}_j) \neq \emptyset$ for some $i \neq j$ if and only if

$$\frac{r}{1-r} \max \Gamma \geq g, \quad (3.18)$$

and $\mathcal{K} - \mathcal{K}$ is an interval if and only if

$$\frac{r}{1-r} \max \Gamma \geq \frac{1}{2}G. \quad (3.19)$$

Suppose for simplicity that $a_1 \leq \dots \leq a_m$. Indeed, for $\text{Conv}(\mathcal{K}_i)$ and $\text{Conv}(\mathcal{K}_j)$ to intersect for some $i \neq j$, we need to have

$$\begin{aligned} a_i + r \sum_{n=0}^{\infty} a_m r^n &\geq a_{i+1} + r \sum_{n=0}^{\infty} a_1 r^n \\ \Leftrightarrow a_i + \frac{r}{1-r} a_m &\geq a_{i+1} + \frac{r}{1-r} a_1 \\ \Leftrightarrow \frac{r}{1-r} (a_m - a_1) &\geq a_{i+1} - a_i \end{aligned}$$

for some $i = 1, \dots, m-1$, and (3.18) follows. For the equation (3.19), consider the set

$$\mathcal{K} - \mathcal{K} = \left\{ \sum_{n=0}^{\infty} c_n r^n : c_n \in \Gamma \right\}$$

which is also a self-similar set. Suppose again that $\Gamma = \{c_1, \dots, c_M\}$, where $c_1 \leq \dots \leq c_M$. Similarly to the first part, if $\mathcal{K} - \mathcal{K}$ is an interval, then

$$c_i + r \sum_{n=0}^{\infty} \max_{j \leq M} \{c_j\} r^n \geq c_{i+1} + r \sum_{n=0}^{\infty} \min_{j \leq M} \{c_j\} r^n$$

for every $i = 1, \dots, M-1$. Because of the fact that sets of the form $A - A$ are symmetrical with respect to the origin, we have $\max_{j \leq M} \{c_j\} = c_M$ and $\min_{j \leq M} \{c_j\} = -c_M$. Therefore we need to have

$$\begin{aligned} c_i + \frac{r}{1-r} c_M &\geq c_{i+1} + \frac{r}{1-r} (-c_M) \\ \Leftrightarrow 2 \frac{r}{1-r} c_M &\geq c_{i+1} - c_i \end{aligned}$$

for every $i = 1, \dots, M-1$. Hence (3.19). On the other hand, let $\mathcal{K} - \mathcal{K}$ be a set with (3.19). Now $\mathcal{K} - \mathcal{K}$ is a self-similar set generated by similitudes $S_i(x) = rx + c_i$, $i = 1, \dots, M$. Evidently $S_M\left(\frac{c_M}{1-r}\right) = \frac{c_M}{1-r}$ and $S_1\left(-\frac{c_M}{1-r}\right) = \frac{-c_M}{1-r}$, since $c_1 = -c_M$. Therefore $S_i\left(\left[-\frac{c_M}{1-r}, \frac{c_M}{1-r}\right]\right) \subset \left[-\frac{c_M}{1-r}, \frac{c_M}{1-r}\right]$ and $S_i\left(\left[-\frac{c_M}{1-r}, \frac{c_M}{1-r}\right]\right)$ is an interval of length $r2\frac{c_M}{1-r}$ for every i . Since we assumed $r2\frac{c_M}{1-r} \geq G$, every two consecutive images intersect each other. Thus $\left[-\frac{c_M}{1-r}, \frac{c_M}{1-r}\right]$ is the unique attractor $\mathcal{K} - \mathcal{K}$.

The following two lemmas sharpens Proposition 3.7 for families of projections.

Lemma 3.10. *Let $\mathcal{K} \subset \mathbb{R}^2$ be a self-similar set generated by an i.f.s. which satisfies (1.5) and the strong separation condition. Then $\dim_{\mathbb{H}} \mathcal{I}P \leq \dim_{\mathbb{H}}(\mathcal{K} - \mathcal{K})$.*

Proof. As \mathcal{K} is compact, strong separation condition guarantees that there is $t > 0$ such that $\mathcal{K}_i(t) \cap \mathcal{K}_j(t) = \emptyset$ for every $i \neq j$. If $x \in \mathcal{K}_i$ and $y \in \mathcal{K}_j$, then $|x - y| \geq t$, so

$$\bigcup_{i \neq j} (\mathcal{K}_i - \mathcal{K}_j) \subset \{x \in \mathbb{R}^2 : |x| \geq t\} =: A.$$

Let $\Upsilon(x) := x/|x|$. If $x, y \in A$ we have

$$\left| \Upsilon(x) - \Upsilon(y) \right| = \left| \Upsilon\left(\frac{x}{t}\right) - \Upsilon\left(\frac{y}{t}\right) \right| \leq \left| \frac{x}{t} - \frac{y}{t} \right| = \frac{1}{t} |x - y|.$$

The inequality holds since x/t and y/t are on the unit circle or outside of it. This means that Υ is a Lipschitz mapping from the set A to the unit circle. Strong separation condition also gives us (2.6), so Lemma 2.6(i) states that

$$\mathcal{I}P = \{ \theta \in [0, \pi) : \text{proj}_\theta \mathcal{K}_i \cap \text{proj}_\theta \mathcal{K}_j \neq \emptyset \text{ for some } i \neq j \}.$$

This means that if $\theta \in \mathcal{I}P$, then for some $i \neq j$ there are two points $x \in \mathcal{K}_i$ and $y \in \mathcal{K}_j$ which are on the same line perpendicular to L_θ . In other words, $x - y$ is on the line $L_{\theta+\pi/2}$ and

$$\left(\cos\left(\frac{\pi}{2} + \theta\right), \sin\left(\frac{\pi}{2} + \theta\right) \right) \in \Upsilon\left(\bigcup_{i \neq j} (\mathcal{K}_i - \mathcal{K}_j)\right).$$

Therefore

$$\mathcal{I}P = \left\{ \theta \in [0, \pi) : \left(\cos\left(\frac{\pi}{2} + \theta\right), \sin\left(\frac{\pi}{2} + \theta\right) \right) \in \Upsilon\left(\bigcup_{i \neq j} (\mathcal{K}_i - \mathcal{K}_j)\right) \right\}$$

and $\mathcal{I}P$ is the same set as $\Upsilon\left(\bigcup_{i \neq j} (\mathcal{K}_i - \mathcal{K}_j)\right)$ apart from a rotation by $\pi/2$. So for the dimension we get

$$\dim_{\mathbb{H}} \mathcal{I}P = \dim_{\mathbb{H}} \Upsilon\left(\bigcup_{i \neq j} (\mathcal{K}_i - \mathcal{K}_j)\right) \leq \dim_{\mathbb{H}} \bigcup_{i \neq j} (\mathcal{K}_i - \mathcal{K}_j) = \max_{i \neq j} \dim_{\mathbb{H}} (\mathcal{K}_i - \mathcal{K}_j) \leq \dim_{\mathbb{H}} (\mathcal{K} - \mathcal{K}),$$

since Lipschitz functions preserve dimension. \square

Lemma 3.11. *Let $\mathcal{K} \subset \mathbb{R}^2$ be a self-similar set generated by an i.f.s. which satisfies (1.5) and the strong separation condition. If $\dim_{\mathbb{H}}(\mathcal{K} - \mathcal{K}) < 1$, then $\mathcal{H}^s(\text{proj}_\theta \mathcal{K}) > 0$ for almost every $\theta \in [0, \pi)$. This happens especially when $s = \dim_{\mathbb{H}} \mathcal{K} < \frac{1}{2}$.*

Proof. The first part follows immediately from Lemma 3.10, Lemma 2.6(iii) and the fact that the Lebesgue measure of a set of Hausdorff dimension less than one is zero. For the latter part note that if $x \in A - A$, then $x = z - y$, $y, z \in A$ and $(y, z) \in A \times A$. Now $\text{proj}_{\frac{3\pi}{4}}(y, z)$ is rotation equivalent (by $\frac{5\pi}{4}$) to $y \cos \frac{3\pi}{4} + z \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}(z - y)$. Therefore $A - A$ and $\text{proj}_{\frac{3\pi}{4}}(A \times A)$ are the same sets up to scaling and rotation for every $A \subset \mathbb{R}^2$. Because \mathcal{K} is self-similar Hausdorff and packing dimensions agree. Therefore by [10, Corollary 8.11] we have

$$\dim_{\mathbb{H}}(\mathcal{K} - \mathcal{K}) = \dim_{\mathbb{H}} \text{proj}_{\frac{3\pi}{4}}(\mathcal{K} \times \mathcal{K}) \leq \dim_{\mathbb{H}}(\mathcal{K} \times \mathcal{K}) = 2\dim_{\mathbb{H}} \mathcal{K}.$$

\square

Chapter 4

Positive packing measure

Recall the notations from the beginning of Chapter 2. In Chapter 3 we proved that the s -dimensional Hausdorff measure of a projection of s -dimensional self-similar set is typically zero, if there is some overlap in the projection. In this chapter we restrict our attention to packing measure in similar conditions. We prove that unlike the Hausdorff measure, packing measure of this kind of projection is typically positive. Also, this is shown for Borel sets, not just for self-similar sets. The main result is Theorem 4.1. This yields useful and interesting Theorems 4.7 and 4.8.

Theorem 4.1. *Let (Ω, d) be a complete separable metric space, $A \subset \Omega$ a Borel set and $\mathcal{H}^\gamma(A) > 0$. Also, let $\Pi_\lambda: A \rightarrow \mathbb{R}$, $\lambda \in J$ be a one-parameter family of continuous maps, where J is a closed interval. Suppose that $\alpha(\lambda)$ is some positive function such that for all $\omega, \tau \in \Omega$, $\omega \neq \tau$ the functions*

$$\Psi(\lambda) = \Psi_{\omega, \tau}(\lambda) = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{d(\omega, \tau)^{\alpha(\lambda)}} \quad (4.1)$$

belong to $C^1(J)$. Suppose also that there are $\delta > 0$ and $M > 0$ independent of ω and τ such that

$$\|\Psi\|_{C^1(J)} \leq M \quad (4.2)$$

and

$$|\Psi(\lambda)| + |\Psi'(\lambda)| > \delta \quad (4.3)$$

for all $\lambda \in J$. Then

$$\dim_{\mathbb{H}}\{\lambda \in J : \mathcal{P}^{\sigma(\lambda)}(\Pi_\lambda(A)) = 0\} \leq \sigma_{\max} \quad (4.4)$$

where $\sigma(\lambda) = \gamma/\alpha(\lambda)$ and $\sigma_{\max} = \sup\{\sigma(\lambda) : \lambda \in J\}$.

Remark 4.2. (a) The theorem has content only when $\sigma_{\max} < 1$.

(b) Since Ψ is bounded, the functions Π_λ are Hölder with exponent $\alpha(\lambda)$. Thus $\mathcal{P}^\gamma(A) < \infty$ implies $\mathcal{P}^{s(\lambda)}(\Pi_\lambda(A)) < \infty$.

(c) The function α is continuous, since $\Psi \in C^1(J)$.

(d) Condition (4.3) is closely related to Definition 2.4 and was mentioned in Lemma 2.6.

Lemma 4.3. *Suppose that $\Psi \in C^1(J)$ and conditions (4.2) and (4.3) holds for Ψ . Then for every $\rho \leq \delta/2$ the set $\{\lambda \in J : |\Psi(\lambda)| \leq \rho\}$ is a union of at most $1 + M|J|/\delta$ intervals of length at most $4\rho/\delta$.*

Proof. If $|\Psi(\lambda)| \leq \rho \leq \delta/2$, then $|\Psi'(\lambda)| \geq \delta/2$. So the set $\{\lambda \in J : |\Psi(\lambda)| \leq \rho\}$ is a union of disjoint intervals $\{A_i\}$ on each of which the function Ψ is monotone. The length of such interval is at most $4\rho/\delta$. To check how many such intervals there is observe that $\{\lambda \in J : |\Psi(\lambda)| \leq \rho\} \subset \{\lambda \in J : |\Psi(\lambda)| \leq \delta/2\}$ and the latter set is also a union of disjoint intervals $\{B_i\}$ on each of which the function Ψ is monotone. Therefore each interval B_i contains exactly one of the sets A_i . Since $|B_i| \geq \frac{\delta}{M}$ for every i we need at most $1 + M|J|/\delta$ such intervals to cover J . \square

Proposition 4.4. *Under the assumptions of Theorem 4.1, let $s_{\max} < s_1 < 1$ and let η be a probability Frostman measure on J such that $\eta(B(x, \rho)) \leq c\rho^{s_1}$ for any $\rho > 0$. Suppose also that μ is a Borel probability measure on Ω such that*

$$(\mu \times \mu)\{(\omega, \tau) \in \Omega^2 : d(\omega, \tau) < r\} \leq C_1 r^\gamma \quad (4.5)$$

for all $r > 0$. Then

$$\mathcal{J} := \int_J \int_{\mathbb{R}} \nu_\lambda(B(x, r^{\alpha(\lambda)})) d\nu_\lambda(x) d\eta(\lambda) \leq C r^\gamma \quad (4.6)$$

for all $r > 0$, where $\nu_\lambda = \Pi_{\lambda*}\mu$ is the image measure.

Proof. Using Fubini's theorem, we get

$$\begin{aligned} \mathcal{J} &= \int_J \int_\Omega \mu(\{\omega \in \Omega : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r^{\alpha(\lambda)}\}) d\mu(\tau) d\eta(\lambda) \\ &= \int_J \int_\Omega \int_\Omega \chi_{\{\omega \in \Omega : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r^{\alpha(\lambda)}\}} d\mu(\omega) d\mu(\tau) d\eta(\lambda) \\ &= \int_\Omega \int_\Omega \int_J \chi_{\{\lambda \in J : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r^{\alpha(\lambda)}\}} d\eta(\lambda) d\mu(\omega) d\mu(\tau) \\ &= \int_\Omega \int_\Omega \eta(\{\lambda \in J : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r^{\alpha(\lambda)}\}) d\mu(\omega) d\mu(\tau). \end{aligned}$$

Decompose this integral as

$$\iint_{d(\omega, \tau) < r} + \sum_{k=1}^{\infty} \iint_{2^{k-1}r \leq d(\omega, \tau) < 2^k r} =: \mathcal{J}_0 + \sum_{k=1}^{\infty} \mathcal{J}_k.$$

Recall that $\Psi_{\omega, \tau}(\lambda) = (\Pi_\lambda(\omega) - \Pi_\lambda(\tau))d(\omega, \tau)^{-\alpha(\lambda)}$ and observe that for $d(\omega, \tau) \geq 2^{k-1}r$ we have

$$\begin{aligned} \eta(\{\lambda : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r^{\alpha(\lambda)}\}) &\leq \eta(\{\lambda : |\Psi_{\omega, \tau}(\lambda)| \leq r^{\alpha(\lambda)}(2^{k-1}r)^{-\alpha(\lambda)}\}) \\ &= \eta(\{\lambda : |\Psi_{\omega, \tau}(\lambda)| \leq 2^{-(k-1)\alpha(\lambda)}\}) \\ &\leq \eta(\{\lambda : |\Psi_{\omega, \tau}(\lambda)| \leq 2^{-(k-1)\alpha_{\min}}\}), \end{aligned}$$

where $\alpha_{\min} = \inf_{\lambda \in J} \alpha(\lambda) > 0$, since α is continuous. Choose $k_0 \in \mathbb{N}$ large enough to $2^{-(k_0-1)\alpha_{\min}} \leq \delta/2$. Now we can use Lemma 4.3 to see that $\{\lambda : |\Psi_{\omega, \tau}(\lambda)| \leq 2^{-(k-1)\alpha_{\min}}\}$ is contained in a finite union of intervals of length $C'2^{-k\alpha_{\min}}$, and since η is Frostman measure we get that

$$\eta(\{\lambda : |\Psi_{\omega, \tau}(\lambda)| \leq 2^{-(k-1)\alpha_{\min}}\}) \leq C_2 2^{-k\alpha_{\min}s_1} \quad (4.7)$$

for every ω, τ and $k \geq k_0$. Now we can continue the estimation of \mathcal{J} with (4.5) and (4.7) and recalling that μ and η are probability measures:

$$\begin{aligned} \mathcal{J} &\leq \sum_{k=0}^{k_0-1} (\mu \times \mu)\{(\omega, \tau) : d(\omega, \tau) < 2^k r\} + C_2 \sum_{k=k_0}^{\infty} 2^{-k\alpha_{\min}s_1} (\mu \times \mu)\{(\omega, \tau) : d(\omega, \tau) < 2^k r\} \\ &\leq C_3 \sum_{k=0}^{k_0-1} (2^k r)^\gamma + C_4 \sum_{k=k_0}^{\infty} 2^{-k\alpha_{\min}s_1} (2^k r)^\gamma. \end{aligned}$$

Observe that for the first sum

$$\sum_{k=0}^{k_0-1} (2^k r)^\gamma = r^\gamma \sum_{k=0}^{k_0-1} 2^{k\gamma} \leq r^\gamma 2^{k_0\gamma}$$

and for the second

$$\sum_{k=k_0}^{\infty} 2^{-k\alpha_{\min}s_1} (2^k r)^\gamma = r^\gamma \sum_{k=k_0}^{\infty} (2^{\gamma-\alpha_{\min}s_1})^k = r^\gamma \frac{(2^{\gamma-\alpha_{\min}s_1})^{k_0}}{1 - 2^{\gamma-\alpha_{\min}s_1}}$$

because we assumed that $s_1 > s_{\max} = \gamma/\alpha_{\min}$. We conclude that there is C such that $\mathcal{J} \leq Cr^\gamma$ for every $r > 0$. \square

Theorem 4.5. *Suppose that we are in the setting of Theorem 4.1 and μ is a Borel probability measure as in Proposition 4.4. Then*

$$\dim_{\mathbb{H}}\{\lambda \in J : \int \underline{D}_{s(\lambda)}(\nu_\lambda, x) d\nu_\lambda(x) = \infty\} \leq s_{\max}, \quad (4.8)$$

where

$$\underline{D}_{s(\lambda)}(\nu_\lambda, x) = \liminf_{r \rightarrow 0} \frac{\nu_\lambda(B(x, r))}{r^{s(\lambda)}}$$

is the lower $s(\lambda)$ -dimensional density of ν_λ at the point x .

Proof. Denote the set in (4.8) by J_0 . Suppose that J_0 has Hausdorff dimension greater than s_{\max} . Then there is $s_1 > s_{\max}$ such that $\mathcal{H}^{s_1}(J_0) > 0$, so Theorem 1.30 implies that there is a measure η as in Proposition 4.4 supported on the set J_0 . Since $\eta(J_0) > 0$, we have

$$\int_J \int \underline{D}_{s(\lambda)}(\nu_\lambda, x) d\nu_\lambda(x) d\eta(\lambda) = \infty.$$

But on the other hand, since $s(\lambda) = \gamma/\alpha(\lambda)$, Proposition 4.4 and Fatou's lemma yields

$$\begin{aligned} \int_J \int \underline{D}_{s(\lambda)}(\nu_\lambda, x) \, d\nu_\lambda(x) \, d\eta(\lambda) &= \int_J \int \liminf_{r \rightarrow 0} \frac{\nu_\lambda(B(x, r^{\alpha(\lambda)}))}{r^{\alpha(\lambda)s(\lambda)}} \, d\nu_\lambda(x) \, d\eta(\lambda) \\ &= \int_J \int \liminf_{r \rightarrow 0} \frac{\nu_\lambda(B(x, r^{\alpha(\lambda)}))}{r^\gamma} \, d\nu_\lambda(x) \, d\eta(\lambda) \\ &\leq \liminf_{r \rightarrow 0} \frac{1}{r^\gamma} \int_J \int \nu_\lambda(B(x, r^{\alpha(\lambda)})) \, d\nu_\lambda(x) \, d\eta(\lambda) \\ &\leq \liminf_{r \rightarrow 0} \frac{1}{r^\gamma} Cr^\gamma < \infty, \end{aligned}$$

which is a contradiction. \square

We will use the following result to prove Theorem 4.1.

Theorem 4.6 (Taylor and Tricot). *For any $\alpha > 0$ and $n \in \mathbb{N}$ there exists a constant $p(\alpha, n) > 0$ with the following property: For any Borel probability measure ν on \mathbb{R}^n , Borel set $A \subset \mathbb{R}^n$ and $C > 0$, if $\underline{D}_\alpha(\nu, x) \leq C$ for every $x \in A$, then*

$$\mathcal{P}^\alpha(A) \geq C^{-1}p(\alpha, n)\nu(A).$$

Proof. See [13]. \square

Proof of Theorem 4.1. Since $\mathcal{H}^\gamma(A) > 0$ Frostman's lemma in metric spaces [7] states that there exists a probability measure μ supported on the set A such that $\mu(B(x, r)) \leq Cr^\gamma$ for any $x \in A$ and $r > 0$. Then

$$\begin{aligned} (\mu \times \mu)\{(\omega, \tau) \in \Omega^2 : d(\omega, \tau) < r\} &= \int_A \mu(\{\omega \in \Omega : d(\omega, \tau) < r\}) \, d\mu(\tau) \\ &= \int_A \mu(B(\tau, r)) \, d\mu(\tau) \leq \mu(A)Cr^\gamma \end{aligned}$$

so condition (4.5) holds for μ . Now ν_λ is a Borel probability measure supported on $\Pi_\lambda(A)$ for every $\lambda \in J$. If $\underline{D}_{s(\lambda)}(\nu_\lambda, x) < \infty$ for almost every $x \in \Pi_\lambda(A)$, then there is $A_0 \subset \Pi_\lambda(A)$ of positive ν_λ measure such that $\underline{D}_{s(\lambda)}(\nu_\lambda, x) \leq C$ for $x \in A_0$. Therefore by Theorem 4.6 we see that $0 < \mathcal{P}^{s(\lambda)}(A_0) \leq \mathcal{P}^{s(\lambda)}(\Pi_\lambda(A))$. By (4.8) this happens for all $\lambda \in J$ except on a set of dimension less than or equal to s_{\max} . \square

Theorem 4.7. *Suppose that there is some i.f.s. satisfying (2.1), (2.2), (2.4) and (2.6) hold, and $r_i(\lambda) = r_i^{\varphi(\lambda)}$ for some positive function $\varphi(\lambda)$ and some reals $r_i \in (0, 1)$. Then $0 < \mathcal{P}^{s(\lambda)}(\mathcal{K}^\lambda) < \infty$ for all $\lambda \in J$ except a set of Hausdorff dimension $s_{\max} = \sup_{\lambda \in J} s(\lambda)$.*

Proof. Let $\Omega = \mathcal{A}^{\mathbb{N}}$ with the metric $d(\omega, \tau) = r_{\omega \wedge \tau}(\lambda_1)$, where $J = [\lambda_1, \lambda_2]$. Let also $A = \Omega$, $\Pi_\lambda = \Pi(\lambda, \cdot)$ be the natural projection map (2.3). Now, with $\alpha(\lambda) = \varphi(\lambda)/\varphi(\lambda_1)$,

$$\Psi_{\omega, \tau}(\lambda) = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{d(\omega, \tau)^{\alpha(\lambda)}} = \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\tau)}{r_{\omega \wedge \tau}(\lambda)} = \Pi_\lambda(\omega') - \Pi_\lambda(\tau'),$$

where $\omega'_1 \neq \tau'_1$. Since (2.6) is assumed (4.3) holds by Lemma 2.6(iv). Also, (4.2) holds by Lemma 2.2. Let $\gamma = \dim_{\mathbb{H}} A$. Then by Lemma 1.35 for $\gamma = s(\lambda_1)$ we have $\mathcal{H}^\gamma(A) > 0$. Moreover

$$\sum_{i=1}^m r_i(\lambda) \frac{s(\lambda_1)}{\alpha(\lambda)} = \sum_{i=1}^m r_i^{\varphi(\lambda_1)s(\lambda_1)} = 1,$$

so $s(\lambda_1)/\alpha(\lambda) = s(\lambda)$ and the claim follows by Theorem 4.1. \square

Theorem 4.8. *Let $\mathcal{K} \subset \mathbb{R}^2$ be any Borel set such that $\mathcal{H}^s(\mathcal{K}) > 0$ for some $s \in (0, 1)$. Then $\mathcal{P}^s(\text{proj}_\theta \mathcal{K}) > 0$ for almost every θ . Moreover,*

$$\dim_{\mathbb{H}}\{\theta \in [0, \pi) : \mathcal{P}^s(\text{proj}_\theta \mathcal{K}) = 0\} \leq s.$$

Proof. Let (Ω, d) be \mathbb{R}^2 with the Euclidean metric. Further, let $A = \mathcal{K}$ and $\Pi_\theta = \text{proj}_\theta$. Choose $\alpha(\theta) = 1$ for all θ whence

$$\Psi_{x,y}(\theta) = \frac{\text{proj}_\theta(x) - \text{proj}_\theta(y)}{|x - y|}.$$

for $x, y \in \mathbb{R}^2$. By Lemma 2.2 we see that condition (4.2) holds. Moreover, with calculations similar to those in the proof of Lemma 2.7 we see that condition (4.3) is satisfied. Also, $s_{\max} = s$, thus the claim follows by Theorem 4.1. \square

Chapter 5

Examples

Here we present two examples in which we use the theorems achieved. The examples are very similar in content. The first one concerns projections of planar Sierpinski gaskets onto lines of different angles. In the second example the same study is made for planar Cantor dusts.

Example 5.1. Let

$$\mathcal{K}_u^r = \left\{ \sum_{n=0}^{\infty} a_n r^n : a_n \in T \right\},$$

with $T = \{0, 1, u\}$, where $0 < r < 1$ is fixed and $u \in \mathbb{R}$ is the parameter. As noted in Example 1.44, this set is affine-equivalent to the family of projections of the s -dimensional Sierpinski gasket \mathcal{G}^r , where $s = \log 3 / \log r^{-1}$. Thus transversality condition holds by Lemma 2.7. Also, as noted in the same example, we may assume, without loss of generality, that $u \geq 2$. Then, by using Lemma 3.8 and Remark 3.9 with the appropriate notations, we have $\Gamma = \{0, \pm 1, \pm u, \pm(u-1)\}$. Now $\max \Gamma = u$, $g = 1$ and $G = \max\{1, u-2\}$. Thus (3.18) and (3.19) implies that $[\frac{1-r}{r}, \frac{2(1-r)}{1-3r}] \subset \mathcal{IP}$. This interval is not empty for $r \in (\frac{1}{5}, \frac{1}{3})$, hence $\mathcal{L}(\mathcal{IP}) > 0$. Therefore Theorem 3.5 is meaningful and states that

$$\mathcal{H}^s(\mathcal{K}_u^r) = 0$$

for every $r \in (\frac{1}{5}, \frac{1}{3})$ and for almost every $u \in [\frac{1-r}{r}, \frac{2(1-r)}{1-3r}]$. On the other hand, Theorem 4.7 implies that

$$0 < \mathcal{P}^s(\mathcal{K}_u^r) < \infty$$

for every $r \in (\frac{1}{5}, \frac{1}{3})$ and almost every $u \in [\frac{1-r}{r}, \frac{2(1-r)}{1-3r}]$, since $s_{\max} < 1$. Moreover \mathcal{G}^r satisfies the strong separation condition and, recalling Section 1.6, $\mathcal{G}^r - \mathcal{G}^r$ is a self-similar set generated by 7 similitudes. Therefore the similarity dimension is $\log 7 / \log r^{-1}$. So, by Lemma 3.11, $\mathcal{H}^s(\mathcal{K}_u^r) > 0$ for every $r < \frac{1}{7}$ and almost every u .

Example 5.2. Let $C_r^2 = C_r \times C_r$, where C_r is the middle- α Cantor set for $\alpha = 1 - 2r$. The family of projections $\{\text{proj}_{\theta} C_r^2\}_{0 \leq \theta < \pi}$ is affine-equivalent to the self-similar sets generated by iterated

function systems $\{rx, rx + 1, rx + u, rx + 1 + u\}_{u > 0}$. We want to have $\dim_{\mathbb{H}} C_r^2 < 1$, so we assume that $r < \frac{1}{4}$. As with the previous Example, we may assume, without loss of generality, that $u \geq 1$. Transversality condition holds by Lemma 2.7. With the appropriate notations we have $\max \Gamma = 1 + u$, $g = \min\{1, u - 1\}$, $G = \max\{1, u - 2\}$, and we can check with conditions (3.18) and (3.19) that $[\frac{1-2r}{r}, \frac{2}{1-3r}] \subset \mathcal{IP}$. This interval is non-empty for $r \in (\frac{1}{6}, \frac{1}{4})$. As in the previous Example we can conclude that for every $r \in (\frac{1}{6}, \frac{1}{4})$ and almost every $u \in [\frac{1-2r}{r}, \frac{2}{1-3r}]$ we have

$$\mathcal{H}^s(\text{proj}_{\theta} C_r^2) = 0 \quad \text{and} \quad 0 < \mathcal{P}^s(\text{proj}_{\theta} C_r^2) < \infty,$$

where s is the similarity dimension $\log 4 / \log r^{-1}$. Moreover, similar to the previous Example, $\mathcal{H}^s(\text{proj}_{\theta} C_r^2) > 0$ for every $r < \frac{1}{9}$ and for almost every θ by Lemma 3.11, since $C_r^2 - C_r^2$ is a planar self-similar set with similarity dimension $\log 9 / \log r^{-r}$.

The case $r \in (\frac{1}{7}, \frac{1}{5})$ in the first example and the case $r \in (\frac{1}{9}, \frac{1}{6})$ in the second example remains open. Anyhow, Peres, Simon and Solomyak [11] suspects that $\mathcal{L}(\mathcal{IP}) > 0$ for almost every such r .

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