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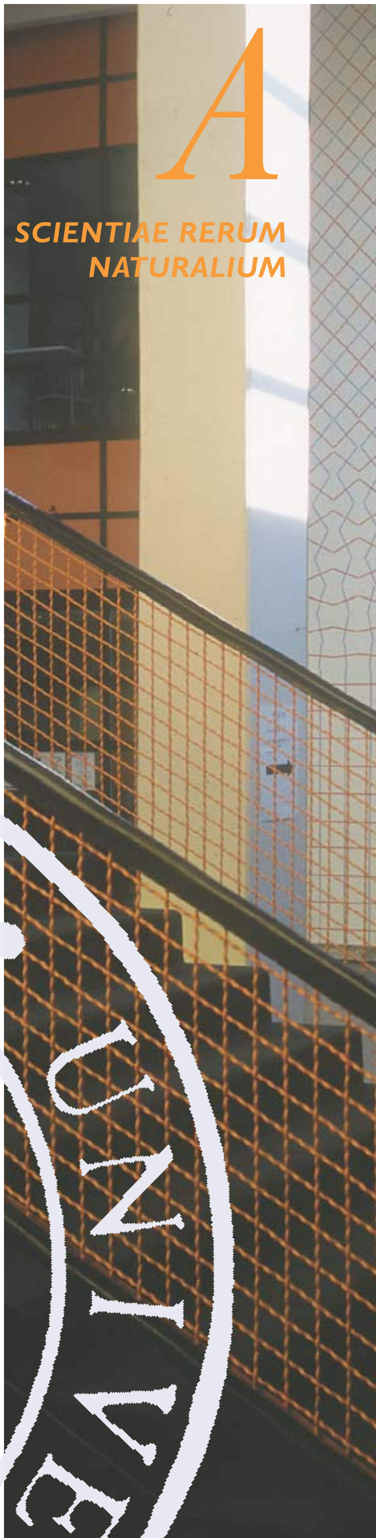

Teemu Tyni

DIRECT AND INVERSE SCATTERING PROBLEMS FOR PERTURBATIONS OF THE BIHARMONIC OPERATOR

*UNIVERSITY OF OULU GRADUATE SCHOOL;
UNIVERSITY OF OULU,
FACULTY OF SCIENCE*

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DIRECT AND INVERSE SCATTERING PROBLEMS FOR PERTURBATIONS OF THE BIHARMONIC OPERATOR

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Abstract

This dissertation is a combination of four articles on the topic of scattering problems for a biharmonic operator. The operator of interest has two coefficients which may be complex-valued and singular. Each of the articles concerns a different aspect of the problem. Namely, the first article discusses the direct scattering problem in higher dimensions and culminates in a proof of Saito's formula, which yields a uniqueness result for the inverse scattering problem. The second paper is about a backscattering problem in two and three dimensions. We prove that the inverse Born approximation can be used to recover the singularities in the coefficients of the operator. The third article fills in an answer to the question about recovering the complex-valued coefficients in three dimensions that was left open in the second article. The final article studies the inverse scattering problem on the line for a quasi-linear operator.

Keywords: biharmonic operator, Born approximation, inverse problem, scattering theory

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Tiivistelmä

Väitöskirjatyö koostuu neljästä artikkelista, jotka käsittelevät sirontaongelmia biharmoniselle operaattorille. Työn kohteena olevalla operaattorilla on kaksi kerrointa, jotka voivat olla kompleksiarvoisia ja singulaarisia. Kukin artikkeli käsittelee sirontaongelmaa eri näkökulmasta. Ensimmäinen artikkeli koostuu pääasiassa suorasta sirontateoriasta korkeammissa ulottuvuuksissa huipentuen lopulta Saiton kaavan todistukseen, jonka seurauksena saadaan yksikäsitteisyystulos käänteiselle sirontaongelmalle. Toisen artikkelin aiheena on takaisinsirontaongelma kahdessa ja kolmessa ulottuvuudessa. Todistamme, että käänteistä Bornin approksimaatiota voidaan käyttää paikantamaan kertoimien mahdolliset singulariteetit. Kolmas artikkeli vastaa toisessa artikkelissa avoimeksi jääneeseen kysymykseen kompleksiarvoisien kertoimien rekonstruoimisesta kolmessa ulottuvuudessa. Viimeisessä artikkelissa tutkitaan käänteistä sirontaongelmaa kvasilineaariselle operaattorille yhdessä ulottuvuudessa.

Asiasanat: biharmoninen operaattori, Bornin approksimaatio, käänteiset ongelmat, sirontateoria

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I wish to express my deepest gratitude to my two supervisors, professor Valery Serov and adjunct professor Markus Harju. Professor Valery Serov brought me into the field of inverse problems and has since then been offering me with invaluable guidance and constant encouragement to try new things. I am sincerely grateful to my second scientific supervisor, adjunct professor Markus Harju, for the numerous discussions and all the support and patience. It has been a great pleasure to complete this thesis with you.

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Oulu, September 2018 Teemu Tyni

List of original articles

This dissertation consists of an introductory part and the following original articles:

I Tyni T & Serov V (2018) Scattering problems for perturbations of the multidimensional biharmonic operator. *Inverse Prob. Imag. 12*(1): 205-227.
II Tyni T & Harju M (2017) Inverse backscattering problem for perturbations of biharmonic operator. *Inverse Probl. 33*: 105002.
III Tyni T (2018) Recovery of singularities from a backscattering Born approximation for a biharmonic operator in 3D. *Inverse Probl. 34*: 045007.
IV Tyni T & Serov V (2018) Inverse scattering problem for quasi-linear perturbation of the biharmonic operator on the line. *In press.*

Author's contribution

Articles I, II and IV were developed jointly by the author and the co-authors, V. Serov and M. Harju, while paper III is author's own research. The majority of article I is written by the author. The author is responsible for the theoretical part of paper II and in small part for the numerics. The author's contribution to article IV is over one half. The numerical example in article IV is done by the author based on M. Harju's earlier work.

Contents

1 Introduction

In inverse scattering problems the objective is to locate and identify the unknown target from the way it scatters incoming waves or particles. This kind of problem has important applications in, for example, medical imaging, radar applications, seismic prospecting, non-destructive material testing and nuclear physics. The waves used in possible applications can be as diverse as acoustic waves in ultra sound imaging, radio waves in radar and electromagnetic and elastic waves in material testing and geophysical applications.

Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ denote the Laplace operator in $\mathbb{R}^n$. Classical examples of operators that are used to describe scattering include the Helmholtz operator $-\Delta - \lambda$, the Schrödinger operator $H := -\Delta + V$ and the magnetic Schrödinger operator $H_{\mathrm{m}} := -(\nabla + \mathrm{i}\vec{W})^2 + V$. These operators are second-order differential operators and they appear in, for example, quantum mechanics and obstacle scattering.

This work concerns scattering problems for perturbations of the fourth-order differential operator, henceforth called the biharmonic operator, Δ^2. Equations involving the biharmonic operator arise naturally, for example, in the study of vibrations of beams and in the theory of elasticity [15]. As an example, in the context of beams the quantities u, ∇u and Δu have certain important physical interpretations. Here u is the displacement of the beam, ∇u is related to the slope of the beam and Δu is the bending moment. The third-order derivative is known as the shear force.

The aim of this dissertation is to study both the direct and the inverse scattering problems for the operator

$$H_4 := \Delta^2 + \vec{q} \cdot \nabla + V,$$

where ∇ is the gradient operator and $\vec{q}$ and V are vector- and scalar-valued functions of the spatial coordinates, respectively. In this work when we speak of a scatterer, we collectively mean the functions $\vec{q}$ and V. Generally, the term scatterer can have different meanings in different contexts. Examples of scatterers in practise might be a tumor in the lungs of a patient or a crack in the supporting pillar of a bridge. The direct scattering problem is to solve what kind of reflection does a given scatterer yield, or more mathematically, to find a suitable solution u to a certain differential equation involving H_4. Conversely, the inverse scattering problem is to recover the operator H_4 (or the coefficients of H_4) from suitable measurements, the scattering data.

A recurring theme in this dissertation is to obtain results for possibly complex-valued coefficients. This generality makes some calculations technically more involved than in the real-valued case.

1.1 Notation

Let $1 \leq p < \infty$. We define the Lebesgue spaces as

$$L^p(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is measurable and } \int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x < \infty \right\}$$

and

$$L^\infty(\mathbb{R}^n) := \{ f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is measurable and essentially bounded} \} .$$

The norms in the Lebesgue spaces are given by

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|$$

and they make these spaces into Banach spaces. In the special case $p = 2$ the Lebesgue space $L^2(\mathbb{R}^n)$ is also a Hilbert space. We often require weighted Lebesgue spaces, which we denote by $L^p_\delta(\mathbb{R}^n)$, where $\delta \in \mathbb{R}$, defined by finiteness of the norm $\|f\|_{L^p_\delta(\mathbb{R}^n)} := \|(1 + |x|^2)^{\frac{\delta}{2}} f\|_{L^p(\mathbb{R}^n)}$. Let S' denote the collection of all tempered distributions (see, e.g. [18, 22]) and let $k \in \mathbb{N}$. The Sobolev spaces $W^k_p(\mathbb{R}^n)$ are the spaces of those tempered distributions, whose weak derivatives up to order k belong to $L^p(\mathbb{R}^n)$, i.e.

$$W^k_p(\mathbb{R}^n) := \left\{ f \in S' \mid \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} < \infty \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

In this dissertation the Fourier transform pair of f is defined by the formulae

$$\widehat{f}(\xi) := F(f)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(x,\xi)} f(x) \mathrm{d}x$$

and

$$F^{-1}(f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,\xi)} f(\xi) \mathrm{d}\xi,$$

where the symbol $(x, \xi) := \sum_{i=1}^n x_i \xi_i$ denotes the usual real inner product of vectors $x, \xi \in \mathbb{R}^n$. These formulae are defined on the Schwartz class of rapidly decaying functions and can be extended to tempered distributions by duality. If $s \in \mathbb{R}$ we define the Sobolev spaces $W_p^s(\mathbb{R}^n)$ as

$$W_p^s(\mathbb{R}^n) := \{f \in S' \mid F^{-1}((1+|\cdot|^2)^{\frac{s}{2}} \widehat{f}) \in L^p(\mathbb{R}^n)\}.$$

We say that f belongs to the weighted Sobolev space $W_{p,\delta}^s(\mathbb{R}^n)$ if $(1+|x|^2)^{\frac{\delta}{2}} f \in W_p^s(\mathbb{R}^n)$. In the case $p=2$ we write $H^s(\mathbb{R}^n) := W_2^s(\mathbb{R}^n)$. Finally, the letter C will be used to denote a generic constant whose value can change from line to line.

1.2 Direct scattering problem

In the rest of this chapter we will work in a space of dimension $n \geq 2$ as the 1D-case requires some minor changes. The classical scattering theory for Schrödinger operators is quite well-understood and some of the basic literature on the subject include the monographs [9, 11, 13]. To learn more about general scattering theory of differential operators with constant coefficients see the classic book of Hörmander [19].

We study the direct scattering problem for the perturbed biharmonic operator given by the equations

$$H_4 u = k^4 u, \quad k > 0, \quad u = u_0 + u_{\mathrm{sc}}, \quad u_0(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)}, \tag{1}$$

where $\theta \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}$ is the incident angle. Here $k \in \mathbb{R}$ is often called the wavenumber. The scattered wave u_{sc} is required to be outgoing in the following sense

$$\begin{cases} \frac{\partial u_{\mathrm{sc}}}{\partial |x|} - \mathrm{i}k u_{\mathrm{sc}} = o\left(|x|^{-\frac{n-1}{2}}\right), \\ \frac{\partial \Delta u_{\mathrm{sc}}}{\partial |x|} - \mathrm{i}k \Delta u_{\mathrm{sc}} = o\left(|x|^{-\frac{n-1}{2}}\right), \end{cases}$$

as $|x| \to +\infty$. These conditions are an analogue of the Sommerfeld radiation condition (see Section 28 of [48]) for the operator H_4. Next, we reduce the scattering problem (1) into an integral equation. We start by rearranging the equation $H_4 u = k^4 u$ as

$$\Delta^2 u - k^4 u = -\vec{q} \cdot \nabla u - V u. \tag{2}$$

Then it is possible to show that the function

$$G_k^+(|x|) = \frac{\mathrm{i}}{8k^2} \left(\frac{|k|}{2\pi|x|}\right)^{\frac{n-2}{2}} \left(H_{\frac{n-2}{2}}^{(1)}(|k||x|) + \frac{2\mathrm{i}}{\pi} K_{\frac{n-2}{2}}(|k||x|)\right)$$

is a fundamental solution to the operator $\Delta^2 - k^4$ in $\mathbb{R}^n$ ($k \neq 0$), that is, $(\Delta^2 - k^4) G_k^+ = \delta_0$ in the sense of distributions. Here $H_{\frac{n-2}{2}}^{(1)}$ and $K_{\frac{n-2}{2}}$ are the Hankel function of first kind and the Macdonald function of orders $\frac{n-2}{2}$ and δ_0 is the delta distribution. This fundamental solution is outgoing in the sense that it satisfies the above radiation conditions. By using theory of differential equations (see e.g. [18]) we write (2) as the integral equation

$$\begin{aligned} u(x,k,\theta) &= u_0(x,k,\theta) \\ &\quad - \int_{\mathbb{R}^n} G_k^+(|x-y|)\,[\vec{q}(y)\cdot\nabla u(y,k,\theta)+V(y)u(y,k,\theta)]\,\mathrm{d}y. \end{aligned} \tag{3}$$

In the context of the Schrödinger operator equation (3) is often called the Lippmann-Schwinger integral equation and we use the same naming convention here.

To find a solution to the direct scattering problem we apply estimates for the resolvent

$$(\Delta^2 - k^4 - \mathrm{i}0)^{-1} := \lim_{\varepsilon\to 0^+}(\Delta^2 - k^4 - \mathrm{i}\varepsilon)^{-1}.$$

In order to define this operator we first note that the spectrum of Δ^2 with domain $H^4(\mathbb{R}^n)$ is $\sigma(\Delta^2) = [0,\infty[$. This fact allows us to define the resolvent operator $(\Delta^2 - k^4 - \mathrm{i}\varepsilon)^{-1}$ as a bounded operator from $L^2(\mathbb{R}^n)$ to $H^4(\mathbb{R}^n)$ for any $\varepsilon > 0$. The limiting procedure as $\varepsilon \to 0^+$ is usually called the limiting absorption principle [1] and there is no reason to expect that the limiting operator exists on $L^2(\mathbb{R}^n)$. However, due to the famous estimates by S. Agmon (see Appendix A of [1]), one can prove that this operator exists in the uniform operator topology of $L^2_\delta \to H^2_{-\delta}$ for $\delta > \frac{1}{2}$ with good norm estimates.

Under certain technical integrability and smoothness assumptions we can show that for a fixed and sufficiently large $k > 0$ the Lippmann-Schwinger integral equation (3) has a unique solution. Further, this solution has the asymptotic form

$$u(x,k,\theta) = u_0(x,k,\theta) + C_n \frac{k^{\frac{n-7}{2}}\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}} A(k,\theta,\theta') + o\left(|x|^{-\frac{n-1}{2}}\right), \quad |x| \to \infty,$$

for fixed $k > 0$. Here $\theta' := x/|x|$ is the direction of the observation (measurement or receiver angle) and

$$A(k,\theta,\theta') = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(y,\theta')}\,[\vec{q}\cdot\nabla u + Vu]\,\mathrm{d}y$$

is known as the scattering amplitude. This scattering amplitude is a quantity which we can measure in practise.

1.3 Inverse scattering problem

Broadly speaking the inverse scattering problem for an operator H_4 is the recovery of its unknown coefficients from measurements made far away from the scatterer. According to J. Hadamard a problem qualifies as well-posed if the problem has a unique solution which depends continuously on the data. Inverse scattering problems are known to be ill-posed, and with some limited measurement data often we cannot hope to fully describe the unknown scatterer. Instead, we settle for partial reconstructions of the unknowns.

There exists a large amount of literature on inverse scattering problems for the Schrödinger operator and the magnetic Schrödinger operator and we mention here [1, 9, 11, 32–34] and the references therein. Some of the popular methods to study inverse scattering problems for the Schrödinger operators include the Born approximation [34], the linear sampling developed by Colton and Kirsch in [10] and continued in [12], the singular source method [35] and the Kirsch factorization method [23].

Along with these different methods, one can also study different data sets and thus obtain very different problems. Here we would like to recover the scatterer which depends on n independent variables. The full data problem, where the scattering amplitude is known in all directions and for all wavenumbers, contains $2n-1$ independent variables and is formally over-determined in dimensions $n>1$ whence one is naturally led to study more limited data. For example, by fixing the measurement or the observation angle one obtains the so-called inverse fixed angle scattering problem [14, 42, 45]. Other data types include fixed energy data, where only one incident wavenumber k (energy) is used [28, 29, 32, 46] and backscattering data, where the measurement is made at the opposing angle of the incident wave [31, 41, 43]. It seems that the study of backscattering data for Schrödinger operators gained popularity after the publication of the 1956 article by Moses [27] and the series of papers by Prosser [36–39] from 1969–1982. Some of the above problems deal with more general operators with non-linear coefficients (e.g. [14, 46]). It is also possible to consider energy-dependent coefficients to model situations where the wave speed depends on location [2, 3].

Recently, inverse problems for bi- and polyharmonic operators have received more attention. Examples include [5, 6, 25, 26] and [24] where the problem of determining the perturbations of the polyharmonic operator $(-\Delta)^m$ and the biharmonic operator Δ^2 from the Dirichlet-to-Neumann map have been studied. In the same spirit, in [47] it is proved that a Dirichlet-to-Neumann map uniquely determines the coefficients of the biharmonic operator up to a second order perturbation. However, to the best of the author's knowledge inverse scattering problems for biharmonic operators are not so common in the literature. The author is aware of [20, 21]. We also mention [4] for a study of the time-evolution of scattering data.

In this thesis we assume that there exists at least one solution to the inverse scattering problem, i.e. the scattering amplitude corresponds to at least one scatterer. Our results then deal mostly with the uniqueness and recovery of this scatterer. We approach the inverse scattering problem via method known as the inverse Born approximation, following [34]. Analysis of the scattering amplitude reveals a connection to the Fourier transform: by substituting in the formula of the scattering amplitude A the first-order Born approximation $u \approx u_0$ we find that

$$A(k,\theta,\theta') \approx \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}k(\theta-\theta',y)} \left[\mathrm{i}k\theta \cdot \vec{q}(y) + V(y)\right] \mathrm{d}y =: A_{\mathrm{B}}(k,\theta,\theta').$$

Especially in the case of backscattering data, where $\theta' = -\theta$, we have by the divergence theorem that

$$A(k,\theta,-\theta) \approx (2\pi)^{\frac{n}{2}} F^{-1}\left(-\frac{1}{2}\nabla\cdot\vec{q} + V\right)(2k\theta).$$

Heuristically, the Fourier transform interchanges the local smoothness of a function with decay at infinity on the frequency side. Thus the large k asymptotics of A should contain the jump discontinuities and singularities of the scatterer. With this interpretation in mind we show in Articles II–III and IV that by using the inverse Born approximation defined as the Fourier transform of the backscattering amplitude indeed the local singularities of the combination $\beta := -\frac{1}{2}\nabla\cdot\vec{q} + V$ can be recovered. The main idea is to show that the difference between the inverse Born approximation and the precise unknown is smoother than the unknown itself, for example in the scale of Sobolev spaces. We remark that the precise definition of the inverse Born approximation depends on the problem at hand, as already seen above.

Another important aspect in inverse scattering problems is the uniqueness of the solution to the problem, that is, whether the scattering amplitude leads to a unique scatterer. For scattering data containing all incoming and observation angles for arbitrarily high frequencies we have Saito's formula (cf. article I) named after Yoshimi Saitō, who used the formula to prove a uniqueness theorem for the inverse scattering problem for the Schrödinger equation [44]. This formula also gives an affirmative answer to the question whether the scattering amplitude corresponds to a unique combination of the coefficients of H_4 when $n \geq 2$.

To conclude this section we mention some limitations of our results. For one, it is known that for Schrödinger operator $-\Delta + V$ the Born approximation A_{B} does not correspond to an exact scattering data set A. This is shown, e.g. in [40] by demanding the equality $A(k,\theta,\theta') = A_{\mathrm{B}}(k,\theta,\theta')$ and then deducing that this is only possible when

$V = 0$. It can also be mentioned that one can try to obtain an analogue of Saito's formula also in the case $n = 1$ by replacing the integrals over the unit spheres with sums over the two possible directions – left and right – on the line and the scattering amplitude with suitable transmission and reflection coefficients. However, for $n = 1$ this formula does not yield a uniqueness result and as such is not very useful. It is also important to keep in mind that all of our recovery results correspond to the unique combination β of the coefficients $\vec{q}$ and V. Without some a priori knowledge about these coefficients $\vec{q}$ and V we cannot tell them apart by simply looking at the reconstruction.

2 Summaries of the original articles

2.1 Article I

The main theme of this article is to study the classical direct scattering theory for operator H_4 in dimensions $n \geq 2$ and finally provide a uniqueness result for the inverse scattering problem in the form of Saito's formula. We start by enforcing the connection between equations (1) and (3).

Theorem 2.1.1. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. If $u = u_0 + u_{\mathrm{sc}}$, $u_{\mathrm{sc}} \in H^4_{\mathrm{loc}}(\mathbb{R}^n) \cap H^2_{-\delta}(\mathbb{R}^n)$, solves* (1) *then it also solves* (3).

The converse of the above theorem can be understood by noting that a solution to the Lippmann-Schwinger integral equation (3) provides a solution to (1) in the sense of distributions. Instead of (3) at this point it is more convenient to study the equivalent integral equation

$$\begin{aligned} u_{\mathrm{sc}} &= -\int_{\mathbb{R}^n} G_k^+(|x-y|)\,[\vec{q}\cdot\nabla(u_0+u_{\mathrm{sc}}) + V(u_0+u_{\mathrm{sc}})]\,\mathrm{d}y \\ &= \widetilde{u_0} - \int_{\mathbb{R}^n} G_k^+(|x-y|)\,[\vec{q}\cdot\nabla u_{\mathrm{sc}} + V u_{\mathrm{sc}}]\,\mathrm{d}y =: \widetilde{u_0} + L_k u_{\mathrm{sc}}, \end{aligned}$$

where $\widetilde{u_0} := L_k u_0$. The solution to this integral equation is obtained as a Neumann series as follows.

Theorem 2.1.2. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. Then there exists a constant $k_0 > 1$ such that the function $u_{\mathrm{sc}}(x,k,\theta)$ defined by the Neumann series*

$$u_{\mathrm{sc}}(x,k,\theta) = \sum_{j=0}^{\infty} L_k^j \widetilde{u_0}(x,k,\theta)$$

solves the integral equation $u_{\mathrm{sc}} = \widetilde{u_0} + L_k u_{\mathrm{sc}}$ uniquely in $H^2_{-\delta}(\mathbb{R}^n)$, when $k > k_0$.

Actually, in the important special cases when $n = 2$ or 3 we can show that the solution u belongs to the Sobolev space $W^1_\infty(\mathbb{R}^n)$. As another consequence we also obtain a mapping property for the resolvent operator of H_4.

Corollary 2.1.3. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. Then the operator*

$$\widehat{G_{\mathrm{p}}} := \lim_{\varepsilon\to 0^+} (H_4 - k^4 - \mathrm{i}\varepsilon)^{-1}$$

exists in the uniform operator topology from $L^2_\delta(\mathbb{R}^n)$ to $H^1_{-\delta}(\mathbb{R}^n)$ with the norm estimates

$$\|\widehat{G_{\mathrm{p}}} f\|_{H^j_{-\delta}(\mathbb{R}^n)} \leq \frac{C}{k^{3-j}} \|f\|_{L^2_\delta(\mathbb{R}^n)}, \quad j = 0, 1,$$

for sufficiently large $k > 0$.

For fixed $k > 0$ the solution u to the Lippmann-Schwinger integral equation has the asymptotic behaviour

$$u(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(\theta,x)} - C_n \frac{k^{\frac{n-7}{2}} \mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}} A(k,\theta,\theta') + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \to \infty,$$

where $\theta' \in \mathbb{S}^{n-1}$ is the angle of measurement and the function

$$A(k,\theta,\theta') = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta',y)} \left[\vec{q} \cdot \nabla u + Vu\right] \mathrm{d}y$$

is called the scattering amplitude. In fact, if the coefficients $\vec{q}$ and V are compactly supported we can improve the behaviour of the remainder term to $O\left(|x|^{-\frac{n+1}{2}}\right)$. While this better behaviour is not needed for our purposes, it may have some independent interest.

To conclude this part we prove an analogue of Saito's formula.

Theorem 2.1.4 (Saito's formula)**.** *Assume that $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. Then the limit*

$$\lim_{k\to\infty} k^{n-1} \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} A(k,\theta,\theta') \mathrm{d}\theta \mathrm{d}\theta' = 2^n \pi^{n-1} \int_{\mathbb{R}^n} \frac{\beta(y)}{|x-y|^{n-1}} \mathrm{d}y$$

holds uniformly in x. Recall that $\beta := -\frac{1}{2}\nabla \cdot \vec{q} + V$.

The proof of this theorem is technical. It is based on the asymptotic behaviour of Bessel functions (see for example [49]) and the mapping properties of the operator $\widehat{G_{\mathrm{p}}}$ combined with the Sobolev embedding theorems.

Observe that the right-hand side of Saito's formula can be considered as a convolution operator which maps $L^p_{2\delta}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. This operator has trivial kernel in the sense of distributions and consequently we obtain the following result.

Corollary 2.1.5 (Uniqueness)**.** *Let $\vec{q}_1, V_1$ and $\vec{q}_2, V_2$ be as in Theorem 2.1.4. If the corresponding scattering amplitudes for these coefficients coincide for some sequence $k_j \to \infty$ then the corresponding coefficients β_1 and β_2 are equal in the sense of distributions.*

Remark 2.1.6. Since β is locally integrable (in fact, integrable) the equality in Corollary 2.1.5 holds not only in the sense of distributions but also almost everywhere.

The inversion of Saito's formula also yields the following representation formula.

Corollary 2.1.7 (Representation formula)**.** *Under the same assumptions as in Theorem 2.1.4 we have*

$$\beta(x) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2^{n+1}\pi^{\frac{3n-1}{2}}} \lim_{k\to\infty} k^{n-1} \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} A(k,\theta,\theta')|\theta-\theta'|\mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\mathrm{d}\theta\mathrm{d}\theta'$$

in the sense of tempered distributions. Here Γ *is the gamma function.*

Remark 2.1.8. The combination β can be discontinuous and it can contain infinite singularities, since locally it is just an L^p-function. In the sequel we consider the recovery of these possible jumps and singularities.

2.2 Article II

The second article is about the backscattering problem for perturbations of the biharmonic operator. The backscattering data is obtained by taking the measurement in the opposing angle of the incident wave, that is, $\theta' = -\theta$. This data type is very natural and important because it corresponds to measuring reflections (or echoes): the receiver is in the same direction as the emitter. The choice $\theta' = -\theta$ also simplifies certain formulae considerably.

We choose to study the inverse Born approximation for this data type. For technical reasons we define the scattering amplitude as $A(k,\theta,-\theta) = 0$ if $0 < k < k_0$, where $k_0 > 0$ is sufficiently large (cf. Theorem 2.1.2). By substituting the first-order Born approximation $u(x,k,\theta) \approx u_0(x,k,\theta)$ into the formula of the scattering amplitude and using $\theta' = -\theta$ we get

$$A(k,\theta,-\theta) \approx \int_{\mathbb{R}^n} \mathrm{e}^{2\mathrm{i}k(\theta,y)}\left[\mathrm{i}k\theta\cdot\vec{q}(y)+V(y)\right]\mathrm{d}y = (2\pi)^{\frac{n}{2}}F^{-1}(\beta)(2k\theta), \tag{4}$$

where we applied the divergence theorem in the first term and used $\beta = -\frac{1}{2}\nabla\cdot\vec{q}+V$. This approximation suggests the backscattering Born approximation q_{B} of β can be defined as

$$q_{\mathrm{B}}(x) := \frac{1}{(2\pi)^n}\int_0^\infty k^{n-1}\int_{\mathbb{S}^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta,x)}A\left(\frac{k}{2},\theta,-\theta\right)\mathrm{d}\theta\mathrm{d}k.$$

By substituting in the definition of the scattering amplitude the Neumann series $u = u_0 + L_k u_0 + \sum_{j=2}^\infty L_k^j u_0$, where L_k is as above and $L_k^j = L_k(L_k^{j-1})$, $j \geq 2$, then the backscattering Born approximation can also be expressed as $q_{\mathrm{B}} = q_0 + q_1 + q_{\mathrm{rest}}$. Here q_1 and q_{rest} correspond to the terms $L_k u_0$ and $\sum_{j=2}^{\infty} L_k^j u_0$, respectively. Then by changing to Cartesian coordinates as $y = k\theta$ and using Fourier inversion in (4) we see that

$$q_0(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{2k_0}^{\infty} k^{n-1} \int_{\mathbb{S}^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta,x)} F^{-1}(\beta)(k\theta)\mathrm{d}\theta\mathrm{d}k = \beta(x) + \widetilde{q}(x),$$

where

$$\widetilde{q}(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{0}^{2k_0} k^{n-1} \int_{\mathbb{S}^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta,x)} F^{-1}(\beta)(k\theta)\mathrm{d}\theta\mathrm{d}k \in C^{\infty}(\mathbb{R}^n)$$

as the Fourier transform of a compactly supported distribution.

Lemma 2.2.1. *Let $n \geq 2$, $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $n < p \leq \infty$ and $2\delta > n - \frac{n}{p}$. Then $q_{\mathrm{rest}} \in H^s(\mathbb{R}^n)$ for any $s < \frac{6-n}{2}$.*

Usually the main difficulty in the inverse Born approximation method is to obtain good enough estimates for the first nonlinear term q_1 (also known as the first quadratic term [7] or the bilinear term [31]). In this text our approach is to split the fundamental solution G_k^+ into two parts as

$$G_k^+ = G_k^{\mathrm{M}} + G_k^{\mathrm{E}},$$

where G_k^{M} is the oscillating main part corresponding to the Hankel function and the exponentially decaying part G_k^{E} corresponds to the Macdonald function. Then we can further split $q_1 = q_{1,\mathrm{M}} + q_{1,\mathrm{E}}$ and obtain the following results.

Lemma 2.2.2. *Let $n \geq 2$ and $\vec{q}$ and V be as in Lemma 2.2.1. Then $q_{1,\mathrm{E}} \in H^s(\mathbb{R})$ for any $s < \frac{8-n}{2}$.*

Our next step is to obtain a more convenient representation for $q_{1,\mathrm{M}}$.

Remark 2.2.3. There is a misprint in article II: the function denoted by χ there should be the characteristic function of $\mathbb{R} \setminus [-k_0, k_0]$, not that of $[-k_0, k_0]$. To emphasize this complementary role in the summary part, we denote by χ_c the characteristic function of $\mathbb{R} \setminus [-k_0, k_0]$. This misprint does not affect the proofs in any way.

Lemma 2.2.4. *Let $n \geq 2$. If $\vec{q}$ and V are as in Lemma 2.2.1, then*

$$\begin{aligned} q_{1,\mathrm{M}}(x) = \frac{(2\pi)^{\frac{n}{2}}}{2} \mathscr{F}^{-1} \left(\frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2} \frac{\widehat{\nabla\cdot\vec{q}}(\eta)\widehat{\nabla\cdot\vec{q}}(\mu)}{(\mu,\eta)+\mathrm{i}0} \right)(x,x) \\ - \frac{(2\pi)^{\frac{n}{2}}}{2} \mathscr{F}^{-1} \left(\frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2} \frac{\sum_{j,k=1}^{n} \widehat{\partial_j q_k}(\eta)\widehat{\partial_k q_j}(\mu)}{(\mu,\eta)+\mathrm{i}0} \right)(x,x) \\ - 2(2\pi)^{\frac{n}{2}} \mathscr{F}^{-1} \left(\frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2} \frac{\widehat{\widetilde{V}}(\eta)\widehat{V}(\mu)}{(\mu,\eta)+\mathrm{i}0} \right)(x,x) \end{aligned}$$

in the sense of distributions. Here $\mathscr{F}^{-1}$ is the $2n$-dimensional inverse Fourier transform and $\widetilde{V} := \nabla\cdot\vec{q} - V$.

Lemma 2.2.4 can be used to show that it suffices to study the behaviour of the bilinear form

$$\begin{aligned} I(f,g)(x) &= \text{p.v.} \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)} \frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2} \frac{\widehat{f}(\eta)\widehat{g}(\mu)}{(\eta,\mu)} \mathrm{d}\eta\mathrm{d}\mu \\ &\quad - \mathrm{i}\pi \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)} \frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2} \widehat{f}(\eta)\widehat{g}(\mu)\delta_0((\eta,\mu)=0)\mathrm{d}\eta\mathrm{d}\mu \\ &=: I' + I''. \end{aligned}$$

Finally, a careful analysis of the two bilinear forms I' and I'' allows us to conclude the recovery of singularities of β in the special cases when $n = 2$ or 3. To the best of the author's knowledge, this approach was first taken in [31] in the context of the backscattering problem for the Schrödinger operator in two dimensions. Our results may be summarized separately in two and three dimensions as follows.

Theorem 2.2.5 (Main theorem, $n = 2$)**.** *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^2)$ and $V \in L^p_{2\delta}(\mathbb{R}^2)$, where $2 < p \leq \infty$ and $2\delta > 2 - \frac{2}{p}$. Then the difference $q_{\mathrm{B}} - \beta$ defines a bounded and continuous function. If, in addition, $\vec{q}$ and V are real-valued, then* $\operatorname{Re} q_{\mathrm{B}} - \beta \in H^s(\mathbb{R}^2)$ $(\operatorname{mod} C^\infty(\mathbb{R}^2))$ *for any $s < 2$.*

Corollary 2.2.6. *If $\vec{q}$ and V are as in Theorem 2.2.5, then the jumps of β over smooth (bounded) curves are uniquely determined by the backscattering data $A(k,\theta,-\theta)$ and can be recovered from q_{B}.*

Theorem 2.2.7 (Main theorem, $n = 3$)**.** *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^3)$ and $V \in L^p_{2\delta}(\mathbb{R}^3)$, where $3 < p \leq \infty$ and $2\delta > 3 - \frac{3}{p}$, be real-valued. Then the difference* $\operatorname{Re} q_{\mathrm{B}} - \beta$ *belongs to the Sobolev space $H^s(\mathbb{R}^3)$* $(\operatorname{mod} C^\infty(\mathbb{R}^3))$ *for any $s < \frac{3}{2}$.*

Corollary 2.2.8. *If $\vec{q}$ and V are as in Theorem 2.2.7, then the main singularities of β over smooth (bounded) domains are uniquely determined by the backscattering data $A(k,\theta,-\theta)$ and can be recovered from q_{B}.*

Remark 2.2.9. In the three dimensional case we use the fact that $\vec{q}$ and V are real-valued in a quite essential way. If the coefficients are real-valued, it can be shown that the delta term in the bilinear form I'' can be cancelled entirely. This simplifies the proofs, but leaves open the question about recovery of singularities in the case of complex-valued coefficients in 3D.

Several examples are also presented to demonstrate the numerical reconstruction of the potentials in two dimensions. We remark that numerically we are able to recover the shape, size and location of β reasonably well, even though our theoretical results guarantee only the recovery of jumps and singularities.

2.3 Article III

This part answers the question posed in Remark 2.2.9 above. Namely, in Article II we assumed that the coefficients are real-valued in three dimensions to obtain convenient cancellation of a certain delta term. In this part we further extend the method of inverse Born approximation for complex-coefficients in 3D. The missing link in 3D is the computation of the bilinear form $I''(f,g)$ of Article II.

Note that we cannot directly generalize the approach of the 2D case to three dimensions. In the two dimensional case our approach was to write

$$\begin{aligned} I'' &= \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)} \frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2} \widehat{f}(\eta)\widehat{g}(\mu)\delta_0((\eta,\mu)=0)\mathrm{d}\eta\mathrm{d}\mu \\ &= \int_{\mathbb{R}^2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x,\eta+t\eta^\perp)} \frac{\chi_c(|\eta+t\eta^\perp|/2)}{|\eta|^2+t^2} \widehat{f}(\eta)\widehat{g}(t\eta^\perp)|\eta|^{-1}\mathrm{d}t\mathrm{d}\eta, \end{aligned}$$

where $\eta^\perp$ is the unit vector perpendicular to η chosen according to any specific orthogonal reference. The similar approach does not, at least directly, work in 3D. The reason is that in three dimensions there is no smooth (even continuous) way to choose for each unit vector η a unique perpendicular $\eta^\perp$ (cf. hairy ball theorem [17]). Instead, we use the mapping properties of the Radon transform. The Radon transform of a suitable measurable function f is defined as

$$R(f)(\theta,t) := \int_{(\theta,x)=t} f(x)\mathrm{d}\sigma(x),$$

where $\theta \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$, see e.g. [16]. The measure $\mathrm{d}\sigma(x)$ is the usual Lebesgue surface measure. Our main tool is the following theorem, which is proved in [30] by methods of complex interpolation.

Theorem 2.3.1. *For $n \geq 3$ the inequality*

$$\int_{\mathbb{S}^{n-1}} \sup_{t\in\mathbb{R}} |R(f)(\theta,t)|^\rho \,\mathrm{d}\theta \leq C\|f\|^{\alpha}_{L^a(\mathbb{R}^n)}\|f\|^{1-\alpha}_{L^b(\mathbb{R}^n)}$$

holds with $\rho \leq n$ whenever $1 \leq a < \frac{n}{n-1} < b \leq \infty$ and

$$\frac{\alpha}{a} + \frac{1-\alpha}{b} = \frac{n-1}{n}.$$

Remark 2.3.2. Any estimate of the type of Theorem 2.3.1 cannot hold in two dimensions [30]. An explicit counterexample can be provided by taking $f = \chi_K$, where χ_K is the characteristic function of a Besicovitch set (or Kakeya set) $K \subset \mathbb{R}^2$. A Besicovitch set K is a set which contains a unit line segment in every direction. Here K can be chosen so that it is compact and has an arbitrarily small Lebesgue measure (such sets were first constructed by Besicovitch in [8]). This means that $\sup_{t\in\mathbb{R}} |R(f)(\theta,t)| \geq 1$ for all $\theta \in \mathbb{S}^1$, while the right-hand side of the inequality in Theorem 2.3.1 can be made as small as we want.

By using Theorem 2.3.1 we prove the following lemma.

Lemma 2.3.3. *Let $f,g \in L^2(\mathbb{R}^3)$. If f or g is also in $L^1(\mathbb{R}^3)$, then $I'' = I''(f,g)(x)$ defines a bounded and continuous function of $x \in \mathbb{R}^3$.*

The main results of this article can be summarized in the following theorem and its corollary.

Theorem 2.3.4 (Recovery of singularities)**.** *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^3)$ and $V \in L^p_{2\delta}(\mathbb{R}^3)$ with $3 < p \leq \infty$ and $2\delta > 3 - \frac{3}{p}$. Then the difference $q_{\mathrm{B}} - \beta$ belongs to the Sobolev space $H^t(\mathbb{R}^3) \pmod{C(\mathbb{R}^3)}$ for all $t < \frac{3}{2}$.*

Corollary 2.3.5. *Under the same assumptions as in Theorem 2.3.4 the infinite singularities of $\beta = -\frac{1}{2}\nabla \cdot \vec{q} + V$ over boundaries of smooth domains in three dimensions are uniquely determined by the backscattering data $A(k,\theta,-\theta)$ and can be recovered from q_{B}.*

2.4 Article IV

The final article discusses an inverse scattering problem for quasi-linear perturbations of the biharmonic operator on the line. The operator is given by

$$Q_4 u := \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + q_1(x,|u|)u' + q_0(x,|u|)u,$$

where the coefficients q_1 and q_0 are required to satisfy several technical regularity conditions.

Assumption 2.4.1. Let us assume that the coefficients q_j, $j = 0,1$, have the following properties.

1. There exist functions $\alpha_j \in L^1(\mathbb{R})$ such that for all $a > 0$ we find $C_j(a) > 0$ with the property that $|q_j(x,s)| \leq C_j(a)\alpha_j(x)$, for all $0 \leq s \leq a$.
2. The coefficients q_j have Lipschitz property in the second variable, that is, there exists $\beta_j \in L^1(\mathbb{R})$ such that for all $a > 0$ we find $C_j'(a) > 0$ with the property that $|q_j(x,s_1) - q_j(x,s_2)| \leq C_j'(a)\beta_j(x)|s_1 - s_2|$ for all $0 \leq s_1 \leq 1+a$ and $0 \leq s_2 \leq 1+a$.
3. Denote $h_1(y) := q_1(y,1)$ and $h_0(y) := q_0(y,1)$. Suppose that the coefficients q_0 and q_1 have the following representations:

$$q_0(x,1+s) = h_0(x) + q_0^*(x,s_0^*)s,$$
$$q_1(x,1+s) = h_1(x) + q_1^*(x,1)s + q_1^{**}(x,s_1^*)\frac{s^2}{2},$$

   where $|s_0^*|, |s_1^*| < |s|$. Here we assume that $h_1 \in W_1^1(\mathbb{R})$, $q_1^*(x,1) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $p > 1$ and $|q_0^*(x,s_0^*)| \leq h_0^*(x)$, $|q_1^{**}(x,s_1^*)| \leq h_1^{**}(x)$ uniformly in $|s| < s_0$ for some $0 < s_0 \leq 1$ and for some $h_0^*, h_1^{**} \in L^1(\mathbb{R})$.

The first two assumptions are smallness and regularity conditions for the nonlinear coefficients and they guarantee that the direct problem has a unique solution if $k > k_0$ is large enough. The proof of this result is a straight-forward application of the Banach fixed-point theorem. As a consequence we find that the scattered wave can be expressed as the limit

$$u_{\mathrm{sc}} = \lim_{j\to\infty} u_{\mathrm{sc}}^{(j)},$$

where the terms are obtained iteratively from $u_{\mathrm{sc}}^{(j)} := T(u_{\mathrm{sc}}^{(j-1)})$, $u_{\mathrm{sc}}^{(0)} = 0$ and

$$T(\widetilde{u}) := -\int_{-\infty}^{\infty} G_k^+(|x-y|)\big(q_1(y,|u_0+\widetilde{u}|)(u_0+\widetilde{u})' + q_0(y,|u_0+\widetilde{u}|)(u_0+\widetilde{u})\big)\mathrm{d}y.$$

The solution u has the asymptotic behaviour

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + b(k)\mathrm{e}^{-\mathrm{i}kx} + o(1), \quad x \to -\infty,$$

where $b(k)$ is called the reflection coefficient and is defined by

$$b(k) = -\frac{\mathrm{i}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{iky}\left(q_1(y,|u|)u' + q_0(y,|u|)u\right)\mathrm{d}y$$

for sufficiently large $k > 0$. By formally defining the solutions $u(x,k) = \overline{u(x,-k)}$ for $k < 0$ also the reflection coefficient can be extended for $k < 0$ and it satisfies $b(k) = \overline{b(-k)}$. For technical simplicity we set $b(k) = 0$ if $-k_0 < k < k_0$.

By using Assumption 2.4.1 for large $k > 0$ we may approximate $u(x,k) \approx u_0(x,k)$ and obtain

$$b(k) \approx -\frac{\mathrm{i}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{2iky}\left(\mathrm{i}kq_1(y,1) + q_0(y,1)\right)\mathrm{d}y.$$

Here an integration by parts yields

$$b(k) \approx -\frac{\mathrm{i}}{4k^3}\int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}ky}\left(-\frac{1}{2}q_1'(y,1) + q_0(y,1)\right)\mathrm{d}y$$

and hence it makes sense to attempt recovery of the special combination of the coefficients $h(x) := -\frac{1}{2}q_1'(x,1) + q_0(x,1)$.

We propose to define the inverse Born approximation by

$$h_{\mathrm{B}}(x) := F\left(\frac{\mathrm{i}k^3}{2\sqrt{2\pi}}b\left(\frac{k}{2}\right)\right)(x)$$

in the sense of distributions. The third condition of Assumption 2.4.1 allows us to linearize the coefficients sufficiently and after some quite technical calculations we can conclude our main result and its corollary. Let us denote the space of continuous functions vanishing at infinity $\{f \in C(\mathbb{R}) \mid f(x) \to 0, \text{ as } x \to \pm\infty\}$ by $\dot{C}(\mathbb{R})$.

Theorem 2.4.2. *The inverse Born approximation h_{B} of h is of the form*

$$h_{\mathrm{B}}(x) = \mathrm{Re}(h)(x) + \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(h)(y)}{x-y}\mathrm{d}y \quad (\mathrm{mod}\ \dot{C}(\mathbb{R})).$$

Corollary 2.4.3. *Let q_0 and q_1 satisfy Assumptions 2.4.1. If* $\mathrm{Im}(h) \in H^r(\mathbb{R})$ *for some $r > \frac{1}{2}$ or if h is just real-valued, then the difference $h_{\mathrm{B}} - \mathrm{Re}(h)$ is a continuous function. In particular, any jumps and singularities contained in* $\mathrm{Re}(h)$ *can be recovered by calculating h_{B}.*

We also present a numerical example demonstrating the inverse Born approximation for nonlinear complex-valued coefficients. The reader may also be interested in the difference between the results of the reconstruction on the line and in higher dimensions. In higher dimensions we expect to recover information about the function β directly, while on the line we recover information about the real part and the Hilbert transform of the imaginary part. A heuristic reason for this is that on the line we have quite limited data (reflection coefficient in one direction) while in higher dimensions we have more room to integrate around the problematic neighbourhood of $k = 0$.

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I Tyni T & Serov V (2018) Scattering problems for perturbations of the multidimensional biharmonic operator. *Inverse Prob. Imag. 12*(1): 205-227.
II Tyni T & Harju M (2017) Inverse backscattering problem for perturbations of biharmonic operator. *Inverse Probl. 33*: 105002.
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