

DOUBLE PHASE IMAGE RESTORATION

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ABSTRACT. In this paper we explore the potential of the double phase functional in an image processing context. To this end, we study minimizers of the double phase energy for functions with bounded variation and show that this energy can be obtained by Γ -convergence or relaxation of regularized functionals. A central tool is a capped fractional maximal function of the derivative of BV functions.

1. INTRODUCTION

The double phase functional was introduced in the 1980s by Zhikov [46], but has only recently become the focus of intense research, starting in 2015 with Baroni, Colombo and Mingione [5, 6, 15, 17]. Subsequently, many other researchers studied double phase problems as well, see, e.g., [9, 18, 20, 22, 39, 40] for regularity theory, [10, 21, 43] for Calderón–Zygmund estimates and [27, 36, 37] for some other topics. Generalizations of the double phase functional have been studied, e.g. in [7, 25, 26, 28, 33, 34, 38, 45].

Zhikov’s original motivation for his functionals with non-standard growth was modelling physical phenomena. Another of his models, the variable exponent functional, was later applied also to the context of image processing, see [1, 14, 31, 35]. In this article, we demonstrate the potential also of the double phase functional in the image processing domain. This is to the best of our knowledge the first paper to consider the double phase functional in the space BV of functions of bounded variation.

In mathematical image processing, we interpret a function $u : \Omega \rightarrow \mathbb{R}$ as the gray-scale intensity at each location. If the function is discretized, we obtain an array of pixels common in computer implementations. Typically, Ω is a rectangle and the image contains different objects whose edges correspond to discontinuities of u . The presence of discontinuities makes this field challenging to approach with tools of analysis, but the BV space has proven useful. We refer to the book [4] by Aubert and Kornprobst for an overview of PDE-based image processing.

The classical ROF-model [41] for image restoration calls for minimizing the energy

$$\inf_u \int_{\Omega} |\nabla u| + |u - f|^2 dx,$$

where f is the given, corrupted input image that is to be restored. Here $|u - f|^2$ is a fidelity term which forces u to be close to f on average, whereas the regularizing term $|\nabla u|$ limits the variation of u . This model is known to be prone to a stair-casing or banding effect whereby piecewise constant minimizers are often produced [13]. On the other hand, replacing $|\nabla u|$

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by $|\nabla u|^2$ leads to a heat-equation type problem, and solutions which are C^∞ . This is not usually desirable in the image processing context, as edges become blurred.

The energy of the double phase functional combines growth with two different powers. It is given by the expression

$$\int_{\Omega} |\nabla u|^p + a(x)|\nabla u|^q dx.$$

Here $a \geq 0$ is a bounded function and $p < q$. All the previously mentioned double-phase references concern super-linear growth (usually $p > 1$, but see also [22]). However, for image processing, the case $p = 1$ and $q = 2$ is especially interesting (see above and the discussion in [14]). Then the first term corresponds to the ROF-model, whereas the second term introduces a smoothing effect when $a > 0$. The parameter a is chosen such that $a = 0$ at the edges in the image and $a > 0$ elsewhere. Usually, the location of the edges is not known, so in applications a is estimated from the initial data f . Then this adaptive model can avoid the stair-casing effect of the ROF-model.

In the case $p = 1$, the double phase energy must naturally be studied in a space of BV -type. It is not difficult to prove existence of the minimizer even in this case (cf. Proposition 2.4). However, the BV -space is quite ill-behaved, so it is useful for practical implementations to approximate the energy by more regular functionals (see, e.g., [44, Section 6] in the image processing context). The notion of Γ -convergence is often employed in this situation [8, 19], and this article is no exception: our main result (Theorems 4.1 and 4.2) shows that the BV double phase functional (with fidelity term)

$$|Du|(\Omega) + \int_{\Omega} (a(x)|\nabla u|)^2 + |u - f|^2 dx$$

can be approximated in the sense of Γ -convergence by both

$$\int_{\Omega} |\nabla u|^{1+\varepsilon} + (a(x)|\nabla u|)^2 + |u - f|^2 dx \quad \text{and} \quad \int_{\Omega} |\nabla u| + (\varepsilon + a(x)^2)|\nabla u|^2 + |u - f|^2 dx.$$

Finally, in Corollary 4.3, we show that the BV double phase functional can be understood as the relaxation of the $W^{1,1}$ double phase functional.

Note that we use a inside the power-function, $(a(x)t)^2$. This is of course equivalent to having another function outside, but it turns out that the condition on a can be more conveniently expressed with this formulation (see Remark 3.3).

2. NOTATION AND EXISTENCE OF MINIMIZERS OF BOUNDED VARIATION

We consider subsets of the Euclidean space \mathbb{R}^n , $n \geq 2$. The most interesting case for image processing is $n = 2$, but we can include higher dimensions without extra complication. By $\Omega \subset \mathbb{R}^n$ we denote a bounded domain, i.e. an open and connected set. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. By c we denote a generic constant whose value may change between appearances. Let $a \in L^\infty(\Omega)$ be non-negative. By $L_a^p(\Omega)$ we denote the weighted Lebesgue space with weight a , given by the norm

$$\|u\|_{L_a^p(\Omega)} := \|a u\|_{L^p(\Omega)} = \left(\int_{\Omega} (a(x)|u|)^p dx \right)^{\frac{1}{p}}.$$

$W_a^{1,p}(\Omega)$ is the corresponding Sobolev space. Note that we use the ‘‘weight as multiplier’’ formulation, so the corresponding weighted measure is $d\mu = a^p dx$, not $d\mu = a dx$. By

\mathcal{H}^k we denote the k -dimensional Hausdorff measure. By $|\mu|$ we denote the total variation measure of a vector measure μ , defined as

$$|\mu|(A) = \sup \left\{ \sum_{i \in \mathbb{N}} |\mu(A_i)| \mid \bigcup_{i \in \mathbb{N}} A_i = A, A_i \text{ disjoint and measurable} \right\}.$$

By Mu we denote the Hardy–Littlewood maximal function of u .

A function $u \in L^1(\Omega)$ has *bounded variation*, denoted $u \in BV(\Omega)$, if

$$|Du|(\Omega) := \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty.$$

Note that this quantity is sometimes denoted by $\|Du\|(\Omega)$. We follow the notation of [3], which is convenient since it turns out that $|Du|$ is the total variation of a vector measure Du . Furthermore, Du can be decomposed as

$$(2.1) \quad Du = \nabla u \mathcal{H}^n + (u_+ - u_-) \nu_u \mathcal{H}^{n-1}|_{J_u} + C_u,$$

where ∇u is the absolutely continuous part of the derivative, $u_+ - u_-$ is the essential point-wise jump of the function, ν_u is the normal of the level-set, J_u is a set of Hausdorff dimension at most $n - 1$ [2, Theorem 2.3] and the Cantor part C_u has the property that $C_u(A) = 0$ if $\mathcal{H}^{n-1}(A) < \infty$ [3, Proposition 3.92]. The space BV has the following precompactness property [3, Proposition 3.13]: if $\sup_i (|Du_i|(\Omega) + \|u_i\|_{L^1(\Omega)}) < \infty$, then there exists a subsequence, denoted again by (u_i) , and $u \in BV(\Omega)$ such that

$$(2.2) \quad u_i \rightarrow u \text{ in } L^1(\Omega) \quad \text{and} \quad |Du|(\Omega) \leq \liminf |Du_i|(\Omega).$$

The derivative of the convolution of a BV -function can be calculated as expected using either the derivative-measure or the function [3, Proposition 3.2 and equation (2.2)]:

$$(2.3) \quad \nabla(u * \eta_\delta)(x) = \int_{\mathbb{R}^n} \eta_\delta(x - y) \, dDu(y) = \int_{\mathbb{R}^n} u(y) \nabla \eta_\delta(x - y) \, dy.$$

We refer to [2, 3, 8] for more information about BV spaces.

We abbreviate $BV_a^{1,2}(\Omega) := BV(\Omega) \cap W_a^{1,2}(\Omega) \cap L^2(\Omega)$ and define for $u \in BV_a^{1,2}(\Omega)$ and initial data $f \in L^2(\Omega)$ the BV double phase functional

$$\mathcal{I}(u, A) := |Du|(A) + \int_A (a(x)|\nabla u|^2 + |u - f|^2) \, dx$$

for measurable $A \subset \Omega$. We can easily show the existence of a minimizer for this functional using the direct method of calculus of variations:

Proposition 2.4. *There exists a unique minimizer $u \in BV_a^{1,2}(\Omega)$, i.e.*

$$\mathcal{I}(u, \Omega) = \inf_{v \in BV_a^{1,2}(\Omega)} \mathcal{I}(v, \Omega).$$

Proof. Let u_i be a minimizing sequence, that is $u_i \in BV_a^{1,2}(\Omega)$ with

$$\lim_{i \rightarrow \infty} \mathcal{I}(u_i, \Omega) = \inf_{v \in BV_a^{1,2}(\Omega)} \mathcal{I}(v, \Omega).$$

By BV -precompactness (2.2) there exists a subsequence, denoted again by (u_i) , such that $u_i \rightarrow u$ in $L^1(\Omega)$ and $|Du|(\Omega) \leq \liminf |Du_i|(\Omega)$. The space $W_a^{1,2}(\Omega)$ is reflexive [29, Theorem 3.6.8], so we can find a weakly convergent subsequence (u_i) . By [24, Theorem 2.2.8],

the modular in $W_a^{1,2}(\Omega)$ is weakly lower semicontinuous, so that

$$\int_{\Omega} (a(x)|\nabla u|)^2 dx \leq \liminf \int_{\Omega} (a(x)|\nabla u_i|)^2 dx.$$

The inequality for the term $|u - f|^2$ follows analogously. Hence u is a minimizer.

Finally, we note that the BV and $W_a^{1,2}$ parts are convex and the $|u - f|^2$ part is strictly convex, so the usual argument yields uniqueness, namely, if u and v are distinct minimizers, then we obtain a contradiction from $\mathcal{I}(\frac{u+v}{2}, \Omega) < \frac{1}{2}(\mathcal{I}(u, \Omega) + \mathcal{I}(v, \Omega))$. \square

3. LOWER ESTIMATES FOR THE BV DOUBLE PHASE FUNCTIONAL

To be able to construct the minimizers of \mathcal{I} with some numerical scheme, we must show that the BV double phase functional can be approximated by some more regular variants. We regularize the functional by adding ε either to the exponent of the first term (so that the problem is in $W^{1,1+\varepsilon}(\Omega)$) or to the weight a (in which case the problem is in $W^{1,2}(\Omega)$). For brevity, we present the proof only for one case which includes both these regularizations:

$$\mathcal{I}_{\varepsilon}(u, A) := \int_A |\nabla u|^{1+\varepsilon} + (\varepsilon + a(x)^2)|\nabla u|^2 + |u - f|^2 dx.$$

We start with a lower bound for \mathcal{I} , which is the more difficult part.

Lemma 3.1. *Let $F \subset \Omega$ be closed and $a \in C^{0,1}(\Omega)$. For $\varepsilon_i \rightarrow 0^+$ and $u \in BV_a^{1,2}(\Omega)$, there exist $u_i \in W^{1,2}(U)$ in a neighborhood U of F such that*

$$\limsup_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i, F) \leq \mathcal{I}(u, F).$$

Proof. Let $u_{\delta} := u * \eta_{\delta}$ be the convolution with the standard mollifier and assume that $\delta < \text{dist}(F, \partial\Omega)$. By [30, Lemma 4.5] and classical L^2 -results

$$\limsup_{\delta \rightarrow 0} \left[|Du_{\delta}|(F) + \int_F |u_{\delta} - f|^2 dx \right] \leq |Du|(F) + \int_F |u - f|^2 dx.$$

For the term with the weight a , we consider two cases and use the different expressions from (2.3). If $0 < a(x) \leq 2a(y)$ for all $y \in B(x, \delta)$, then

$$a(x)|\nabla u_{\delta}| \leq 2 \int_{\mathbb{R}^n} a(y)|\nabla u(y)| \eta_{\delta}(x - y) dy \lesssim M(a|\nabla u|)(x);$$

note that the condition $0 < a(y)$ with $u \in W_a^{1,2}(\Omega)$ ensures that $Du = \nabla u$ is absolutely continuous in $B(x, \delta)$ and note also that the last inequality follows from elementary estimates (e.g. [24, Lemma 4.6.3]). Furthermore, since $a|\nabla u| \in L^2(\Omega)$ and the maximal operator is bounded on $L^2(\Omega)$, we see that the function on the right-hand side is in $L^2(\Omega)$, as well. If $a(x) = 0$, then the estimate trivially holds. Suppose then that $a(x) > 2a(y)$ for some $y \in B(x, \delta)$. Since $a \in C^{0,1}(\Omega)$, we obtain the inequality

$$a(x) - c|x - y| \leq a(y) \leq \frac{1}{2}a(x),$$

so that $a(x) \lesssim |x - y| \leq \delta$. Therefore

$$(3.2) \quad a(x)|\nabla u_{\delta}| \lesssim \int_{\mathbb{R}^n} \delta |u(y)\nabla \eta_{\delta}(x - y)| dy \lesssim \frac{1}{|B(x, \delta)|} \int_{B(x, \delta)} |u(y)| dy \leq Mu(x),$$

where we used that $|\delta \nabla \eta_\delta| \lesssim \delta^{-n} \chi_{B(x, \delta)}$ for the middle step. Again, since $u \in L^2(\Omega)$, we obtain an upper bound independent of δ in the space $L^2(\Omega)$. In the set $\{a > 0\}$ we have $\nabla u_\delta \rightarrow \nabla u$ almost everywhere. Thus it follows by dominated convergence in $L^2(\Omega)$ that

$$\lim_{\delta \rightarrow 0} \int_F (a(x) |\nabla u_\delta|)^2 dx = \int_F (a(x) |\nabla u|)^2 dx.$$

We have so far shown that

$$\limsup_{\delta \rightarrow 0} \mathcal{I}(u_\delta, F) \leq \mathcal{I}(u, F).$$

It remains to change the first functional from \mathcal{I} to $\mathcal{I}_{\varepsilon_i}$. Equation (2.3) implies $|\nabla u_\delta| \leq \frac{c}{\delta^n}$, where c depends on $|Du|(\Omega)$. Therefore

$$\int_F |\nabla u_\delta|^{1+\varepsilon} + (\varepsilon + a(x)^2) |\nabla u_\delta|^2 dx \leq \left(\frac{c}{\delta^n}\right)^\varepsilon |Du_\delta|(F) + \int_F (a(x) |\nabla u_\delta|)^2 dx + \varepsilon \left(\frac{c}{\delta^n}\right)^2 |F|.$$

We choose $\delta_i := \varepsilon_i^{1/(3n)}$ so that $\left(\frac{c}{\delta_i^n}\right)^{\varepsilon_i} \rightarrow 1$ and $\varepsilon_i \left(\frac{c}{\delta_i^n}\right)^2 \rightarrow 0$ and set $u_i := u_{\delta_i}$. Then

$$\begin{aligned} & \limsup_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i, F) \\ & \leq \limsup_{i \rightarrow \infty} \left[\left(\frac{c}{\delta_i^n}\right)^{\varepsilon_i} |Du_i|(F) + \int_F (a(x) |\nabla u_i|)^2 + |u_i - f|^2 dx + \varepsilon_i \left(\frac{c}{\delta_i^n}\right)^2 |F| \right] \\ & = \limsup_{i \rightarrow \infty} \left[|Du_i|(F) + \int_F (a(x) |\nabla u_i|)^2 + |u_i - f|^2 dx \right] \leq \mathcal{I}(u, F). \quad \square \end{aligned}$$

Remark 3.3. From the previous proof we can see that the exact condition used for a is not $C^{0,1}(\Omega)$, but rather the inequality

$$a(x) \lesssim \max\{|x - y|, a(y)\} \quad \text{for all } x, y \in \Omega.$$

This means that we could replace $a(x)^2$ in the double phase functional with $a(x)^q$ for $a \in C^{0,\alpha}(\Omega)$ as long as $q\alpha \geq 2$. This kind of condition was first identified for the double phase functional in [29, Section 7.2].

With the method of the previous proof, one can obtain from (3.2) that $a(x) |\nabla u_\delta|$ is bounded by $M_\alpha(Du)$ when $a \in C^{0,\alpha}(\Omega)$ and M_α denotes the fractional maximal operator (cf. Lemma 3.5). This will allow us to prove the result for bounded functions u with a larger class of weights a . A number of recent studies, e.g. [11, 12], deal with the question of the Sobolev regularity of the maximal function $M_\alpha u$ of a Sobolev or BV function u . However, we have not found any results on the maximal function of the derivative of a BV function. Therefore, the following result may be of independent interest.

Proposition 3.4. *Let μ be a vector Borel measure in Ω with finite total variation $|\mu|(\Omega) < \infty$, $\sigma \in (0, n)$ and $\alpha \in (0, n - \sigma)$. Then the capped fractional maximal function*

$$M_\alpha^\sigma \mu(x) := \sup_{r \leq \text{diam } \Omega} \frac{\min\{|\mu|(B(x, r)), r^\sigma\}}{|B(x, r)|^{1-\frac{\alpha}{n}}}$$

belongs to $L^p(\Omega)$ if $p < 1 + \frac{\alpha}{n-\sigma-\alpha}$.

Furthermore, the bound is sharp since the claim does not hold for $p \geq 1 + \frac{\alpha}{n-\sigma-\alpha}$.

Proof. We consider dyadic cubes intersecting Ω with side-length at most $\text{diam } \Omega$. Specifically, we assume that the cubes are of the form $[a_1, b_1) \times \cdots \times [a_n, b_n)$ and denote by \mathcal{D}_k the set of such cubes with side-length 2^k . Let $D_k^x \in \mathcal{D}_k$ be the cube which contains x and $3D_k^x$

be its threefold dilate. We define $\mu_k(A) := \min\{|\mu|(A \cap \Omega), 2^{\sigma k}\}$. If $2^{k-1} \leq r < 2^k$, then $B(x, r) \subset 3D_k^x$. Thus

$$M_\alpha^\sigma \mu(x) \lesssim \sup_{k \in K_0} \frac{\mu_k(3D_k^x)}{2^{(n-\alpha)k}},$$

where $K_0 := \{-\infty, \dots, k_0\}$ and k_0 is the smallest integer with $2^{k_0} > \text{diam } \Omega$. We raise this to the power p and estimate the supremum by a sum:

$$M_\alpha^\sigma \mu(x)^p \lesssim \sup_{k \in K_0} \left(\frac{\mu_k(3D_k^x)}{2^{(n-\alpha)k}} \right)^p \leq \sum_{k \in K_0} \left(\frac{\mu_k(3D_k^x)}{2^{(n-\alpha)k}} \right)^p.$$

Next we integrate over Ω and use that $\mu_k(3D_k^x)$ can be estimated by the sum of 3^n terms of the form $\mu_k(D_k)$ with $D_k \in \mathcal{D}_k$. Thus we obtain that

$$\begin{aligned} \int_\Omega M_\alpha^\sigma \mu(x)^p dx &\lesssim \sum_{k \in K_0} 2^{-(n-\alpha)pk} \int_\Omega \mu_k(D_k^x)^p dx \\ &\leq \sum_{k \in K_0} 2^{-(n-\alpha)pk} \sum_{D \in \mathcal{D}_k} \mu_k(D)^p |D|. \end{aligned}$$

Let us maximize the sum $\sum_{D \in \mathcal{D}_k} \mu_k(D)^p$ separately for each k . Since $\mathcal{D}_k \cap \Omega$ is a partition of Ω , we can write this optimization problem as

$$S_k := \sup \left\{ \sum_i a_i^p \mid \sum_i a_i \leq |\mu|(\Omega), a_i \in [0, 2^{\sigma k}] \right\}$$

where $a_i = \mu_k(D_i)$ for $D_i \in \mathcal{D}_k$; the last restriction holds since $\mu_k(\Omega) \leq 2^{\sigma k}$ by the definition of μ_k . We consider what values of the a_i 's leads to a maximally large sum. If $0 < a_i < a_j < 2^{\sigma k}$, then

$$a_i^p + a_j^p < (a_i - t)^p + (a_j + t)^p$$

for $0 < t < \min\{a_i, 2^{\sigma k} - a_j\}$. Therefore the sum is maximized subject to the constraints when $a_i = 2^{\sigma k}$ for as many indices as possible and zero for the rest. There are no more than $\lceil 2^{-\sigma k} |\mu|(\Omega) \rceil$ such maximal indices. Thus

$$S_k \approx 2^{-\sigma k} |\mu|(\Omega) 2^{\sigma k p} \approx 2^{(p-1)\sigma k}.$$

We use this estimate in our previous inequality, and conclude that

$$\int_\Omega M_\alpha^\sigma \mu(x)^p dx \lesssim \sum_{k \in K_0} 2^{-(n-\alpha)pk} 2^{(p-1)\sigma k} 2^{nk} = \sum_{k \in K_0} 2^{[-(n-\alpha)p + (p-1)\sigma + n]k}.$$

The last sum is finite if $-(n-\alpha)p + (p-1)\sigma + n > 0$, which is equivalent to the condition in the proposition.

It remains to prove sharpness. For simplicity we consider only the case when σ is an integer. We let E be a σ -dimensional plane and define $\mu(A) := \mathcal{H}^\sigma(E \cap A)$. Denote $d(x) := \text{dist}(x, E)$. Then

$$M_\alpha^\sigma \mu(x) \gtrsim \frac{\mu(B(x, 2d(x)))}{|B(x, 2d(x))|^{1-\frac{\alpha}{n}}} \approx d(x)^{\sigma-n+\alpha}.$$

We raise this to the power p and integrate over x :

$$\int_\Omega M_\alpha^\sigma \mu(x)^p dx \gtrsim \int_\Omega d(x)^{(\sigma-n+\alpha)p} dx \approx \int_0^1 r^{(\sigma-n+\alpha)p} r^{n-\sigma-1} dr.$$

This integral diverges if $(\sigma - n + \alpha)p + n - \sigma \leq 0$, which gives the claimed bound for p . In the case of non-integer σ , we instead choose our set as the Cartesian product of a plane and a Cantor set, and estimate as before. \square

With the fractional maximal operator we can extend Lemma 3.1 in the case of bounded functions. Bounded functions are very natural in the context of image processing, since the grey-scale values are usually taken in some compact interval such as $[0, 255]$ or $[0, 1]$. Note that to use the previous proposition, we cannot directly move to the total variation measure $|Du|$, since this is not in general going to satisfy the appropriate decay r^{n-1} when u is bounded. Rather, we have to first estimate the absolute value of the measure of a ball, $|Du(B(x, r))|$, and only afterward move to $|Du|$. In the next result we therefore work with the vector measure Du rather than its total variation, which makes the estimates slightly more difficult.

Lemma 3.5. *Let $F \subset \Omega$ be closed and $a \in C^{0,\alpha}(\Omega)$ for some $\alpha > \frac{1}{2}$. For $\varepsilon_i \rightarrow 0^+$ and $u \in BV_a^{1,2}(\Omega) \cap L^\infty(\Omega)$, there exist $u_i \in W^{1,2}(U) \cap L^\infty(\Omega)$ in a neighborhood U of F such that*

$$\limsup_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i, F) \leq \mathcal{I}(u, F).$$

Proof. The proof is identical to that of Lemma 3.1, except for the estimate of $a(x)|\nabla u_\delta|$ in the second case, $a(x) < \frac{1}{2}a(y)$. Let us show that we can use Proposition 3.4 to handle this case. By the construction of the measure Du ,

$$\int_{B(x,r)} \varphi \cdot dDu = - \int_{B(x,r)} u \operatorname{div} \varphi \, dy$$

for all $\varphi \in C_0^1(B(x, r); \mathbb{R}^n)$, cf. [3, Proposition 3.6]. We choose $\varphi(y) = b\xi(|x - y|)$ where $b \in B(0, 1)$ and $\xi \in C^1([0, \infty))$ with $\xi|_{[0, r-\varepsilon-\varepsilon^2]} = 1$, $\xi|_{[r-\varepsilon^2, \infty)} = 0$ and $|\xi'| \leq \frac{2}{\varepsilon}$. Then $|\operatorname{div} \varphi| \leq \frac{2}{\varepsilon} \chi_{B(x, r-\varepsilon^2) \setminus B(x, r-\varepsilon-\varepsilon^2)}$ and so

$$\left| \int_{B(x,r)} u \operatorname{div} \varphi \, dy \right| \leq \|u\|_\infty \frac{2}{\varepsilon} |B(x, r - \varepsilon^2) \setminus B(x, r - \varepsilon - \varepsilon^2)| \approx r^{n-1}$$

since u is bounded. It follows by monotone convergence as $\varepsilon \rightarrow 0^+$ that

$$|Du(B(x, r))| = \sup_{|b|=1} b \cdot Du(B(x, r)) \lesssim r^{n-1}.$$

Therefore, $|Du(B(x, r))| \lesssim \min\{|Du|(B(x, r)), r^{n-1}\}$ and so

$$|Du(B(x, r))| \lesssim M_\alpha^{n-1}(Du)(x)r^{n-\alpha}.$$

On the other hand, we can estimate for the derivative of the convolution using (2.3), the distribution function of Du [42, Theorem 8.16] and the estimate $|\frac{d}{dr}\eta_\delta(re_1)| \lesssim \delta^{-n-1}$. For a unit vector e_1 , it follows that

$$\begin{aligned} |\nabla u_\delta| &\leq \left| \int_{\mathbb{R}^n} \eta_\delta(x - y) dDu(y) \right| = \left| \int_0^\delta \frac{d}{dr}\eta_\delta(re_1) Du(B(x, r)) \, dr \right| \\ &\leq \int_0^\delta \left| \frac{d}{dr}\eta_\delta(re_1) \right| M_\alpha^{n-1}(Du)(x)r^{n-\alpha} \, dr \\ &\lesssim M_\alpha^{n-1}(Du)(x)\delta^{-n-1} \int_0^\delta r^{n-\alpha} \, dr \approx \delta^{-\alpha} M_\alpha^{n-1}(Du)(x). \end{aligned}$$

As in Lemma 3.1, we conclude now from $a \in C^{0,\alpha}(\Omega)$ in the second case that $a(x) \leq \delta^\alpha$. Thus $a(x)|\nabla u_\delta| \lesssim M_\alpha^{n-1}(Du)(x)$. By Proposition 3.4, the right-hand side is in $L^2(\Omega)$ provided $2 < 1 + \frac{\alpha}{n-(n-1)-\alpha} = \frac{1}{1-\alpha}$, which holds since $\alpha > \frac{1}{2}$. Thus we can use this as the bound for dominated convergence. The rest of the proof is as before. \square

Remark 3.6. If we consider a double phase functional $t^p + a(x)t^q$ in “normal” form, then the condition from the previous results can be written $q < p + \alpha$. This condition has proved to be of central importance when considering bounded solutions, cf. [6, 16, 32]. In this sense, the assumption in Lemma 3.5 is probably essentially sharp.

However, more precise research has established that one may even take $q \leq p + \alpha$ for bounded minimizers [6, 33] (see also [21, 34] for the borderline case with unbounded minimizers). The borderline is handled using additional Hölder continuity obtained via De Giorgi technique, which in this case implies that $u \in C^{0,\gamma}(\Omega)$ for some $\gamma > 0$. Indeed, from the previous proof we can see that $a \in C^{0,1/2}(\Omega)$ would suffice if we had $u \in C^{0,\gamma}(\Omega)$ for some positive $\gamma > 0$ (as one has when $p, q > 1$) instead of $u \in L^\infty(\Omega)$. However, for BV problems, such higher regularity of the function cannot be expected. Therefore, the borderline $q = p + \alpha$ remains a problem for future research.

Let us also note that Ok [40] has considered double phase functionals under additional a priori integrability assumptions other than $L^\infty(\Omega)$. If one could prove decay estimates $|Du(B(x, r))| \lesssim r^\sigma$ for $\sigma \in (n-1, n)$ when $u \in L^s(\Omega)$, we could cover also this case. We do not know about such of results, so this, likewise, remains for a topic for another study.

4. UPPER ESTIMATES FOR THE BV DOUBLE PHASE FUNCTIONAL

The concept of Γ -convergence, introduced by De Giorgi and Franzoni [23], has been systematically presented in [8, 19]. A family of functionals $\mathcal{I}_\varepsilon : X \rightarrow \overline{\mathbb{R}}$ is said to Γ -converge (in topology τ) to $\mathcal{I} : X \rightarrow \overline{\mathbb{R}}$ if the following hold for every positive sequence (ε_i) converging to zero:

- (a) $\mathcal{I}(u) \leq \liminf_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i)$ for every $u \in X$ and every $(u_i) \subset X$ τ -converging to u ;
- (b) $\mathcal{I}(u) \geq \limsup_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i)$ for every $u \in X$ and some $(u_i) \subset X$ τ -converging to u .

Let us remark that the somewhat strange assumption $\mathcal{H}^{n-1}(\{a = 0\} \cap \partial\Omega) = 0$ in the next theorem is actually quite natural: since $\{a = 0\}$ is the set where the image edges occur, we cannot identify the edge if it coincides with the image boundary $\partial\Omega$. On the other hand, we also have no need for the jump in the function at this location, since the other part of the jump will be outside the image, and thus cannot be seen.

Theorem 4.1. *Suppose that Ω is a rectangular cuboid, $a \in C^{0,1}(\overline{\Omega})$, and assume that $a > 0$ \mathcal{H}^{n-1} -a.e. on the boundary $\partial\Omega$. Then \mathcal{I}_ε Γ -converges to \mathcal{I} in $L^1(\Omega)$ topology with $X := BV_a^{1,2}(\Omega)$.*

Proof. Let us start with condition (a) in the definition of Γ -convergence. Let (ε_i) be a positive sequence converging to zero. Let $u \in BV_a^{1,2}(\Omega)$ and let $(u_i) \subset BV_a^{1,2}(\Omega)$ be a sequence converging to u in $L^1(\Omega)$. If $\liminf_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i) = \infty$, then there is nothing to prove, so we assume that $K := \liminf_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i) < \infty$. We restrict our attention to a subsequence with $\lim_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i) = K$ and $u_i \in W^{1,2}(\Omega)$. Then (u_i) is a bounded sequence in $BV_a^{1,2}(\Omega)$. By precompactness of BV there exists a limit function for a subsequence such that $|Du^b|(\Omega) \leq \liminf |Du_i|(\Omega)$; by reflexivity of $W_a^{1,2}(\Omega)$ and $L^2(\Omega)$, we obtain subsequences with $\nabla u_i \rightharpoonup$

∇u^w , $u_i \rightharpoonup u^w$ in $L_a^2(\Omega)$ and $u_i - f \rightharpoonup u^l - f$ in $L^2(\Omega)$. By $u_i \rightarrow u$ in $L^1(\Omega)$ and the uniqueness of the limit, we conclude that $u^b = u^w = u^l = u$.

The weak lower semi-continuity of the Lebesgue integral yields that

$$\int_{\Omega} |u - f|^2 dx \leq \liminf_{i \rightarrow \infty} \int_{\Omega} |u_i - f|^2 dx$$

and, since $\varepsilon_i \geq 0$,

$$\int_{\Omega} (a(x)|\nabla u|^2) dx \leq \liminf_{i \rightarrow \infty} \int_{\Omega} (a(x)|\nabla u_i|^2) dx \leq \liminf_{i \rightarrow \infty} \int_{\Omega} (\varepsilon_i + a(x)^2)|\nabla u_i|^2 dx.$$

Finally, for the BV part we use the estimate from the previous paragraph, Young's inequality and $(\frac{1}{1+\varepsilon_i})^{1/\varepsilon_i} \rightarrow \frac{1}{e}$:

$$\begin{aligned} |Du|(\Omega) &\leq \liminf_{i \rightarrow \infty} |Du_i|(\Omega) = \liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i| dx \\ &\leq \liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i|^{1+\varepsilon_i} + \left(\frac{1}{1+\varepsilon_i}\right)^{\frac{1}{\varepsilon_i}} \frac{\varepsilon_i}{1-\varepsilon_i} dx = \liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i|^{1+\varepsilon_i} dx. \end{aligned}$$

By combining the above inequalities we obtain condition (a). Note that for this part we do not need the assumptions on Ω and a .

Let us then move to condition (b). Since Ω is a rectangular cuboid, we can extend both the function u and the weight a by reflections to the rectangular cuboid with the same center but 3 times the side-lengths. Then we use Lemma 3.1 with $F := \overline{\Omega}$ to conclude that there exist $u_i \in W^{1,2}(U)$ such that

$$\limsup_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i, \overline{\Omega}) \leq \mathcal{I}(u, \overline{\Omega}).$$

We need this inequality with Ω instead of $\overline{\Omega}$. Since $|\partial\Omega| = 0$ and u_i is a Sobolev function, $\mathcal{I}_{\varepsilon_i}(u_i, \overline{\Omega}) = \mathcal{I}_{\varepsilon_i}(u_i, \Omega)$. On the right-hand side, the same reason implies that

$$\int_{\overline{\Omega}} (a(x)|\nabla u|^2) + |u - f|^2 dx = \int_{\Omega} (a(x)|\nabla u|^2) + |u - f|^2 dx.$$

The singular set of Du is contained in $\{a = 0\}$ because $u \in W_a^{1,2}(U)$. Since $\{a = 0\} \cap \partial\Omega$ has Hausdorff $(n-1)$ -measure zero by assumption, it follows by the decomposition (2.1) that $|Du|(\partial\Omega) = 0$ and so $|Du|(\overline{\Omega}) = |Du|(\Omega)$. Thus we have established condition (b) of Γ -convergence. \square

In the previous theorem we could consider a Lipschitz domain instead of a rectangular cuboid. In this case, the extension of both u and a would be done by flattening the boundary with the Lipschitz map. If we use Lemma 3.5 instead of Lemma 3.1, we obtain the following variant.

Theorem 4.2. *Suppose that Ω is a bounded Lipschitz domain, $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{1}{2}$, and assume that $a > 0$ \mathcal{H}^{n-1} -a.e. on the boundary $\partial\Omega$. Then $\mathcal{I}_{\varepsilon}$ Γ -converges to \mathcal{I} in $L^1(\Omega)$ topology with $X := BV_a^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

We use the following formulation for relaxation, which emphasizes the connection with Γ -convergence. A functional $\overline{\mathcal{J}} : X \rightarrow \overline{\mathbb{R}}$ is the *relaxation* of $\mathcal{J} : X \rightarrow \overline{\mathbb{R}}$ in topology τ if

- (a) $\overline{\mathcal{J}}(u) \leq \liminf_{i \rightarrow \infty} \mathcal{J}(u_i)$ for every $u \in X$ and every $(u_i) \subset X$ τ -converging to u ;
- (b) $\overline{\mathcal{J}}(u) \geq \limsup_{i \rightarrow \infty} \mathcal{J}(u_i)$ for every $u \in X$ and some $(u_i) \subset X$ τ -converging to u .

The relaxation is the greatest lower-semicontinuous minorant of \mathcal{J} . See [8, Proposition 1.31, p. 33]. Let us write for $u \in BV(\Omega)$ that

$$\mathcal{J}(u) := \begin{cases} \int_{\Omega} |\nabla u| + (a(x)|\nabla u|)^2 + |u - f|^2 dx, & \text{if } u \in W^{1,1}(\Omega) \\ \infty, & \text{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega). \end{cases}$$

We show that the relaxation $\overline{\mathcal{J}}$ of this functional equals \mathcal{I} . The proof is identical to Theorem 4.1, we simply take $\mathcal{I}_{\varepsilon} = \mathcal{J}$ for every $\varepsilon > 0$ and $\mathcal{I} = \overline{\mathcal{J}}$. Naturally, we could also prove an analogue to Theorem 4.2.

Corollary 4.3. *Suppose that Ω is a rectangular cuboid, $a \in C^{0,1}(\overline{\Omega})$, and assume that $a > 0$ \mathcal{H}^{n-1} -a.e. on the boundary $\partial\Omega$. Then $\overline{\mathcal{J}} = \mathcal{I}$ in $L^1(\Omega)$ topology.*

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