An Exact Expression for the Average AoI in a Multi-Source M/M/1 Queueing Model

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***Abstract*—Information freshness is crucial in a wide range of wireless applications where a destination needs the most recent measurements of a remotely observed random process. In this paper, we study the information freshness of a single-server multi-source M/M/1 queueing model under a first-come first-served (FCFS) serving policy. The information freshness of the status updates of each source is evaluated by the average age of information (AoI). We derive an exact expression for the average AoI for the multi-source M/M/1 queueing model. Simulation results are provided to validate the derived exact expression for the average AoI.**

Index Terms– Information freshness, age of information (AoI), multi-source queueing model.

I. Introduction

There has been a growing interest in services that require time-sensitive information updates of a random process such as temperature of a specific environment (room, greenhouse, etc.) [1], and a vehicular status (position, acceleration, etc.) [2]. To enable these services, various sensors may be assigned to send status updates about a random process to intended destinations [1], [3]–[5]. One critical factor for these services is high freshness of the sensors' information at the destination. The traditional metrics such as throughput and delay cannot fully characterize the information freshness [3], [6]. Recently, the age of information (AoI) was proposed as a destination-centric metric to measure the information freshness [6], [7]. A status update packet contains the measured value of a monitored process and a time stamp representing the time when the sample was generated. Due to wireless channel access, fading, etc., communicating a status update packet through the network experiences a random delay. If at a time instant t, the most recently received status update packet contains the time stamp $U(t)$, AoI is defined as the random process $\Delta(t) = t - U(t)$. Thus, the AoI measures for each sensor the time elapsed since the last received status update packet was generated.

The first queueing theoretic work on AoI is [6] where the authors derived the average AoI for a single-source M/M/1 first-come first-served (FCFS) queueing model. The work [8] was the first to investigate the average AoI in a multi-source setup. The authors of [8] derived the average AoI for a multi-source M/M/1 FCFS queueing model. The closed-form expressions for the average AoI in a multi-source M/G/1/1 preemptive queueing model were derived in [9].

In this paper, we analyze the average AoI of the different sources in a single-server multi-source M/M/1 queueing model under an FCFS service policy. We derive an exact expression for the average AoI for the multi-source M/M/1 queueing model. The setup was earlier addressed in [3], [8], where the authors derived an approximate expression for the average AoI by neglecting the statistical dependency between certain random variables (see Section III-C). In [10], we made a first attempt to correct the error in [3], [8]. Unfortunately, the average AoI expression derived there is still inaccurate (see Remark 2 in Section III-C). Thus, this paper provides the first exact expression of the average AoI in a multi-source M/M/1 queueing model. We present simulation results to validate the exact average AoI expression derived here.

II. System Model and Definitions

We consider a system consisting of a set of independent sources denoted by $\mathcal{C} = \{1, \ldots, C\}$ and one server. Each source observes a random process, representing, e.g., temperature, vehicular speed or location at random time instants. A remote destination is interested in timely information about the status of these random processes. Status updates are transmitted as packets, containing the measured value of the monitored process and a time stamp representing the time when the sample was generated. We assume that the packets of source c are generated according to the Poisson process with rate λ_c, $c \in \mathcal{C}$, and the packets are served according to an exponentially distributed service time with mean $1/\mu$.

Definition 1 (AoI). Let $t_{c,i}$ denote the time instant at which the ith packet of source c was generated, and $t'_{c,i}$ denote the time instant at which this packet arrives at the destination. At a time instant τ, the index of the most recently received packet of source c is given by $N_c(\tau) = \max\{i' | t'_{c,i'} \leq \tau\}$, and the time stamp of the most recently received packet of source c is $U_c(\tau) = t_{c,N_c(\tau)}$. The AoI of source c at the destination is defined as the random process $\Delta_c(t) = t - U_c(t)$.

From here onwards, we refer to the ith packet from source c simply as packet c, i. Let $X_{c,i}$ denote the ith interarrival time of source c; $W_{c,i}$ denote the waiting time of packet c, i; $S_{c,i}$ denote the service time of packet c, i; and $T_{c,i}$ denote the system time of packet c, i, i.e., the time interval the packet spends in the system which consists of the sum of the waiting time and the service time, $T_{c,i} = S_{c,i} + W_{c,i}$. Then, the average AoI of source c, denoted as Δ_c, is given as [6]

$$\begin{aligned}\Delta_c &= \lambda_c\left(\frac{\mathbb{E}[X_{c,i}^2]}{2} + \mathbb{E}[X_{c,i}T_{c,i}]\right) \qquad (1)\\ &= \lambda_c\left(\frac{\mathbb{E}[X_{c,i}^2]}{2} + \mathbb{E}[X_{c,i}(S_{c,i}+W_{c,i})]\right).\end{aligned}$$

III. Average AoI in a Multi-Source M/M/1 Queueing Model

To evaluate the AoI of one source in a queueing model with multiple sources of Poisson arrivals, we can consider two sources without loss of generality. Thus, we proceed to evaluate the AoI of source 1 by aggregating the other $C-1$ sources into source 2 having the Poisson arrival rate $\lambda_2 = \sum_{c' \in \mathcal{C}\setminus\{c\}} \lambda_{c'}$. Let $\rho_1 = \lambda_1/\mu$ and $\rho_2 = \lambda_2/\mu$ be the load of source 1 and 2, respectively. Since packets of each source are generated according to the Poisson process and the sources are independent, the packet generation in the system follows the Poisson process with rate $\lambda = \lambda_1 + \lambda_2$, and the overall load in the system is $\rho = \rho_1 + \rho_2 = \lambda/\mu$.

In the following, we derive the average AoI (1) for source 1, denoted as Δ_1. The first term in (1) is easy to compute. Namely, since the interarrival time of source 1 follows the exponential distribution with parameter λ_1, we have $\mathbb{E}[X_{1,i}^2] = 2/\lambda_1^2$. However, because the random variables $X_{1,i}$ and $T_{1,i}$ are dependent, the most challenging part in calculating (1) is $\mathbb{E}[X_{1,i}T_{1,i}] = \mathbb{E}[X_{1,i}(W_{1,i}+S_{1,i})]$ which is derived next.

Since the interarrival time and service time of the packet $1, i$ are independent, we have

$$\mathbb{E}[X_{1,i}(W_{1,i}+S_{1,i})] = \mathbb{E}[X_{1,i}W_{1,i}] + \frac{1}{\mu}\mathbb{E}[X_{1,i}]. \qquad (2)$$

To calculate $\mathbb{E}[X_{1,i}W_{1,i}]$, we follow the approach of [8] and characterize the waiting time $W_{1,i}$ by means of two events $E_{1,i}^{\mathrm{B}}$ and $E_{1,i}^{\mathrm{L}}$ as

$$E_{1,i}^{\mathrm{B}} = \{T_{1,i-1} \geq X_{1,i}\}, \quad E_{1,i}^{\mathrm{L}} = \{T_{1,i-1} < X_{1,i}\}. \qquad (3)$$

Here, *brief event* $E_{1,i}^{\mathrm{B}}$ is the event where the interarrival time of packet $1, i$ is shorter than the system time of packet $1, i-1$. On the contrary, *long event* $E_{1,i}^{\mathrm{L}}$ refers to the complementary event where the interarrival time of packet $1, i$ is longer than the system time of packet $1, i-1$.

Under the event $E_{1,i}^{\mathrm{B}}$, the waiting time of packet $1, i$ ($W_{1,i}$) contains two terms: 1) the residual system time to complete serving packet $1, i-1$, and 2) the sum of service times of the source 2 packets that arrived during $X_{1,i}$ and must be served before packet $1, i$ according to the FCFS policy. Under the event $E_{1,i}^{\mathrm{L}}$, the waiting time of packet $1, i$ contains two terms: 1) the possible residual service time of a source 2 packet that is under service at the arrival instant of packet $1, i$, and 2) the sum of service times of source 2 packets in the queue that must be served before packet $1, i$ according to the FCFS policy. Thus, by means of the two events in (3), the waiting time for packet $1, i$ can be expressed as

$$W_{1,i} = \begin{cases} T_{1,i-1} - X_{1,i} + \sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{B}}} S_{2,i'}, & E_{1,i}^{\mathrm{B}} \\ \sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{L}}} S_{2,i'} + R_{2,i}^{\mathrm{L}}, & E_{1,i}^{\mathrm{L}}, \end{cases} \qquad (4)$$

where $\mathcal{M}_{2,i}^{\mathrm{B}}$ is the set of indices of queued packets of source 2 that must be served before packet $1, i$ under the event $E_{1,i}^{\mathrm{B}}$, where $|\mathcal{M}_{2,i}^{\mathrm{B}}| = M_{2,i}^{\mathrm{B}}$; $\mathcal{M}_{2,i}^{\mathrm{L}}$ is the set of indices of packets of source 2 that are in the queue (but not under service) at the arrival instant of packet $1, i$ conditioned on the event $E_{1,i}^{\mathrm{L}}$ and, thus, must be served before packet $1, i$, where $|\mathcal{M}_{2,i}^{\mathrm{L}}| = M_{2,i}^{\mathrm{L}}$; $R_{2,i}^{\mathrm{L}}$ is a random variable that represents the possible residual service time of the packet of source 2 that is under service at the arrival instant of packet $1, i$ conditioned on the event $E_{1,i}^{\mathrm{L}}$.

For the case $E_{1,i}^{\mathrm{B}}$, let us further divide the waiting time $W_{1,i}$ in (4) into two terms $R_{1,i}^{\mathrm{B}}$ and $S_{1,i}^{\mathrm{B}}$ as follows. Let

$$R_{1,i}^{\mathrm{B}} = T_{1,i-1} - X_{1,i} \qquad (5)$$

represent the residual system time to complete serving packet $1, i-1$ and let

$$S_{1,i}^{\mathrm{B}} = \sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{B}}} S_{2,i'} \qquad (6)$$

represent the sum of service times of source 2 packets that arrived during $X_{1,i}$ and must be served before packet $1, i$. Similarly for the event $E_{1,i}^{\mathrm{L}}$, let

$$S_{1,i}^{\mathrm{L}} = \sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{L}}} S_{2,i'} \qquad (7)$$

represent the sum of service times of source 2 packets that must be served before packet $1, i$. Based on (5), (6), and (7), $\mathbb{E}[X_{1,i}W_{1,i}]$ in (2) can be expressed as

$$\begin{aligned}\mathbb{E}[X_{1,i}W_{1,i}] = &\left(\mathbb{E}[R_{1,i}^{\mathrm{B}}X_{1,i}|E_{1,i}^{\mathrm{B}}] + \mathbb{E}[S_{1,i}^{\mathrm{B}}X_{1,i}|E_{1,i}^{\mathrm{B}}]\right)\\ &P(E_{1,i}^{\mathrm{B}}) + \mathbb{E}[(S_{1,i}^{\mathrm{L}} + R_{2,i}^{\mathrm{L}})X_{1,i}|E_{1,i}^{\mathrm{L}}]P(E_{1,i}^{\mathrm{L}}), \qquad (8)\end{aligned}$$

where $P(E_{1,i}^{\mathrm{B}})$ and $P(E_{1,i}^{\mathrm{L}})$ denote the probabilities of the events $E_{1,i}^{\mathrm{B}}$ and $E_{1,i}^{\mathrm{L}}$, respectively.

Next, we derive the expressions for $P(E_{1,i}^{\mathrm{B}})$ and $P(E_{1,i}^{\mathrm{L}})$ in (8). Then, by referring to $\mathbb{E}[R_{1,i}^{\mathrm{B}}X_{1,i}|E_{1,i}^{\mathrm{B}}]$, $\mathbb{E}[S_{1,i}^{\mathrm{B}}X_{1,i}|E_{1,i}^{\mathrm{B}}]$, and $\mathbb{E}[(S_{1,i}^{\mathrm{L}} + R_{2,i}^{\mathrm{L}})X_{1,i}|E_{1,i}^{\mathrm{L}}]$ in (8) as the first, the second, and the third conditional expectation terms of (8), we derive the first, second, and the third terms in Sections III-A, III-B, and III-C respectively.

Lemma 1. *The probabilities of the events* $E_{1,i}^{\mathrm{B}}$ *and* $E_{1,i}^{\mathrm{L}}$

in (3) *are calculated as follows:*

$$P(E^{\mathrm{B}}_{1,i}) = \frac{\rho_1}{(1-\rho_2)}, \quad P(E^{\mathrm{L}}_{1,i}) = \frac{(1-\rho)}{(1-\rho_2)}. \tag{9}$$

Proof. See Appendix A. □

A. The First Conditional Expectation in (8)

Let us now focus on the first conditional expectation term $\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}]$ in (8). According to (5), this term is expressed as follows:

$$\begin{aligned}&\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}] = \mathbb{E}[T_{1,i-1}X_{1,i}|E^{\mathrm{B}}_{1,i}] - \mathbb{E}[X^2_{1,i}|E^{\mathrm{B}}_{1,i}]\\&= \int_0^\infty \int_0^\infty xt f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t -\\&\int_0^\infty x^2 f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)\mathrm{d}x,\end{aligned} \tag{10}$$

where $f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)$ is the PDF of the interarrival time $X_{1,i}$ given the event $E^{\mathrm{B}}_{1,i}$ and $f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)$ is the joint PDF of $X_{1,i}$ and system time $T_{1,i-1}$ given the event $E^{\mathrm{B}}_{1,i}$. They are given by the following two lemmas.

Lemma 2. *The conditional PDF $f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)$ is given by*

$$f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x) = \mu(1-\rho_2)e^{-\mu(1-\rho_2)x}. \tag{11}$$

Proof. See Appendix A. □

Lemma 3. *The PDF $f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)$ is given by*

$$\begin{aligned}&f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)\\&= \begin{cases} 0 & x>t \\ \mu^2(1-\rho)(1-\rho_2)e^{-\lambda_1 x}e^{-\mu(1-\rho)t} & x \le t. \end{cases}\end{aligned}$$

Due to the space limitations, the proof is given in [11].

Now, having introduced the conditional PDFs in Lemma 2 and Lemma 3, we have

$$\begin{aligned}&\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}] = \int_0^\infty\int_0^\infty xt f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t -\\&\int_0^\infty x^2 f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)\mathrm{d}x = \frac{1}{\mu^2(1-\rho_2)(1-\rho)}.\end{aligned} \tag{12}$$

B. The Second Conditional Expectation in (8)

Next, we derive the second term $\mathbb{E}[S^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}]$ in (8). First, let us elaborate the quantity $M^{\mathrm{B}}_{2,i}$ which is an integral part of calculating (8). Recall that $M^{\mathrm{B}}_{2,i}$ is defined as the number of queued packets of source 2 that must be served before packet $1,i$ according to the FCFS policy under the event $E^{\mathrm{B}}_{1,i} = \{T_{1,i-1} \ge X_{1,i}\}$. Thus, $M^{\mathrm{B}}_{2,i}$ is equal to the number of arrived (and thus, queued) packets of source 2 during the (brief) interarrival time $X_{1,i}$. Consequently, we have a Markov chain $T_{1,i-1} \leftrightarrow X_{1,i} \leftrightarrow M^{\mathrm{B}}_{2,i}$ conditioned on the event $E^{\mathrm{B}}_{1,i}$, i.e., $M^{\mathrm{B}}_{2,i}$ is independent of $T_{1,i-1}$ given $X_{1,i}$ under the event $E^{\mathrm{B}}_{1,i}$. Accordingly, the expectation $\mathbb{E}[S^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}]$ is given as

$$\begin{aligned}&\mathbb{E}[S^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}] = \int_0^\infty x\mathbb{E}\big[\textstyle\sum_{i'\in\mathcal{M}^{\mathrm{B}}_{2,i}} S_{2,i'}|E^{\mathrm{B}}_{1,i}, X_{1,i}=x\big]\\&f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)\mathrm{d}x\\&\overset{(a)}{=} \frac{1}{\mu}\int_0^\infty x\mathbb{E}\big[M^{\mathrm{B}}_{2,i}|X_{1,i}=x\big] f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)\mathrm{d}x\\&\overset{(b)}{=} \rho_2\int_0^\infty x^2\mu(1-\rho_2)e^{-\mu(1-\rho_2)x}\mathrm{d}x = \frac{2\rho_2}{\mu^2(1-\rho_2)^2},\end{aligned} \tag{13}$$

where equality (a) follows because (i) the service time $S_{2,i'}$ is independent of all other random variables in the system and (ii) by the Markov chain property $T_{1,i-1} \leftrightarrow X_{1,i} \leftrightarrow M^{\mathrm{B}}_{2,i}$ conditioned on $E^{\mathrm{B}}_{1,i}$, $M^{\mathrm{B}}_{2,i}$ is independent of $T_{1,i-1}$ given $X_{1,i}=x$ under the event $E^{\mathrm{B}}_{1,i}$; equality (b) comes from Lemma 2 and the fact that $\mathbb{E}[M^{\mathrm{B}}_{2,i}|X_{1,i}=x] = \lambda_2 x$.

C. The Third Conditional Expectation in (8)

The third term $\mathbb{E}[(S^{\mathrm{L}}_{1,i}+R^{\mathrm{L}}_{2,i})X_{1,i}|E^{\mathrm{L}}_{1,i}]$ in (8) can be calculated as

$$\begin{aligned}&\mathbb{E}[(S^{\mathrm{L}}_{1,i}+R^{\mathrm{L}}_{2,i})X_{1,i}|E^{\mathrm{L}}_{1,i}] = \int_0^\infty\int_0^\infty x\mathbb{E}\big[\textstyle\sum_{i'\in\mathcal{M}^{\mathrm{L}}_{2,i}} S_{2,i'}|\cdots\\&X_{1,i}=x, T_{1,i-1}=t, E^{\mathrm{L}}_{1,i}\big] f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t\\&+\int_0^\infty\int_0^\infty x\mathbb{E}\left[R^{\mathrm{L}}_{2,i}|X_{1,i}=x, T_{1,i-1}=t, E^{\mathrm{L}}_{1,i}\right]\cdots\\&\qquad f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{14}$$

where the first term on the right hand side is calculated as

$$\begin{aligned}&\int_0^\infty\int_0^\infty x\mathbb{E}\big[\textstyle\sum_{i'\in\mathcal{M}^{\mathrm{L}}_{2,i}} S_{2,i'}|X_{1,i}=x, T_{1,i-1}=t, E^{\mathrm{L}}_{1,i}\big]\cdots\\&f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t \overset{(a)}{=} \frac{1}{\mu}\int_0^\infty\int_0^\infty x\mathbb{E}\big[M^{\mathrm{L}}_{2,i}|\cdots\\&X_{1,i}=x, T_{1,i-1}=t, E^{\mathrm{L}}_{1,i}\big] f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t\\&= \frac{1}{\mu}\int_0^\infty\int_0^\infty x \textstyle\sum_{m=0}^\infty m\Pr[M^{\mathrm{L}}_{2,i}=m|X_{1,i}=x,\cdots\\&\quad T_{1,i-1}=t, E^{\mathrm{L}}_{1,i}] f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{15}$$

where equality (a) follows because the service time $S_{2,i'}$ is independent of all other random variables in the system.

Due to the memoryless property of the exponentially distributed service time, the possible residual service time of the packet of source 2 that is under service at the arrival instant of packet $1,i$ for event $E^{\mathrm{L}}_{1,i}$ is also exponentially distributed; thus, the waiting time is the sum of $\hat{M}^{\mathrm{L}}_{2,i}$ exponentially distributed random variables, where $\hat{M}^{\mathrm{L}}_{2,i}$ is the total number of source 2 packets in the system (either in the queue or under service) at the arrival instant of packet $1,i$ conditioned on the event $E^{\mathrm{L}}_{1,i}$ [12, p. 168]. Therefore, the waiting time $W_{1,i}$ can be expressed as

$$W_{1,i} = S^{\mathrm{L}}_{1,i} + R^{\mathrm{L}}_{2,i} = \textstyle\sum_{i'\in\hat{\mathcal{M}}^{\mathrm{L}}_{2,i}} S_{2,i'}, \tag{16}$$

where $\hat{\mathcal{M}}^{\mathrm{L}}_{2,i}$ is the set of indices of source 2 packets that are in the system at the arrival instant of packet $1,i$ for event $E^{\mathrm{L}}_{1,i}$, with $|\hat{\mathcal{M}}^{\mathrm{L}}_{2,i}| = \hat{M}^{\mathrm{L}}_{2,i}$. By (16), $\mathbb{E}[W_{1,i}X_{1,i}|E^{\mathrm{L}}_{1,i}]$

(cf. (14)) can be calculated as

$$\mathbb{E}[W_{1,i}X_{1,i}|E^{\rm L}_{1,i}] = \int_0^\infty \int_0^\infty x\mathbb{E}\big[\textstyle\sum_{i'\in\hat{\mathcal{M}}^{\rm L}_{2,i}} S_{2,i'}| \cdots$$
$$X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}\big] f_{X_{1,i}T_{1,i-1}|E^{\rm L}_{1,i}}(x,t){\rm d}x{\rm d}t$$
$$=\frac{1}{\mu}\int_0^\infty \int_0^\infty x\mathbb{E}\big[\hat{M}^{\rm L}_{2,i}|X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}\big]\cdots$$
$$f_{X_{1,i}T_{1,i-1}|E^{\rm L}_{1,i}}(x,t){\rm d}x{\rm d}t$$
$$=\frac{1}{\mu}\int_0^\infty \int_0^\infty x\textstyle\sum_{m=0}^\infty m{\rm Pr}[\hat{M}^{\rm L}_{2,i}=m|X_{1,i}=x,\cdots$$
$$T_{1,i-1}=t, E^{\rm L}_{1,i}] f_{X_{1,i}T_{1,i-1}|E^{\rm L}_{1,i}}(x,t){\rm d}x{\rm d}t. \quad (17)$$

Next, we calculate ${\rm Pr}[\hat{M}^{\rm L}_{2,i}=m|X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}]$ in (17) by introducing an auxiliary random variable $J^{\rm L}_{2,i}$ that represents the number of source 2 packets in the system at the departure instant of packet $1,i-1$ for event $E^{\rm L}_{1,i}$. Using the law of total expectation, we have

$${\rm Pr}[\hat{M}^{\rm L}_{2,i}=m|X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}] = \textstyle\sum_{j=0}^\infty\cdots$$
$${\rm Pr}[\hat{M}^{\rm L}_{2,i}=m|J^{\rm L}_{2,i}=j, X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}]\cdots$$
$${\rm Pr}[J^{\rm L}_{2,i}=j|X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}], \quad (18)$$

where

$${\rm Pr}[J^{\rm L}_{2,i}=j|X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}] \quad (19)$$
$$\overset{(a)}{=} {\rm Pr}[J^{\rm L}_{2,i}=j|T_{1,i-1}=t, E^{\rm L}_{1,i}] \overset{(b)}{=} e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!},$$

where equality (a) follows because $J^{\rm L}_{2,i}$ is conditionally independent of $X_{1,i}$ given $T_{1,i-1}$ and $E^{\rm L}_{1,i}$; equality (b) follows because (i) under the long event $E^{\rm L}_{1,i}$, all $J^{\rm L}_{2,i}$ source 2 packets that are in the system at the departure instant of packet $1,i-1$ must have arrived during the system time $T_{1,i-1}$, and (ii) the probability of having j Poisson arrivals of rate λ_2 during the time interval $T_{1,i-1}=t$ is $e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!}$ [12, Eq. (2.119)].

Note that during the time interval between the departure of packet $1,i-1$ and the arrival of packet $1,i$ the queue receives packets only from source 2 and, therefore, the system behaves as a single-source M/M/1 queue. Thus, ${\rm Pr}[\hat{M}^{\rm L}_{2,i}=m|J^{\rm L}_{2,i}=j, X_{1,i}=x, T_{1,i-1}=t, E^{\rm L}_{1,i}]$ in (18) represents the probability that a single-source M/M/1 queueing system with arrival rate λ_2 and which initially holds j packets (either in the queue or under service) ends up holding m packets after $\tau=x-t$ seconds. We denote this probability compactly by $\bar{P}_{m|j}(\tau)$ and it is given by the transient analysis of an M/M/1 queueing system as [12, Eq. (2.163)]

$$\bar{P}_{m|j}(\tau) = e^{-(\lambda_2+\mu)\tau}\Bigg[\rho_2^{(m-1)/2} I_{m-1}(2\sqrt{\mu\lambda_2}\tau) \quad (20)$$
$$+\rho_2^{(m-j-1)/2} I_{m+j+1}(2\sqrt{\mu\lambda_2}\tau)\Bigg] + \rho_2^m(1-\rho_2)\cdots$$
$$\Bigg(1-Q_{m+j+2}(\sqrt{2\lambda_2\tau},\sqrt{2\mu\tau})\Bigg),$$

where $I_k(\cdot)$ is the modified Bessel function of the first kind of order k, and $Q_k(a,b)$ is the generalized Q-function. Substituting (18), (19), and (20) into (17), we have

$$\mathbb{E}[W_{1,i}X_{1,i}|E^{\rm L}_{1,i}]=\frac{1}{\mu}\int_0^\infty \int_0^\infty x\textstyle\sum_{m=0}^\infty\sum_{j=0}^\infty m\cdots$$
$$\bar{P}_{m|j}(x-t)e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!} f_{X_{1,i}T_{1,i-1}|E^{\rm L}_{1,i}}(x,t){\rm d}x{\rm d}t$$
$$\overset{(a)}{=} \frac{\lambda_1(1-\rho)}{P(E^{\rm L}_{1,i})}\int_0^\infty\int_0^\infty (t+\tau)e^{-\mu(t+\rho_1\tau)}\cdots$$
$$\Bigg(\textstyle\sum_{m=0}^\infty\sum_{j=0}^\infty m\bar{P}_{m|j}(\tau)\frac{(\lambda_2 t)^j}{j!}\Bigg){\rm d}\tau{\rm d}t \quad (21)$$
$$\triangleq \frac{\lambda_1(1-\rho)}{P(E^{\rm L}_{1,i})}\Psi(\mu,\rho_1,\lambda_2),$$

where (a) follows substitution $\tau=x-t$ and Lemma 4 below which derives the conditional PDF $f_{X_{1,i},T_{1,i-1}|E^{\rm L}_{1,i}}(x,t)$. Note that the integral in $\Psi(\mu,\rho_1,\lambda_2)$ needs to be in general numerically calculated.

Lemma 4. *The PDF $f_{X_{1,i},T_{1,i-1}|E^{\rm L}_{1,i}}(x,t)$ is given by*

$$f_{X_{1,i},T_{1,i-1}|E^{\rm L}_{1,i}}(x,t)$$
$$=\begin{cases}0 & x<t\\ \mu^2\rho_1(1-\rho_2)e^{-\lambda_1 x}e^{-\mu(1-\rho)t} & x\geq t.\end{cases}$$

Due to the space limitations, the proof is given in [11].

By substituting the probabilities $P(E^{\rm B}_{1,i})$ and $P(E^{\rm L}_{1,i})$ given by Lemma 1 and the three derived conditional expectation terms (12), (13), and (21) into (8), $\mathbb{E}[X_{1,i}W_{1,i}]$ is calculated. Finally, by substituting the result of $\mathbb{E}[X_{1,i}W_{1,i}]$ and (2) into (1), the average AoI of source 1 can be expressed as:

$$\Delta_1 = \lambda_1^2(1-\rho)\Psi(\mu,\rho_1,\lambda_2)+ \quad (22)$$
$$\frac{1}{\mu}\Bigg(\frac{1}{\rho_1}+\frac{\rho}{1-\rho}+\frac{(2\rho_2-1)(\rho-1)}{(1-\rho_2)^2}+\frac{2\rho_1\rho_2(\rho-1)}{(1-\rho_2)^3}\Bigg).$$

Remark 1. Note that (22) does not coincide with the prior result [3, Theorem 1] and [8, Eq. (16)]. The dissimilarity is explained in the following. The authors of [3], [8] considered a similar two-source FCFS M/M/1 queueing model, with the aim of deriving a closed-form expression for the average AoI of source 1 (Δ_1). Let us focus on relation (33) of [8] where the authors compute a conditional expectation equivalent to our $\mathbb{E}[W_{1,i}X_{1,i}|E^{\rm L}_{1,i}]$ given by (21), which by (16) can be expressed as

$$\mathbb{E}[W_{1,i}X_{1,i}|E^{\rm L}_{1,i}] = \mathbb{E}\Big[\textstyle\sum_{i'\in\hat{\mathcal{M}}^{\rm L}_{2,i}} S_{2,i'}X_{1,i}|E^{\rm L}_{1,i}\Big]. \quad (23)$$

The authors of [8] tacitly assumed conditional independency between $\sum_{i'\in\hat{\mathcal{M}}^{\rm L}_{2,i}} S_{2,i'}$ and $X_{1,i}$ under the event $E^{\rm L}_{1,i}=\{T_{1,i-1}<X_{1,i}\}$, and calculated (23) as a multiplication of two expectations as

$$\mathbb{E}[W_{1,i}X_{1,i}|E^{\rm L}_{1,i}] = \mathbb{E}\Big[\textstyle\sum_{i'\in\hat{\mathcal{M}}^{\rm L}_{2,i}} S_{2,i'}|T_{1,i-1}<X_{1,i}\Big]\cdots$$

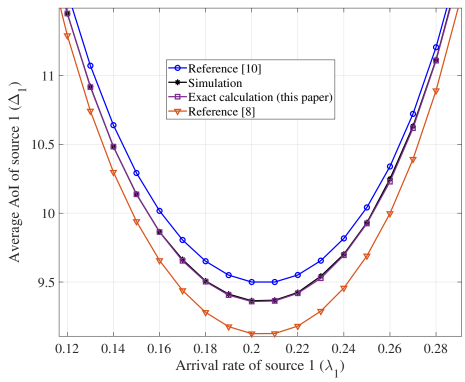

Fig. 1: The average AoI of source 1 as a function of λ_1 with $\mu = 1$.

$$\mathbb{E}\Big[X_{1,i}|T_{1,i-1} < X_{1,i}\Big]. \tag{24}$$

The critical point is that even if $X_{1,i}$ is independent of $T_{1,i-1}$, they become *dependent* when conditioned on the event $E^{\mathrm{L}}_{1,i} = \{T_{1,i-1} < X_{1,i}\}$, as in (23). This conditional dependency is violated by the separation of the expectations in (24) because the quantity $\hat{M}^{\mathrm{L}}_{2,i}$ in general depends on both $T_{1,i-1}$ and $X_{1,i}$, and, thus, the multiplicative quantities $\sum_{i' \in \hat{\mathcal{M}}^{\mathrm{L}}_{2,i}} S_{2,i'}$ and $X_{1,i}$ are dependent under the event $E^{\mathrm{L}}_{1,i}$. Note that we incorporate this conditional dependency in calculating $\mathbb{E}[W_{1,i}X_{1,i}|E^{\mathrm{L}}_{1,i}]$ by using $f_{X_{c,i},T_{c,i-1}|E^{\mathrm{L}}_{c,i}}(x,t)$.

Remark 2. It is worth to note that (22) neither coincides with our prior result [10, Eq. (25)]. The dissimilarity comes from the fact that in [10], we wrongly used steady-state properties of a queueing system in calculating $\mathbb{E}[\hat{M}^{\mathrm{L}}_{2,i}|X_{1,i} = x, T_{1,i-1} = t, E^{\mathrm{L}}_{1,i}]$ in (17).

IV. Validation and Simulation Results

In this section, we evaluate the average AoI in a multi-source M/M/1 queueing model and compare our exact expression in (22) with the results in existing works [8] and [10]. Fig. 1 depicts the average AoI of source 1 (Δ_1) as a function of λ_1 with $\lambda_2 = 0.6$ and $\mu = 1$. As it can be seen, the simulation result and our proposed solution overlap perfectly. Due to the calculation errors in [8] and [10], both curves have a gap to the correct average AoI value. In addition, this figure illustrates that generating the status update packets too frequently or too rarely does not minimize the average AoI.

Fig. 2 depicts the average delay of source 1 as a function of λ_1 for different values of λ_2 with $\mu = 1$. The average delay is defined as the summation of the average waiting time and average service time i.e., $\mathbb{E}[W] + 1/\mu$. As the number of arrivals of source 2 packets increases, the queue becomes more congested and the average delay of source 1 increases. By comparing Figs. 1 and 2 one can see that the delay does not fully capture the information freshness, i.e., minimizing the average system delay does not necessarily lead to a good performance in terms of AoI and, reciprocally, minimizing the average AoI does not minimize the average system delay.

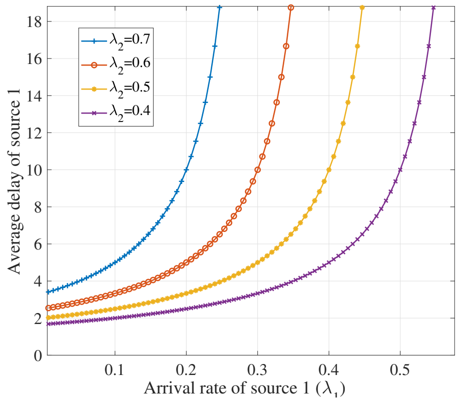

Fig. 2: The average delay of source 1 as a function of λ_1 for different values of λ_2 with $\mu = 1$.

V. Conclusions

We considered a single-server multi-source M/M/1 queueing model with FCFS serving policy and analyzed the average AoI of each source. We derived an exact expression for the average AoI for a multi-source M/M/1 queueing model. The simulation results validated the exact expression for the average AoI for the considered queueing model. In addition, the simulations illustrated that generating the status update packets too frequently or too rarely does not minimize the average AoI.

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Appendix A
Proof of Lemma 1, 2, and 3

A. *Proof of Lemma 1*

Using the facts that $T_{1,i-1}$ and $X_{1,i}$ are independent and $f_{X_{1,i}}(x) = \lambda_1 e^{-\lambda_1 x}$, $P(E^{\mathrm{B}}_{1,i})$ can be written as

$$P(E^{\mathrm{B}}_{1,i}) = \int_0^\infty P(T_{1,i-1} \geq X_{1,i}|T_{1,i-1} = t) f_{T_{1,i-1}}(t)\mathrm{d}t$$

$$= \int_0^\infty F_{X_{1,i}}(t) f_{T_{1,i-1}}(t) \mathrm{d}t = 1 - \int_0^\infty e^{-\lambda_1 t} f_{T_{1,i-1}}(t) \mathrm{d}t$$

$$\stackrel{(a)}{=} 1 - \mathbb{E}[e^{-\lambda_1 T}] = 1 - L_T(\lambda_1), \tag{25}$$

where equality (a) follows because the system times of different packets are stochastically identical, i.e., $T_{1,i} =^{\mathrm{st}} T_{2,i} =^{\mathrm{st}} T$, $\forall i$ [8]; $L_T(\lambda_1) = \mathbb{E}[e^{-\lambda_1 T}]$ is the Laplace transform of the PDF of the system time T at λ_1. Because $E^{\mathrm{L}}_{1,i}$ is the complementary event of $E^{\mathrm{B}}_{1,i}$, we have $P(E^{\mathrm{L}}_{1,i}) = 1 - P(E^{\mathrm{B}}_{1,i}) = L_T(\lambda_1)$. The relation between the Laplace transforms of the system time T and service time S is given as [13, Sect. 5.1.2]

$$L_T(a) = \frac{(1-\rho) a L_S(a)}{a - \lambda(1 - L_S(a))}, \tag{26}$$

where $L_S(a)$ is the Laplace transform of the PDF of the service time S at a; note that the service times of all packets are stochastically identical as $S_{1,i} =^{\mathrm{st}} S_{2,i} =^{\mathrm{st}} S$, $\forall i$. Since the service time is an exponentially distributed random variable with mean $1/\mu$, we have

$$L_S(a) = \int_0^\infty \mu e^{-(\mu+a)s} \mathrm{d}s = \frac{\mu}{\mu + a}. \tag{27}$$

By substituting (27) into (26), $L_T(a)$ is given as

$$L_T(a) = \frac{\mu(1-\rho)}{\mu(1-\rho) + a}. \tag{28}$$

Finally, by substituting λ_1 into (28) we have

$$P(E^{\mathrm{L}}_{1,i}) = L_T(\lambda_1) = \frac{1-\rho}{1-\rho_2}, \tag{29}$$

$$P(E^{\mathrm{B}}_{1,i}) = 1 - P(E^{\mathrm{L}}_{1,i}) = \frac{\rho_1}{1-\rho_2}.$$

B. Proof of Lemma 2

The conditional PDF $f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)$ can be obtained by taking the derivative of the cumulative distribution function (CDF) $F_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)$, i.e., $f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x) = \frac{\mathrm{d}(F_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x))}{\mathrm{d}x}$, such that we have

$$f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x) = \lim_{h \to 0} \frac{F_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x+h) - F_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)}{h}$$

$$= \lim_{h \to 0} \frac{P\big((X_{1,i} \le x+h) \cap E^{\mathrm{B}}_{1,i}\big) - P\big((X_{1,i} \le x) \cap E^{\mathrm{B}}_{1,i}\big)}{P\big(E^{\mathrm{B}}_{1,i}\big) h}$$

$$= \lim_{h \to 0} \frac{P\big((x \le X_{1,i} \le x+h) \cap (T_{c,i-1} \ge X_{1,i})\big)}{P\big(E^{\mathrm{B}}_{1,i}\big) h}$$

$$\stackrel{(a)}{=} \lim_{h \to 0} \frac{\int_x^{x+h} \int_x^\infty f_{X_{1,i}}(\eta) f_{T_{1,i-1}}(t) \mathrm{d}t \mathrm{d}\eta}{P(E^{\mathrm{B}}_{1,i}) h}$$

$$= \frac{1 - F_{T_{1,i-1}}(x)}{P(E^{\mathrm{B}}_{1,i})} \lim_{h \to 0} \frac{\int_x^{x+h} f_{X_{1,i}}(\eta) \mathrm{d}\eta}{h}$$

$$\stackrel{(b)}{=} \frac{\big(1 - F_{T_{1,i-1}}(x)\big) f_{X_{1,i}}(x)}{P(E^{\mathrm{B}}_{1,i})}, \tag{30}$$

where equality (a) follows because $X_{1,i}$ and $T_{1,i-1}$ are independent, i.e., $f_{X_{1,i},T_{1,i-1}}(x,t) = f_{X_{1,i}}(x) f_{T_{1,i-1}}(t)$; equality (b) follows from the definition of the derivative of an integral [14, Sect. 6.3]. From (30), we need to calculate $F_{T_{1,i-1}}(x)$ to obtain $f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)$. To derive $F_{T_{1,i-1}}(x)$, first we calculate the PDF of the system time $f_{T_{1,i-1}}(x)$ by calculating the inverse Laplace transform of the Laplace transform of the system time T (28). Thus, the inverse Laplace transform of (28) is given as [14, Sect. 13.5]

$$f_T(x) = L^{-1}(L_T(a)) = \mu(1-\rho) e^{-\mu(1-\rho)x}, \tag{31}$$

where $L^{-1}(\cdot)$ is the inverse Laplace transform. Consequently, the CDF of the system time $F_T(x)$ is given as

$$F_T(x) = \int_0^x f_T(a) \mathrm{d}a = 1 - e^{-\mu(1-\rho)x}. \tag{32}$$

Finally, substituting (32), $P(E^{\mathrm{B}}_{1,i})$, and $f_{X_{1,i}}(x) = \lambda_1 e^{-\lambda_1 x}$ in (30) gives (11).

REFERENCES

[1] P. Corke, T. Wark, R. Jurdak, W. Hu, P. Valencia, and D. Moore, "Environmental wireless sensor networks," *Proc. IEEE*, vol. 98, no. 11, pp. 1903–1917, Nov. 2010.

[2] P. Papadimitratos, A. D. L. Fortelle, K. Evenssen, R. Brignolo, and S. Cosenza, "Vehicular communication systems: Enabling technologies, applications, and future outlook on intelligent transportation," *IEEE Commun. Mag.*, vol. 47, no. 11, pp. 84–95, Nov. 2009.

[3] R. D. Yates and S. K. Kaul, "The age of information: Real-time status updating by multiple sources," *IEEE Trans. Inform. Theory*, vol. 65, no. 3, pp. 1807–1827, Mar. 2019.

[4] M. Moltafet, M. Leinonen, M. Codreanu, and N. Pappas, "Power minimization in wireless sensor networks with constrained AoI using stochastic optimization," in *Proc. Annual Asilomar Conf. Signals, Syst., Comp.*, Pacific Grove, USA, Nov. 3–6, 2019, pp. 406–410.

[5] M. Moltafet, M. Leinonen, and M. Codreanu, "Worst case age of information in wireless sensor networks: A multi-access channel," *IEEE Wireless Commun. Lett.*, vol. 9, no. 3, pp. 321–325, Mar. 2020.

[6] S. Kaul, R. Yates, and M. Gruteser, "Real-time status: How often should one update?" in *Proc. IEEE Int. Conf. on Computer. Commun. (INFOCOM)*, Orlando, FL, USA, Mar. 25–30, 2012, pp. 2731–2735.

[7] S. K. Kaul, R. D. Yates, and M. Gruteser, "Status updates through queues," in *Proc. Conf. Inform. Sciences Syst. (CISS)*, Princeton, NJ, USA, Mar. 21–23, 2012, pp. 1–6.

[8] R. D. Yates and S. Kaul, "Real-time status updating: Multiple sources," in *Proc. IEEE Int. Symp. Inform. Theory*, Cambridge, MA, USA, Jul. 1–6, 2012, pp. 2666–2670.

[9] E. Najm and E. Telatar, "Status updates in a multi-stream M/G/1/1 preemptive queue," in *Proc. IEEE Int. Conf. on Computer. Commun. (INFOCOM)*, Honolulu, HI, USA, Apr. 15–19, 2018, pp. 124–129.

[10] M. Moltafet, M. Leinonen, and M. Codreanu, "Closed-form expression for the average age of information in a multi-source M/G/1 queueing model," in *Proc. IEEE Inform. Theory Workshop*, Visby, Gotland, Sweden, Aug. 25–28, 2019.

[11] ——, "On the age of information in multi-source queueing models," *Submitted to IEEE Trans. Commun*, [Online]. https://arxiv.org/abs/1911.07029v1, 2019.

[12] L. Kleinrock, *Queueing Systems, Volume 1: Theory*. New York: John Wiley and Sons, 1995.

[13] J. N. Daigle, *Queueing Theory with Applications to Packet Telecommunication*. New York: Springer Science, 2005.

[14] L. Rade and B. Westergren, *Mathematics Handbook for Science and Engineering*. Berlin, Germany: Springer, 2005.