ON EFFECTIVE IRRATIONALITY MEASURES FOR SOME VALUES OF CERTAIN HYPERGEOMETRIC FUNCTIONS

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Abstract

The dissertation consists of three articles in which irrationality measures for some values of certain special cases of the Gauss hypergeometric function are considered in both archimedean and non-archimedean metrics.

The first presents a general result and a divisibility criterion for certain products of binomial coefficients upon which the sharpenings of the general result in special cases rely. The paper also provides an improvement concerning the values of the logarithmic function. The second paper includes two other special cases, the first of which gives irrationality measures for some values of the arctan function, for example, and the second concerns values of the binomial function. All the results of the first two papers are effective, but no computation of the constants for explicit presentation is carried out. This task is fulfilled in the third article for logarithmic and binomial cases. The results of the latter case are applied to some Diophantine equations.

Keywords: Diophantine equations, Padé approximation, p-adic, divisibility

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Oulu, March 1997 Ari Heimonen

List of the original articles

I Heimonen A. & Matala-aho T. & Vnnen K. (1993) On irrationality measures of the values of Gauss hypergeometric function. Manuscripta Math. 81: 183–202

II Heimonen A. & Matala-aho T. & Vnnen K. (1994) An application of Jacobi type polynomials to irrationality measures. Bull. Austral. Math. Soc. 50: 225–243

III Heimonen A. (1996) Effective irrationality measures for some values of Gauss hypergeometric function. Math. Univ. Oulu, Preprint 1–39

Contents

1. Introduction

The Gauss hypergeometric function is defined by the series

$$F(z) = {}_2F_1\left(\begin{matrix} a\ ,\ b \\ c \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, \tag{1}$$

where $a, b, c \neq 0, -1, -2, \ldots$ are parameters, and $(a)_0 = 1$, $(a)_n = a(a+1)\ldots(a+n-1)$, $n = 1, 2, \ldots$. For the general properties of $F(z)$, we refer the reader to [17].

There are several important special cases of this function. With the parameters $a = b = 1$, $c = 2$ we obtain the Maclaurin series of the function $\log(1-z)/z$ and the parameters $a = c = 1$, $b = 1/k$ give the binomial series, i.e. the series expansion of $(1-z)^{-1/k}$. In the case $a = 1$, $b = 1/k$, $c = 1 + 1/k$ the values of the series are closely related to the definite integrals

$$\frac{1}{z}\int_0^z \frac{dt}{1 \pm t^k}.$$

These numbers are in many cases representable using arctan, logarithm and square root functions. The case $k = 2$ in particular enables us to consider the properties of some values of the arctan function. We shall refer these cases in the following as *the logarithmic case*, *the binomial case* and *the arctan case*, respectively.

We denote by $|\ |_p$ a valuation of $\mathbb{Q}$, where p is either ∞, corresponding to the usual absolute value, or a prime corresponding to the p-adic valuation normalized by $|p| = p^{-1}$. We shall denote by $\mathbb{Q}_p$ the p-adic completion of $\mathbb{Q}$. In particular $\mathbb{Q}_\infty = \mathbb{R}$. Let $\vartheta \in \mathbb{Q}_p$ be an irrational number, the irrationality of which is usually measured by determining lower bounds

$$|\vartheta - M/N|_p > 1/H^{m(\vartheta)+\varepsilon} \tag{2}$$

which hold for all $\varepsilon > 0$ and $H \geq H_0(\varepsilon)$, where $H = N$, if $p = \infty$, and $H = \max\{|N|, |M|\}$, if p is a prime. The number $m(\vartheta)$ is called an *(asymptotic) irrationality measure* of ϑ and is denoted by $m_{\text{asymp}}(\vartheta)$. A result of this kind is called *effective* if the dependence of H_0 on ε can (at least in principle) be calculated. Alternatively, an effective irrationality measure can be given in the form

$$|\vartheta - M/N|_p > c/H^{m_{\mathrm{eff}}(\vartheta)} \tag{3}$$

for all $H > H_0$, where c and H_0 are explicitly given constants. If the dependences and constants of an irrationality measure result are explicitly given, we say that it is *in effective form.*

It follows from the Dirichlet box principle that the theoretically smallest possible value for $m_{\mathrm{asymp}}(\vartheta)$ is 2. This measure is obtained in some cases, for example $m_{\mathrm{asymp}}(e) = 2$, and is effective (see [7], pp. 107). On the other hand, for Liouville numbers (2) is not true for any finite number $m(\vartheta)$. As examples of measures for transcendental numbers obtained so far we may select $m_{\mathrm{asymp}}(\pi) \leq 8.0161$ [20] and $m_{\mathrm{asymp}}(\zeta(3)) \leq 8.831$ [18].

Liouville proved in 1844 that the irrationality measure of an algebraic number of degree k is $\leq k$, a result that allows explicit effectivization. In the following we shall term an effective irrationality measure of an algebraic number smaller than Liouville's bound *non-trivial.* It is well-known that these non-trivial irrationality measures correspond to the algorithmic solvability of certain Diophantine equations. Thue, Siegel and Roth in turn improved the result of Liouville until $m_{\mathrm{asymp}}(\vartheta) = 2$ was obtained for every algebraic number ϑ, but all these improvements were substantially ineffective (see [7], pp. 66).

The main method used to prove irrationality results is based on finding sequences of rational approximations for numbers such that the sequences converge to these numbers sufficiently fast. More exactly, if we are considering the irrationality of a number ϑ, we try to find a rational approximation sequence (p_n/q_n) such that we have an upper bound for p_n and q_n that does not grow too fast. We should also have a decreasing upper bound for the non-zero remainder term $r_n = |q_n\vartheta - p_n|$. With these tools we can prove the irrationality of a number and also determine an irrationality measure for it.

Most irrationality considerations nowadays handle classes of values of functions represented by power series. In this case the approximation sequences are usually found using rational approximations for these functions. The classical Padé approximation is found by determining a polynomial of degree n such that the product of the series and the polynomial includes a gap of length n, i.e. the coefficients of n successive powers are zero. If we have an approximation with a gap of length $\leq n$ we speak about *Padé type approximation.*

In the papers presented here the main idea is to generalize the classical Padé approximation of the function F in an appropriate way to obtain "arithmetically" better approximations. This means that we try to find a rational approximation with coefficients such that they could be multiplied to integers with as small a multiplier as possible. The improvement obtained in this way is based on the fact that for an appropriate choice of parameters of the generalization the coefficients of the approximation polynomials have a non-trivial common factor. This often involves an "analytical" weaking of the approximation, which means that the analytical bounds for the approximations increase. Thus, this procedure leads to an optimization problem. On account of the complexity of the situation, this is done for each value numerically with a computer.

2. Summary of the original articles

2.1. Notation and the general result

All the papers presented here consider the problem of computing irrationality measures for the values of the function

$$F(z) = {}_2F_1\left(\begin{matrix} 1\ ,\ b \\ c \end{matrix} \middle| \, z \right),$$

the method being based on the Padé type approximation

$$Q_{l,m,n}(z)F(z) - P_{l,m,n}(z) = R_{l,m,n}(z)$$

of the function F derived from the following generalization of classical Jacobi polynomials:

$$A_{l,m,n}(z) = \frac{1}{z^{b-1}(1-z)^{c-b-1}} \left(\frac{d}{dx}\right)^l (z^{n+b-1}((1-z)^{m+c-b-1}).$$

In paper I we first considered the function F in general terms. We now gather together the basic assumptions and notation needed to state our theorems.

We assume that $c > b > 0$, $b = a/f$, $c = g/h$, where a, f, g, h are natural numbers such that $(a, f) = (g, h) = 1$. Let us denote $B = b - 1 = E/F$, $C = c - b - 1 = G/H$ with $E, G \in \mathbb{Z}$, $F, H \in \mathbb{Z}_+$, $(E, F) = (G, H) = 1$. Further, let $L = \text{l.c.m.}(F, H)$, and let H^* denote the denominator of h/H (therefore $H^* \mid H$). We also use the notations

$$\mu_F = \prod_{p \mid F} p^{\frac{1}{p-1}}, \quad \lambda(h) = \frac{h}{\phi(h)} \sum_{\substack{i=1 \\ (i,h)=1}}^{h} \frac{1}{i}.$$

Our general theorem in the complex case reads as follows:

Theorem 1. *If $r/s \in (-1,1)$ is a non-zero rational number satisfying*

$$(r,s)=1, \quad LH^*\mu_L\mu_{H^*}e^{\lambda(h)}\left(\sqrt{s}-\sqrt{s-r}\right)^2<1,$$

then

$$m_{\text{asymp}}\left(F\left(\frac{r}{s}\right)\right) \le 1-\frac{2\ln\left(\sqrt{s}+\sqrt{s-r}\right)+\lambda(h)+\ln(LH^*\mu_L\mu_{H^*})}{2\ln\left|\sqrt{s}-\sqrt{s-r}\right|+\lambda(h)+\ln(LH^*\mu_L\mu_{H^*})}.$$

2.2. Sharpening in the logarithmic and arctan cases based on a divisibility criterion

Theorem 1 is sharpened in the special cases referred to in the Introduction using a technical lemma concerning primes dividing the numbers

$$\binom{n+B}{i}\binom{m+C}{l-i}, \quad i=0,1,\dots,l.$$

With this tool and an appropriate choice of the parameters l, m and n we are able to find cases where the "arithmetic improvement" obtained overcomes the "analytic decline". We are also able to present explicit formulae for the asymptotics of the common factors of the coefficients of the approximation polynomials $Q_{l,m,n}$ and $P_{l,m,n}$ in all three cases. We shall later refer to this technical lemma as the Divisibility Criterion and to the factors described above simply as common factors. The binomial case has some particular features of its own and thus we devote a separate section to it.

In order to present these results, we define for $\beta \ge \alpha$

$$A(\alpha,\beta,z)=\min_{0<\rho<|z|+\frac{1}{2}(1-\operatorname{sgn} z)}\left(\frac{(\rho+|z|)(\rho+|z|-\operatorname{sgn} z)^\beta}{\rho^\alpha}\right)$$

for all $z \ge 1$ or $z<0$,

$$G(t)=G(\alpha,\beta,z,t)=\frac{t(1-t)^\beta}{(1-zt)^\alpha},$$

where $0 \le t \le 1$ and $|z|<1$ or $z=-1$, and

$$R(\alpha,\beta,z)=\max_{0\le t\le 1} G(\alpha,\beta,z,t).$$

We rephrase our result for the logarithmic function by first choosing $n=m$ and $l=[\alpha n]$, where $0<\alpha\le 1$, $\alpha=u/v$, $(u,v)=1$ is a rational optimization parameter. We denote by $D_n(\alpha)$ a common factor of all the numbers

$$\binom{n}{i}\binom{n}{[\alpha n]-i}, \qquad i=0,\dots,[\alpha n].$$

Using our Divisibility Criterion we are able in paper I to determine such a common factor with an asymptotic $D_n(\alpha)\sim e^{n\tau_1(\alpha)}$, where $\tau_1(\alpha)$ has an explicit formula.

Theorem 2. *If*

$$Q(\alpha) = e^{2-\alpha-\tau_1(\alpha)}|r|^{2-\alpha}A\left(\alpha, 1, \frac{s}{r}\right), \quad R(\alpha) = e^{2-\alpha-\tau_1(\alpha)}|r|^{2}s^{-\alpha}R\left(\alpha, 1, \frac{r}{s}\right),$$

then

$$m_{\text{asymp}}\left(\log\left(1-\frac{r}{s}\right)\right) \leq \inf_{\alpha}{}^{*}\left\{1-\frac{\ln Q(\alpha)}{\ln R(\alpha)}\right\},$$

where $\inf^*_\alpha$ *means that for a given non-zero rational* $r/s \in [-1,1)$ *the infimum is taken over all rationals* $\alpha \in (0,1]$ *satisfying* $R(\alpha) < 1$.

The result is effective in principle and is worked it out in an explicitly effective form in paper III, i.e in the form (3). The main point is to use effective knowledge of the prime number functions

$$\theta(x) = \sum_{\substack{p\leq x \\ p \text{ prime}}} \log p, \quad \psi(x) = \sum_{\substack{p^k\leq x \\ p \text{ prime}}} \log p.$$

The transforming of a result of type (2) to one of type (3) corresponds to a fixed choice of ε in (2) and the determination of the corresponding constant and the lower bound for the denominator N involved in (3). In practice not all the choices of ε are possible, for we do not at this moment have effective results on distribution of primes to cover very small epsilons. Effective results concerning distribution of primes are given in [28], [30], [31] and [33].

Now let $0 \leq t_0 \leq 1$ be such that $R(\alpha, 1, z) = G(\alpha, 1, z, t_0)$. With this notation, let

$$R_\delta(\alpha, z) = \min G(t_0 \pm \delta).$$

We denote by L_n a sequence of integers such that the coefficients of the approximation polynomials multiplied by L_n are integers, and we show in paper III that L_n can be chosen such that

$$L_1(\alpha, n_0)^n \leq L_n \leq L_2(\alpha, n_0)^n$$

for all $n \geq n_0$, where n is of the form $vi + \nu$ and $L_i(\alpha, n_0)$ correspond to the effectivization of the term $e^{2-\alpha-\tau_1(\alpha)}$ in Theorem 2, thus involving the common factor and the behaviour of the prime number functions θ and ψ. We set

$$Q\left(\frac{r}{s}, \alpha, n_0\right) = L_2(\alpha, n_0)|r|^{2-\alpha}A\left(\alpha, 1, \frac{s}{r}\right),$$
$$R\left(\frac{r}{s}, \alpha, n_0\right) = L_2(\alpha, n_0)|r|^{2}s^{-\alpha}R\left(\alpha, 1, \frac{r}{s}\right).$$

Theorem 3. *Let a rational number* $0 < \alpha \leq 1$ *be chosen and let* L_n *and* $n_0 \in \mathbb{Z}_+$ *be chosen as described above. Suppose that* $R\left(\frac{r}{s}, \alpha, n_0\right) < 1$ *and that for* r/s *it is possible to choose the numbers* $0 < \varepsilon_1$ *and* $0 < \delta < \min\{t_0, 1-t_0\}$ *such that*

$$L_1(\alpha, n_0)|r|^2 s^{-\alpha} R_\delta\left(\alpha, \frac{r}{s}\right) \geq \left(R\left(\frac{r}{s}, \alpha, n_0\right)\right)^{1+\varepsilon_1}.$$

Then

$$m_{\text{eff}}\left(\frac{s}{r}\log\left(1-\frac{r}{s}\right)\right) = 1 - \frac{\ln Q\left(\frac{r}{s},\alpha,n_0\right)}{\ln R\left(\frac{r}{s},\alpha,n_0\right)} + \varepsilon_1,$$

and the corresponding constants c and N_0 are explicitly determined.

We also determine the numbers $L_i(\alpha, n_0)$ in paper III and show that with certain choices of α and n_0 we obtain improvements of previous effective results for rationals with large denominators. In particular, we observe that $m_{\text{eff}}(\log 2) \leq 4.01$. The corresponding constant in this case is $c = 10^{-20}$ and the bound for the denominator is $N_0 = 10^{4100000}$.

Paper II considers the arctan case and the binomial case. In the arctan case we choose $n = l$ and $m = [\beta n]$, where $\beta \geq 1$ is a rational optimization parameter. We then have $b = 1/k$, $c = 1 + 1/k$, where $k \geq 2$ is a natural number. For a given rational r/s, $(r,s) = 1$, we denote the denominators of $(s-r)/k$ and $(s-r)/(k\prod_{p\,|\,k} p)$ by k^* and k^{**}, respectively. Denote

$$\omega_1(\beta) = k\mu_k \min\left\{\left(\frac{k^*}{k}\right)^{\beta}, \left(\frac{k^{**}}{k\mu_k}\right)^{\beta}\right\} e^{\beta\lambda(k)-\sigma(\beta,k)},$$

$$Q(\beta,k) = \omega_1(\beta)|r|^{\beta} A\left(1,\beta,\frac{s}{r}\right), \quad R(\beta,k) = \omega_1(\beta)|r|^{1+\beta}s^{-1}R\left(1,\beta,\frac{r}{s}\right),$$

where $\sigma(\beta,k)$ corresponds to the common factor of the coefficients of the approximation polynomials, as given in paper II, formula (19).

Theorem 4. *Let $r/s \in (-1,1)$ be a rational number satisfying $(r,s) = 1$. Then*

$$m_{\text{asymp}}\left({}_2F_1\left(\begin{matrix}1\ ,\ \frac{1}{k}\\ 1+\frac{1}{k}\end{matrix}\,\middle|\,\frac{r}{s}\right)\right) \leq \inf_{\beta}{}^{*}\left\{1 - \frac{\ln Q(\beta,k)}{\ln R(\beta,k)}\right\},$$

where $\inf_{\beta}^{*}$ means that for a given r/s the infimum is taken over all rationals $\beta \geq 1$ satisfying $R(\beta,k) < 1$.

In order to compare the results and methods we must review to the history of these cases. The numbers $\log 2$ and $\pi/\sqrt{3}$ have become established as standards for comparison, and follow this tradition. The first results for the logarithm were obtained by Morduchai-Boltowskoj (1923), while Mahler (1932) gave more accurate general results regarding the approximation of logarithms of algebraic numbers by means of algebraic numbers (see [4]). The first explicit irrationality measures for the values of the logarithmic function was obtained by Baker [4], who observed that $m_{\text{eff}}(\log 2) \leq 12.5$ with an explicit constant $c = 10^{-10^5}$. Danilov [13] improved the result to $m_{\text{asymp}}(\log 2) \leq 6.58$ using Laplace transforms, and also found that $m_{\text{asymp}}(\pi/\sqrt{3}) \leq 9.35$.

Alladi and Robinson [1] used Legendre polynomials to obtain results for all three cases. In particular, they achieved the results

$$m_{\text{asymp}}(\log 2) \leq 4.63 \qquad \text{and} \qquad m_{\text{asymp}}(\pi/\sqrt{3}) \leq 8.31.$$

They also give all their results in effective form, from which we select $m_{\text{eff}}(\log 2) \leq 4.871$ with the constant $c = (2000)^{-1}$. Nikišin [27] showed that all the previous considerations could be gathered together and the proofs reduce to a consideration of approximations obtained from Jacobi polynomials.

Huttner [22] considered the arctan case and obtained a general asymptotic result, and also explicitly effective results for some numbers. In particular, he gave new results for some numbers with $k = 3$ and $k = 4$.

Chudnovsky gave several improvements for both cases by generalizing the approximation polynomials. His main idea was to allow a certain degree of freedom for the approximation and then optimize numerically in the cases considered. All the later sharpenings have essentially been variations on this idea. He announced in [11] that $m_{\text{asymp}}(\log 2) \leq 4.135$, while his asymptotic measure for $\pi/\sqrt{3}$ was 5.8174. Rhin [29] gave quite a simple explanation for Chudnovsky's procedure and improved the asymptotic measure of $\log 2$ to 4.0765 by employing more complicated polynomials. His results also covered numbers not lying directly within the scope of hypergeometric treatment, for example $m(\log 3) \leq 13.3$.

In considering the binomial case, Chudnovsky [12] also introduced the important method of common factors. This idea combined with that set out in the preceding paragraph made it possible for Rukhadze and Dubitskas to obtain $m_{\text{asymp}}(\log 2) \leq 3.893$ and $m_{\text{asymp}}(\pi/\sqrt{3}) \leq 5.516$, respectively.

Hata [18] generalized the classical Legendre polynomials and applied them to the values of the hypergeometric function and to some numbers involving higher dimensional integral representations. He formulated his results only for special numbers such as $\log 2$ and $\pi/\sqrt{3}$, for which he obtained the asymptotic measures 3.8914 and 5.0875. The essence of these sharpenings is the same as in Ruchadze and Dubitskas. In [19] he used closely related Legendre type polynomials and considered the numbers

$$ {}_2F_1\left(\begin{matrix} 1\ ,\ 1-a/b \\ 2-a/b \end{matrix} \middle| \, 1/s \right) $$

with s satisfying certain conditions. In this case he also gave an explicit formula for the asymptotics of the common factor of the coefficients of the approximation polynomials.

Our paper I reformulates the idea of finding the common factor of the coefficients of approximation polynomials. Our Divisibility Criterion gives a general method for determining this common factor for the Jacobi type polynomials $A_{l,m,n}$. Thus it is possible for us to give an explicit formula for the asymptotic behaviour of this common factor in logarithmic and arctan cases. We obtain essentially the same results for the numbers $\log 2$ and $\pi/\sqrt{3}$ as Hata [18]. The effective measure for $\log 2$ in paper III improves the result of [1] for rationals with large denominators.

Hata's case in [19] is closely related to a special case of our arctan case, and all the common numerical examples coincide.

We note that our function is a special case of Gauss's hypergeometric ratio

$$G(z) = \frac{{}_2F_1\left(\begin{matrix} a+1\ ,\ b \\ c+1 \end{matrix}\middle| z\right)}{{}_2F_1\left(\begin{matrix} a\ ,\ b \\ c \end{matrix}\middle| z\right)}.$$

The Padé approximation method has been applied to irrationality considerations of the values of this function (see [21] and [41]). Theorem 1 provides sharper measures for certain class of values of the function G with $a = 0$ than do these more general results. The recent paper [23] employs our Divisibility Criterion to obtain sharper results in some other special cases of the function G.

2.3. Sharpening in the binomial case and an application

In our binomial case we have $b = 1/k,\ c = 1$, where $k \geq 3$ is a natural number. For a given $r \in \mathbb{Z}$, let us denote the denominators of

$$r/k \qquad \text{and} \qquad r/(k\prod_{p\,|\,k} p)$$

by k_* and k_{**}, respectively. We assume that $\hat{D}_n$ is a common factor of the numbers

$$\binom{n+\frac{1}{k}}{i}\binom{n-\frac{1}{k}}{n-i}, \qquad i = 0, \ldots, n$$

satisfying $\hat{D}_n \geq e^{n\sigma(n_0,k)}$ for all $n \geq n_0$. Applying our Divisibility Criterion we show in paper II that there exists a sequence $\hat{D}_n$ with an asymptotic $\hat{D}_n \sim e^{n\sigma(\infty,k)}$, where

$$\sigma(\infty,k) = \frac{2}{\phi(k)} \sum_{\substack{k/2<q<k \\ (q,k)=1}} \left(\Psi(1) - \Psi\left(\frac{q}{k}\right)\right)$$

(here Ψ means the digamma function). In paper III we also derive a formula for $\sigma(n_0, k)$ with $n_0 \in \mathbb{Z}_+$.

For $n_0 \in \mathbb{Z}_+ \cup \{\infty\}$ we define

$$\hat{Q}(n_0,k) = \left(\sqrt{s} + \sqrt{s-r}\right)^2 e^{-\sigma(n_0,k)} \min\{k_*\mu_k, k_{**}\},$$

$$\hat{R}(n_0,k) = \left(\sqrt{s} - \sqrt{s-r}\right)^2 e^{-\sigma(n_0,k)} \min\{k_*\mu_k, k_{**}\}.$$

Theorem 5. *If $k \geq 3$ is a natural number and r/s satisfies $\hat{R}(n_0, k) < 1$ for some $n_0 \in \mathbb{Z}_+ \cup \{\infty\}$, then*

$$m\left((1 - r/s)^{-1/k}\right) \leq 1 - \frac{\ln \hat{Q}(n_0, k)}{\ln \hat{R}(n_0, k)},$$

where m means m_{asymp} if $n_0 = \infty$, and m_{eff} if $n_0 \in \mathbb{Z}_+$. In the latter case the constants c and N_0 are explicitly specified for every r/s, k and $n_0 \in \mathbb{Z}_+$ satisfying the conditions of the theorem.

Paper III gives an extensive list of effective irrationality measures and corresponding constants for numbers of the form $\sqrt[k]{D}$. This involves finding an appropriate solution to the Diophantine equation $x^k - Dy^k = K$, i.e. a solution such that K is small with respect to x and y. We show in paper III that solutions to this equation that exceed a certain bound are convergents of the continued fraction expansion of $\sqrt[k]{D}$, and employ this fact in a systematic search for appropriate solutions, finding some cases not explicitly considered in preceding papers on this subject. Direct use of Theorem 5 gives results with N_0 quite large. Easton [15] has presented by means of an example a method involving the continued fractions to eliminate the bound N_0. We provide a general formulation of this method in paper III and employ it to obtain results that are true from $N_0 = 0$ on. Increasing n_0 in Theorem 5 has the effect of improving the measure, but the bound N_0 grows at the same time and the method involves computation of the continued fraction expansion up to N_0. Thus we should choose n_0 in such a way that we obtain as good a measure as possible, while N_0 remains within a range where we can apply the method described above. The new algorithm of Shiu [34] for the calculation of the continued fractions of algebraic numbers made it possible for us to allow the bound N_0 to be of the order of magnitude of 10^{20000}.

These results can be applied to the solution of the Diophantine equation

$$ax^k - by^k = K, \tag{4}$$

usually called *the Thue equation*. More exactly, if we have a non-trivial irrationality measure m for $\sqrt[k]{a/b}$ with an explicit constant c, we obtain an upper bound for the solutions x, y in terms of a, b, k, K, m and c. Some examples of bounds and solutions are also given. For instance, our results give the upper bound 1.3×10^{875} for the solutions of $|x^3 - 5y^3| \leq 100$.

In fact, the history of the approximation of the binomial function is so closely connected with the problem of solving equation (4) that the two questions could not be handled separately. Thue [36], [37], [38], [39] was the first to deduce classical Padé approximations for the binomial function and to apply it to the solution of (4). Thue's ineffective improvement of the theorem of Liouville made it possible for him to restrict the number of solutions of (4) in certain cases. For later results concerning the number of solutions of (4) we refer the reader to [26] and [35].

The other line of investigation also originated by Thue is an attempt to determine upper bounds for the size of the solutions of (4). In fact, Thue's paper [40] includes results which are equivalent to non-trivial effective irrationality measures for the

k:th roots of certain rationals, e.g. for $\sqrt[3]{(a+1)/a}$ with $a \geq 17$ and for $\sqrt[7]{17}$. Our method in paper III for the solution of (4) is essentially that of Thue refined in certain respects, especially by the use the common factor of the approximation polynomials as proposed by Chudnovsky [12].

It was almost half a century before Baker [2], [3] essentially rediscovered the method of Thue based on approximation of the hypergeometric function. Baker found, for example, that $m_{\mathrm{eff}}(\sqrt[3]{2}) \leq 2.955$. Chudnovsky [12] was able to improve on the results of Baker, but he gives his measures only in asymptotic form, and these do not allow immediate effectivization, for this depends on effective knowledge of the distribution of primes in arithmetic progressions, and the polynomial bounds used are deduced from the Lemma of Poincaré. For example, he observed that $m_{\mathrm{asymp}}(\sqrt[3]{2}) \leq 2.4298$. Chudnovsky also gave asymptotic bounds for some equations (4) with higher degrees, but without explicit proofs. The results of McCurley [25] made it possible for Easton [15], [16] to work out some cases handled by Chudnovsky in explicitly effective form. One example of his results is $m_{\mathrm{eff}}(\sqrt[3]{2}) \leq 2.795$ with the constant $c = 2.2 \times 10^{-8}$ in (3).

A second method for computing irrationality measures for algebraic numbers or, equivalently, for determining upper bounds for the solutions of (4), is Baker's method involving linear forms of logarithms. With this approach Baker became the first to obtain a general effective improvement of the result of Liouville. First he [5] observed that the bound

$$\left|\vartheta - \frac{M}{N}\right| > cN^{-k}e^{(\log N)^{1/\kappa}},$$

where $\kappa > k+1$ and $c = c(\vartheta, \kappa)$, holds for every algebraic number ϑ of degree $k \geq 3$ and finally he [6] gave an improvement with a constant diminution of the Liouville bound. These results imply that the solutions of (4) are below a certain bound, and thus give a general algorithm for its solution. In many cases the algorithm works only in principle since the bounds are very large. All the improvements for special classes of algebraic numbers imply better bounds for solution of (4) in corresponding cases. This is perhaps the main ground for interest in this case. Baker and Stewart [8] also obtained using the linear forms of logarithms that $m(\sqrt[3]{5}) \leq 2.99999999999998$.

A third possible approach to this problem may be attributed to Bombieri and Mueller [9], who used their method involving the box principle to obtain a result for certain numbers of the form $\sqrt[k]{a/b}$. If a and b are large numbers satisfying conditions of a certain type, their result compares favourably to those obtained by the hypergeometric method.

Our paper II generalizes the results of Chudnovsky [12], and agrees with them in all the examples given by him. In particular, we give an explicit formula for the common factor in both the asymptotic and the effective case for all $k \geq 3$. We also give the result $m_{\mathrm{asymp}}(\sqrt[3]{5}) \leq 2.7636$ in paper III, which improves on that of Baker [8]. A systematic search with continued fractions also leads to some other new non-trivial measures, but there are still numbers for which the hypergeometric method seems not to give non-trivial measures, for example $\sqrt[3]{14}$.

In paper III we use the recent results of Ramaré and Rumely [28] on the distribution of primes, which are almost throughout sharper than those of McCurley [25]. On account of this and on our more accurate treatment of the common factor and related matters we are able to improve on the results of Easton for all the numbers presented in [15] and [16] and are also able to consider some cube roots not occuring in Easton's papers. As for higher roots of integers, our results provide the first explicitly presented irrationality measures in both asymptotic (paper II) and effective form (paper III).

2.4. The p-adic case

We also obtained in paper I a general p-adic result on the effective irrationality measures of the values of $F(z)$.

Theorem 6. *Suppose that p is a prime such that $p \nmid fh$. If $r/s > 1$ is a rational number satisfying*

$$|r/s|_p < 1, \quad (r,s) = 1, \quad LH^* \mu_L \mu_{H^*} e^{\lambda(h)} r |r|_p^2 < 1,$$

then

$$m_{\text{asymp}}\left(F\left(\frac{r}{s}\right)\right) \leq \frac{2 \ln |r|_p}{2 \ln |r|_p + \ln r + \lambda(h) + \ln(LH^* \mu_L \mu_{H^*})}$$

Theorem 6 enables us to deduce p-adic results for the logarithmic and arctan cases. We note that attempts to improve these results by the method used in the complex case lead in these cases to difficulties in proving that the remainder is not zero.

On the other hand, we were able to make use of the common factor in the p-adic binomial case. We recall that the asymptotic of this common factor relates to $\sigma(\infty, k)$ given above. The result is effective.

Theorem 7. *Let $p \nmid k$ be a prime and let us assume $|r/s|_p < 1$.*
1) If $r/s > 1$ and

$$\min\{k_* \mu_k, k_{**}\} e^{-\sigma(\infty,k)} r |r|_p^2 < 1,$$

then

$$m_{\text{asymp}}\left(\left(1 - \frac{r}{s}\right)^{-1/k}\right) \leq \frac{2 \ln |r|_p}{2 \ln |r|_p + \ln r - \sigma(\infty, k) + \ln \min\{k_* \mu_k, k_{**}\}}.$$

In particular, we have

$$m_{\text{asymp}}\left(\left(1 - p^l\right)^{-1/k}\right) \leq \frac{2l \ln p}{l \ln p + \sigma(\infty, k) - \ln k \mu_k}$$

for all $p^l > k \mu_k e^{-\sigma(\infty,k)}$.

2) If $r/s<1$ *and*

$$\min\{k_*\mu_k, k_{**}\}e^{-\sigma(\infty,k)}\left(\sqrt{s}+\sqrt{s-r}\right)^2|r|_p^2<1,$$

then

$$m_{\text{asymp}}\left(\left(1-\frac{r}{s}\right)^{-1/k}\right)\le \frac{2\ln|r|_p}{2\ln|r|_p+2\ln|\sqrt{s}+\sqrt{s-r}|-\sigma(\infty,k)+\ln\min\{k_*\mu_k, k_{**}\}}.$$

The p-adic parts of paper I and II are among the first works to discuss p-adic Diophantine approximation of the values of the hypergeometric function, and the results represent a major improvement of those of the earlier general paper on this subject [24]. Likewise, our theorem improves on the earlier results of Bundschuh [10] for the p-adic binomial function. As a numerical example, we may mention $m_{\text{asymp}}(\sqrt[3]{3})\le 2.4597$ in the 2-adic case.

3. References

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