On algebraic independence of a class of infinite products

Masaaki Amou and Keijo Väänänen

Abstract

Algebraic independence of values of certain infinite products is proved, where the transcendence of such numbers was already established by Tachiya. As applications explicit examples of algebraically independent numbers are also given.

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1 Introduction and the main results

Let K be an algebraic number field and $\mathcal{O}_K$ the ring of integers of K. For $\alpha \in K$ we shall denote the size of α by $\|\alpha\| = \max(|\overline{\alpha}|, \operatorname{den}(\alpha))$, where $|\overline{\alpha}|$ is the maximum of the absolute values of its conjugates and $\operatorname{den}(\alpha)$ is the least positive integer such that $\operatorname{den}(\alpha)\alpha \in \mathcal{O}_K$. In the present paper we are interested in infinite products of the form

$$\Phi(z) = \prod_{k=0}^{\infty} \frac{E_k(z^{r^k})}{F_k(z^{r^k})}, \tag{1}$$

where $r \geq 2$ is an integer and

$$E_k(z) = 1 + a_{k,1}z + \cdots + a_{k,L}z^L,\ F_k(z) = 1 + b_{k,1}z + \cdots + b_{k,L}z^L \in K[z] \tag{2}$$

with an integer $L \geq 1$.

In [7] Tachiya proved that under some conditions $\Phi(\alpha)$ with algebraic α, $0 < |\alpha| < 1$, is algebraic if and only if $\Phi(z) \in K(z)$. For this proof he applied the method developed in [3], which is based on inductive argument introduced in [2] and a variant of Mahler's method created by Loxton and van der Poorten in [5]. Here it is essential that a sequence of functions

$$\Psi_n(z) = \prod_{k=0}^{\infty} \frac{E_{n+k}(z^{r^k})}{F_{n+k}(z^{r^k})}, \quad n \geq 0,$$

(note that $\Psi_0(z) = \Phi(z)$) satisfies a chain of Mahler type functional equations

$$\Psi_{n+1}(z^r) = \frac{F_n(z)}{E_n(z)}\Psi_n(z), \quad n \geq 0. \tag{3}$$

The results of [7] were further developed in [1], in particular a quantitative refinement of Tachiya's result was obtained in [1, Theorem 4] if the irrationality measure of the function $\Phi(z)$ is finite. Recall that the irrationality measure $\mu(f)$ of $f(z) \in K[[z]]$ is defined to be the infimum of μ such that

$$\mathrm{ord}(A(z)f(z) - B(z)) \leq \mu M,$$

holds for all nonzero $(A(z), B(z)) \in K[z]^2$ satisfying $\max(\deg A, \deg B) \leq M$ provided that $M \geq M_0$ with some sufficiently large M_0 depending only on $f(z)$, where for $g(z) \in K[[z]]$ we denote by $\mathrm{ord}\, g(z)$ the zero order of $g(z)$ at $z = 0$. If there does not exist such a μ, we define $\mu(f) := \infty$. Note that the condition $\mu(\Phi) < \infty$ can be verified in some cases by using [1, Lemma 9].

Remark 1. Under the above notations, if $f(z) \in K(z)$, then $\mu(f) = \infty$, but $\mathrm{ord}(A(z)f(z) - B(z)) \leq 2M$ when $A(z)f(z) - B(z) \neq 0$ and M is at least the maximum of the degrees of the numerator and the denominator of $f(z)$.

Our main aim here is to study the algebraic independence of values of several products of type (1). Let

$$\Phi_j(z) = \prod_{k=0}^{\infty} \frac{E_{j,k}(z^{r^k})}{F_{j,k}(z^{r^k})}, \quad j = 1, \ldots, m, \tag{4}$$

with

$$E_{j,k}(z) = 1 + a_{j,k,1}z + \cdots + a_{j,k,L}z^L,\ F_{j,k}(z) = 1 + b_{j,k,1}z + \cdots + b_{j,k,L}z^L \in K[z]. \tag{5}$$

We assume that the coefficients $a_{j,k,i}$ and $b_{j,k,i}$ satisfy

$$\log |\overline{a_{j,k,i}}|,\ \log \left|\overline{b_{j,k,i}}\right| = o(r^k)\ (k \to \infty), \quad j = 1, \ldots, m;\ i = 1, \ldots, L, \tag{6}$$

and, for each $j = 1, \ldots, m$, there exists a positive integer D_j such that

$$D_j E_{j,k}(z),\ D_j F_{j,k}(z) \in \mathcal{O}_K[z], \quad k \geq 0. \tag{7}$$

For $\underline{f}(z) = (f_1(z), \ldots, f_m(z)) \in K[[z]]^m$ and $I = (i_1, \ldots, i_m) \in \mathbb{Z}^m$ we define

$$\underline{f}^I(z) = f_1^{i_1}(z) \cdots f_m^{i_m}(z).$$

Let $\underline{\Phi}(z) = (\Phi_1(z), \ldots, \Phi_m(z))$. In the following considerations we assume that for all $I = (i_1, \ldots, i_m) \in \mathbb{Z}^m \setminus \{\underline{0}\}$ there exists a positive constant $c(I)$ depending on $\underline{\Phi}(z)$ and I such that

$$\mu(\underline{\Phi}^I) < c(I). \tag{8}$$

To introduce our main results we denote, for each place w of K, by $|\ |_w$ the absolute value of K normalized in the usual way. The absolute height $H(\alpha)$ of $\alpha \in K$ is defined by

$$H(\alpha) = \prod_w \mathrm{M}_w(\alpha), \quad \mathrm{M}_w(\alpha) := \max(1, |\alpha|_w^{\kappa_w/\kappa}),$$

where $\kappa = [K : \mathbb{Q}]$, $\kappa_w = [K_w : \mathbb{Q}_w]$. By the product formula we have $H(\alpha) = H(\alpha^{-1})$ for all nonzero $\alpha \in K$. Further, the absolute height of the vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in K^k$ is defined by

$$H(\underline{\alpha}) = \prod_w \mathrm{M}_w(\underline{\alpha}), \quad \mathrm{M}_w(\underline{\alpha}) := \max(1, |\underline{\alpha}|_w^{\kappa_w/\kappa})$$

with $|\underline{\alpha}|_w = \max(|\alpha_1|_w, \ldots, |\alpha_k|_w)$. Then, for any nonzero vector $\underline{\alpha} \in K^k$, the inequality

$$H(\underline{\alpha})^{-\kappa/\kappa_w} \le |\underline{\alpha}|_w \le H(\underline{\alpha})^{\kappa/\kappa_w} \tag{9}$$

holds. In the following we fix an infinite place v of K, denote $|\ |_v = |\ |$ and assume that $\alpha \in K$ satisfies

$$0 < |\alpha| < 1, \quad \prod_{j=1}^{m} E_{j,k}(\alpha^{r^k}) F_{j,k}(\alpha^{r^k}) \neq 0, \quad k \ge 0 \tag{10}$$

and

$$\lambda(\alpha) = \lambda_K(\alpha) := \frac{\kappa \log H(\alpha^{-1})}{\kappa_v \log |\alpha^{-1}|} < \frac{d+1}{d} \tag{11}$$

with a positive integer d. Then we have

Theorem 1. *Let $\Phi_1(z), \ldots, \Phi_m(z)$ be infinite products (4) with (5) satisfying (6), (7) and (8). Let α be an element of K satisfying (10) and (11). Then, for a finite subset $\Lambda \subset \mathbb{Z}^m \setminus \{\underline{0}\}$ with $d = |\Lambda|$, the number of elements of Λ, there exist positive constants $\phi(\Lambda)$ and h_0 depending only on $\alpha, \underline{\Phi}$ and Λ such that*

$$\left| a_{\underline{0}} + \sum_{I \in \Lambda} a_I \underline{\Phi}^I(\alpha) \right| > h^{-\phi(\Lambda)}$$

holds for any $a_I \in O_K$, $I \in \Lambda \cup \underline{0}$, not all zero, with $h = \max(h_0, H((a_{\underline{0}}, \ldots, a_I, \ldots)))$.

In particular, if $\lambda(\alpha) = 1$, then the numbers $\Phi_1(\alpha), \ldots, \Phi_m(\alpha)$ are algebraically independent.

If the functions $\Phi_j(z)$, $j = 1, ..., m$, satisfy single functional equations of Mahler type, that is $E_{j,k}(z) = E_j(z)$ and $F_{j,k}(z) = F_j(z)$ for all $k \ge 0$, then there exist general results on algebraic independence for values of $\Phi_j(z)$. In fact, we have the following result as a particular case of a result by Nishioka [6, Theorem 4.4.1], where the qualitative part of it is due to Kubota [4] (see also [6, Theorem 3.5]) : Assume that $E_1(z)/F_1(z), ..., E_m(z)/F_m(z)$ are multiplicatively independent modulo $\mathcal{H}$, where $\mathcal{H} = \{g(z^r)/g(z) \mid g(z) \in K(z) \backslash 0\}$. Then the functions $\Phi_1(z), ..., \Phi_m(z)$ are algebraically independent over $K(z)$ and, for any algebraic number α with $0 < |\alpha| < 1$ satisfying (10), the numbers $\Phi_1(\alpha), ..., \Phi_m(\alpha)$ are algebraically independent. Moreover, for any H and $s \ge 1$ and for any polynomial $R \in \mathbb{Z}[x_1, ..., x_m]$ whose degree does not exceed s and whose coefficients are not greater than H in absolute values, we have

$$|R(\Phi_1(\alpha), ..., \Phi_m(\alpha))| > \exp\left(-\gamma s^m (\log H + s^{m+2})\right),$$

where γ is a positive constant depending only on α and the functions $\Phi_1, ..., \Phi_m$.

On the other hand, as far as we know, Theorem 1 seems to be the first one which gives algebraic independence for values of $\Phi_j(z)$ satisfying a chain of functional equations of Mahler type as above. We note that our proof of Theorem 1 uses the original inductive method by Duverney given in [2]. The reason why it works is based on the fact that the set of infinite products form a group under the usual multiplication, by which $\underline{\Phi}^I(z)$ belong to this set for all $I \in \mathbb{Z}^m$. Note also that our previous result [1, Theorem 4] implies, without assuming (11), the measure given in Theorem 1 in the case $d = 1$. Therefore, it is ensured that the starting point of our proof of Theorem 1 by using induction on d is correct. However, we shall give an alternative proof of this result for self-containdness as well as for its own interest.

With the aid of [1, Lemma 9] (which is Lemma 1 in Section 2) we have the following corollary to Theorem 1, which allows us to give explicit examples of algebraically independent numbers $\Phi_1(\alpha), \ldots, \Phi_m(\alpha)$.

Corollary. *Let $\Phi_1(z), \ldots, \Phi_m(z)$ be infinite products (4) with (5) satisfying (6) and (7). Assume that each $E_{j,k}(z)/F_{j,k}(z)$ has only real zeros or poles, if $r \geq 3$, and has only positive zeros or poles, if $r \geq 2$. Assume further that for any $I = (i_1, \ldots, i_m) \in \mathbb{Z}^m \setminus \{\underline{0}\}$ there exists a positive constant $C = C(I)$ such that, for any sufficiently large n,*

$$\prod_{j=1}^{m} E_{j,N}(z)^{i_j} \neq \prod_{j=1}^{m} F_{j,N}(z)^{i_j}$$

holds for some $N = N(n)$ with $n \leq N < n + C$. Let α be an element of K satisfying (10) and $\lambda(\alpha) = 1$. Then the numbers $\Phi_1(\alpha), \ldots, \Phi_m(\alpha)$ are algebraically independent having the measure given in Theorem 1.

Example 1. For a sequence $(a_k)_{k \geq 0}$ of non-zero integers such that

$$\log |a_k| = o(r^k),$$

and for a partition $\mathcal{S}_j = \{n_{j,k}\ ;\ k \in \mathbb{N}\}$, $j = 1, ..., m$, of $\mathbb{N}$ such that $n_{j,k+1} - n_{j,k}$ are uniformly bounded from above for all j and k, we define

$$\Phi_j(r, z) = \prod_{k \in \mathcal{S}_j} (1 - a_k z^{r^k}), \quad j = 1, \ldots, m.$$

Let α, $0 < |\alpha| < 1$, be an algebraic number with $\lambda(\alpha) = 1$ in $\mathbb{Q}(\alpha)$ such that

$$a_k \alpha^{r^k} \neq 1, \quad k \geq 0.$$

If $r \geq 3$, or if $r = 2$ and all a_k are also positive, then the numbers

$$\Phi_1(r, \alpha), \ldots, \Phi_m(r, \alpha)$$

are algebraically independent.

In the following two examples we denote by $(n(k))_{k\geq 0}$ a sequence of strictly increasing positive integers such that $n(k+1)-n(k)$ are uniformly bounded from above for all k.

Example 2. For sequences $(a_{j,k})_{k\geq 0}$, $j=1,\dots,m$, of non-zero integers such that

$$\log|a_{j,k}| = o(r^k), \quad j=1,\dots,m,$$

and that

$$a_{i,k} \neq a_{j,k}, \quad i\neq j,\ k\geq 0,$$

we define

$$\Phi_j(r,z) = \prod_{k=0}^{\infty}(1-a_{j,n(k)}z^{r^{n(k)}}), \quad j=1,\dots,m.$$

Let α, $0<|\alpha|<1$, be an algebraic number with $\lambda(\alpha)=1$ in $\mathbb{Q}(\alpha)$ such that

$$a_{j,n(k)}\alpha^{r^{n(k)}} \neq 1, \quad j=1,\dots,m,\ k\geq 0.$$

If $r\geq 3$, or if $r=2$ and all $a_{j,k}$ are also positive, then the numbers

$$\Phi_1(r,\alpha),\dots,\Phi_m(r,\alpha)$$

are algebraically independent.

Example 3. Let $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ denote Fibonacci and Lucas sequences, respectively. Let c, d_1 and d_2 be positive integers such that $cr+d_1$ is even and $cr+d_2$ is odd. Further, let $(b_{j,k})_{k\geq 0}$, $j=1,2$, be sequences of positive integers such that

$$\log b_{j,k} = o(r^k), \quad j=1,2.$$

If $r\geq 3$, then the numbers

$$\prod_{k=0}^{\infty}\left(1+\frac{b_{1,n(k)}}{F_{cr^{n(k)}+d_1}}\right), \quad \prod_{k=0}^{\infty}\left(1+\frac{b_{2,n(k)}}{L_{cr^{n(k)}+d_2}}\right)$$

are algebraically independent.

We organize this paper as follows. In Section 2, we state and prove Theorem 2, a functional analogue of Theorem 1. Though we do not use this theorem for proving Theorem 1, its proof, which is easier than the proof of Theorem 1, will show us the algebraic structure of the proof of Theorem 1. Then, in Section 3, we prove Theorem 1 along the same line of the proof of Theorem 2, but with necessary arithmetical estimations. The proof of the corollary to Theorem 1 and the examples are also given in Section 3.

2 Algebraic independence of our functions

In this section we denote by K an arbitrary field with characteristic zero, and consider infinite products (4) with (5) as elements of the formal power series ring $K[[z]]$. Under the assumption (8) they are multiplicatively independent modulo $K(z)$. In fact (8) gives much more, namely the following algebraic independence result generalizing [7, Lemma 1] to the case $m > 1$.

Theorem 2. *Let $\Phi_1(z), \ldots, \Phi_m(z)$ be infinite products (4) with (5) satisfying (8). Then, for any finite subset Λ of $\mathbb{Z}^m \setminus \{\underline{0}\}$, there exists positive constants $c(\Lambda)$ and M_0 depending on $\underline{\Phi}(z)$ and Λ such that*

$$\operatorname{ord}\Big(A_{\underline{0}}(z) + \sum_{I \in \Lambda} A_I(z) \underline{\Phi}^I(z)\Big) \leq c(\Lambda) M$$

holds for all $A_I(z) \in K[z], I \in \Lambda \cup \{\underline{0}\}$, not all zero, of degrees not greater than M provided that $M \geq M_0$.

Before proving Theorem 2 we note that the conditions $\mu(f) < \infty$ and $\mu(g) < \infty$ for our infinite products $f(z)$ and $g(z)$ do not always ensure (8), even if they are multiplicatively independent modulo $K(z)$. To give an example we refer the following lemma ([1, Lemma 9]), which will be used also for the proof of the corollary to Theorem 1.

Lemma 1. *Let $\Phi(z)$ be an infinite product (1) with (2), where K is any subfield of the field of complex numbers. Assume that there exists a positive integer C such that, for any n, $E_N(z)/F_N(z) \neq 1$ with some $N = N(n)$ satisfying $n \leq N < n + C$. Assume further that all the quotients $E_k(z)/F_k(z) \neq 1$ have only real zeros or poles if $r \geq 3$, and only positive zeros or poles if $r = 2$. Then $\mu(\Phi) \leq C^*$, where C^* is a positive constant depending only on c, r and L.*

Let us take our infinite products $f(z)$ and $g(z)$ as

$$f(z) = \prod_{k=0}^{\infty} (1 - z^{r^k}), \quad g(z) = \prod_{\substack{k=0 \\ k \neq n!}}^{\infty} (1 - z^{r^k})^{-1}.$$

For $(i, j) \in \mathbb{Z}^2 \setminus \{\underline{0}\}$ it follows from Lemma 1 that $\mu(f^i g^j) < \infty$ when $i \neq j$. On the other hand, under the assumption $i = j > 0$, by denoting

$$P_n(z) = (1 - z^{r^{1!}}) \cdots (1 - z^{r^{n!}}), \quad n \geq 1,$$

we see that $\operatorname{ord}(f(z)^i g(z)^i - P_n(z)^i) = i r^{(n+1)!}$. Since $\deg P_n(z) \leq r^{1! + \cdots + n!} \leq r^{2n!}$, we obtain $\mu(f^i g^i) = \infty$. Hence we have also $\mu(f^{-i} g^{-i}) = \mu(f^i g^i) = \infty$. Moreover, the above facts together with Remark 1 imply that $f(z)$ and $g(z)$ are multiplicatively independent modulo $K(z)$.

Proof of Theorem 2. Starting from (4) we now define sequences of functions

$$\Phi_{j,n}(z) = \prod_{k=0}^{\infty} \frac{E_{j,n+k}(z^{r^k})}{F_{j,n+k}(z^{r^k})}, \quad j = 1, \ldots, m;\ n \geq 0.$$

By denoting

$$\underline{\Phi}_n(z) = (\Phi_{1,n}(z), \ldots, \Phi_{m,n}(z)), \quad n \geq 0,$$

and

$$\underline{E}_k(z) = (E_{1,k}(z), \ldots, E_{m,k}(z)), \quad \underline{F}_k(z) = (F_{1,k}(z), \ldots, F_{m,k}(z)), \quad k \geq 0,$$

we have

$$\underline{\Phi}_n^I(z) = \prod_{k=0}^{\infty} \frac{\underline{E}_{n+k}^I(z^{r^k})}{\underline{F}_{n+k}^I(z^{r^k})}, \quad n \geq 0.$$

Since the components of I may have negative values, we write

$$\frac{\underline{E}_k^I(z)}{\underline{F}_k^I(z)} = \frac{e_{k,I}(z)}{f_{k,I}(z)},$$

where $e_{k,I}(z)$ and $f_{k,I}(z)$ are polynomials of degree $\leq c_I L$ with a positive constant c_I depending on I. Then

$$\underline{\Phi}_n^I(z) = \prod_{k=0}^{\infty} \frac{e_{n+k,I}(z^{r^k})}{f_{n+k,I}(z^{r^k})}, \quad n \geq 0, \tag{12}$$

and analogously to (3) we have a chain of functional equations

$$\underline{\Phi}_{n+1}^I(z^r) = \frac{f_{n,I}(z)}{e_{n,I}(z)} \underline{\Phi}_n^I(z), \quad n \geq 0. \tag{13}$$

In Theorem 2 we consider a finite subset $\Lambda \subset \mathbb{Z}^m \setminus \{\underline{0}\}$, so we may assume that for all $I \in \Lambda$ the degrees of $e_{k,I}(z)$ and $f_{k,I}(z)$ in the expression (12) are $\leq L^* \leq c_\Lambda L$, where $c_\Lambda = \max(c_I)$.

We shall prove Theorem 2 by using induction on $d = |\Lambda|$. Our assumption (8) gives Theorem 2 if $d = 1$. We take an arbitrary Λ with $d = |\Lambda| \geq 2$, and assume that our claim holds for all Λ with a number of elements less that d. Let

$$\Omega(z) = A_{\underline{0}}(z) + \sum_{I \in \Lambda} A_I(z) \underline{\Phi}^I(z), \tag{14}$$

where $A_I(z) \in K[z]$, $I \in \Lambda \cup \{\underline{0}\}$, not all zero, have degrees $\leq M$. To estimate the order of zero at $z = 0$ for $\Omega(z)$ we use Pade approximations. For every integer $n \geq 0$ there exist $Q_n(z)$, $P_{n,I}(z) \in K[z] \setminus \{0\}$, $I \in \Lambda$, such that

$$Q_n(z) \underline{\Phi}_n^I(z) - P_{n,I}(z) = z^{d^2 L^* + dL^* + 1} G_{n,I}(z), \tag{15}$$

where

$$\deg Q_n(z) \le d^2 L^*, \quad \deg P_{n,I}(z) \le d^2 L^*,$$

and

$$G_{n,I}(z) = \sum_{\ell=0}^{\infty} g_{n,I,\ell} z^\ell \in K[[z]] \setminus \{0\}.$$

We replace z by z^{r^n} in (15) and use (13) to get

$$Q_n(z^{r^n}) \underline{\Phi}_0^I(z) - T_{n,I}(z) P_{n,I}(z^{r^n}) = z^{(d^2L^*+dL^*+1)r^n} T_{n,I}(z) G_{n,I}(z^{r^n}), \tag{16}$$

where

$$T_{n,I}(z) = \prod_{k=0}^{n-1} \frac{e_{k,I}(z^{r^k})}{f_{k,I}(z^{r^k})}.$$

By using (14) we then obtain an equality

$$D_n(z) Q_n(z^{r^n}) \Omega(z) = D_n(z) P_n(z) + D_n(z) z^{(d^2L^*+dL^*+1)r^n} G_n(z), \tag{17}$$

where

$$D_n(z) = \prod_{I \in \Lambda} \prod_{k=0}^{n-1} f_{I,k}(z^{r^k}), \quad P_n(z) = Q_n(z^{r^n}) A_{\underline{0}}(z) + \sum_{I \in \Lambda} A_I(z) T_{n,I}(z) P_{n,I}(z^{r^n}),$$

and

$$G_n(z) = \sum_{I \in \Lambda} A_I(z) T_{n,I}(z) G_{n,I}(z^{r^n}).$$

Here $D_n(z)P_n(z)$ is a polynomial of degree $\le M + (d^2L^* + dL^*)r^n$, and therefore

$$\operatorname{ord} \Omega(z) \le M + (d^2L^* + dL^*)r^n \tag{18}$$

if $P_n(z) \ne 0$ and $M < r^n$.

Assume now that $P_n(z) = 0$. Let q be the least integer ℓ such that $g_{n,I,\ell} \ne 0$ for some $I \in \Lambda$, and let J be an element of Λ such that $g_{n,J,q} \ne 0$. By (16),

$$D_{n,J}(z) Q_n(z^{r^n}) \underline{\Phi}_0^J(z) - C_{n,J}(z) P_{n,J}(z)(z^{r^n}) = z^{(d^2L^*+dL^*+1)r^n} C_{n,J}(z) G_{n,I}(z^{r^n})$$

with

$$C_{n,J}(z) = \prod_{k=0}^{n-1} e_{J,k}(z^{r^k}), \quad D_{n,J}(z) = \prod_{k=0}^{n-1} f_{J,k}(z^{r^k}).$$

This implies, by (8), the quantity $(d^2L^* + dL^* + 1 + q)r^n$ is estimated from above by

$$C(J) \max(\deg D_{n,J}(z) Q_n(z^{r^n}), \deg C_{n,I}(z) P_{n,I}(z)(z^{r^n})) \le C(J)(d^2L^* + dL^*),$$

and so

$$q \le (C(J) - 1)(d^2 + d)L^*. \tag{19}$$

Let $\Lambda' = \{I - J : I \in \Lambda \setminus \{\underline{0}\}\}$. Since $T_{n,I}(z) \equiv \underline{\Phi}_0^I(z) \pmod{z^{r^n}}$ we have

$$\begin{aligned} G_n(z) &\equiv \sum_{I\in\Lambda} A_I(z)\underline{\Phi}_0^I(z) g_{n,I,q} z^{qr^n} \\ &\equiv \underline{\Phi}_0^J(z) z^{qr^N}\Big(A_J(z)g_{n,J,q} + \sum_{I\in\Lambda\setminus\{J\}} A_I(z)\underline{\Phi}_0^{I-J}(z)g_{n,I,q}\Big) \pmod{z^{(1+q)r^n}}. \end{aligned}$$

By the induction hypothesis the order of the term in the parenthesis above is estimated from above by $c(\Lambda')M$. We now fix n in such a way that

$$r^{n-1} \le c(\Lambda')M < r^n.$$

Then $G_n(z) \ne 0$ and $\operatorname{ord} G_n(z) < (1+q)r^n$. Thus, by (17),

$$\operatorname{ord}\Omega(z) \le (d^2L^* + dL^* + 2 + q)r^n. \tag{20}$$

From the choice of n we get $r^n \le rc(\Lambda')M$. This together with (18), (19) and (20) gives the truth of our Theorem 2.

3 Proof of the main results

In the proof of Theorem 1 below we shall again use induction on $d = |\Lambda|$. Further, p and n are positive integer parameters and ϵ is a positive constant all to be specified later in order of ϵ, p and n. We shall denote by $c_1, c_2, \dots$ positive constants independent on ϵ, p, n. For the parameter n, we always assume

$$n \ge n(\epsilon, p),$$

where $n(\epsilon, p)$ denotes a positive integer depending on ϵ and p which will be chosen suitably in the course of the proof. Moreover we use the notation $\hat{H}(P)$ to denote the maximum of the conjugates of the coefficients of any polynomial $P(z) \in K[z]$.

Let us write

$$\underline{\Phi}_n^I(z) = \sum_{\ell=0}^{\infty} a_{n,I,\ell} z^\ell.$$

By using the assumptions (6) and (7) we see, analogously to the proof of [7, Lemma 1], that, for any given $\epsilon > 0$,

$$\log|\overline{a_{n,I,\ell}}| \le \epsilon r^n(1+\ell), \quad \operatorname{den}(a_{n,I,\ell}) \,\Big|\, (D_1\cdots D_m)^{c_1(1+\ell)} \tag{21}$$

for all $I \in \Lambda$, $n\,(\ge n(\epsilon))$ and $\ell \ge 0$, where c_1 is a positive constant.

We prepare our proof of Theorem 1 by giving the following lemma.

Lemma 2. *Let $\epsilon > 0$ and $\Lambda \subset \mathbb{Z}^m \setminus \{\underline{0}\}$ with $|\Lambda| = d \ge 1$ be given. Under the notations and assumptions of Theorem 1 there exist auxiliary functions*

$$g_{n,I}(z) = q_n(z)\underline{\Phi}_n^I(z) - p_{n,I}(z) \in K[[z]], \quad I \in \Lambda,$$

with polynomials $q_n(z), p_{n,I}(z) \in O_K[z] \setminus \{0\}$ *of degrees at most* dp*, such that*

$$\tau_I := \operatorname{ord} g_{n,I}(z) \geq dp + p + 1, \tag{22}$$

$$\hat{H}(q_n),\ \hat{H}(p_{n,I}) \leq e^{\epsilon c_2 p^2 r^n}, \tag{23}$$

$$|g_{n,I,\ell}| \leq e^{\epsilon(c_3 p^2 + \ell) r^n}, \quad |g_{n,I,\tau_I}| \geq e^{-\epsilon c_4 p^2 r^n}, \tag{24}$$

where we denote

$$g_{n,I}(z) = \sum_{\ell=0}^{\infty} g_{n,I,\ell} z^{\ell}, \quad I \in \Lambda.$$

Proof. Let us denote

$$Q_n^*(z) = \sum_{k=0}^{dp} \mu_k^* z^k.$$

Then

$$Q_n^*(z)\underline{\Phi}_n^I(z) = \sum_{\ell=0}^{\infty} \nu_{n,I,\ell}^* z^{\ell}, \quad \nu_{n,I,\ell}^* = \sum_{k=0}^{\min(\ell,dp)} \mu_k^* a_{n,I,\ell-k}.$$

We define

$$P_{n,I}^*(z) = \sum_{\ell=0}^{dp} \nu_{n,I,\ell}^* z^{\ell}, \quad I \in \Lambda,$$

and consider a system of homogeneous linear equations

$$\nu_{n,I,\ell}^* = 0, \quad \ell = dp+1, \ldots, dp+p,\ I \in \Lambda. \tag{25}$$

Here we have dp equations in $dp+1$ unknown coefficients μ_k^*. By using Siegel's lemma (see [6, Lemma 1.4.2]) under (21) we find a non-zero polynomial $Q_n^*(z) \in O_K[z]$ satisfying (25). The coefficients of $P_{n,I}^*(z)$ are not necessarily in O_K, but the use of (21) gives a common denominator $D_{n,\Lambda}$ such that the polynomials

$$q_n(z) = \sum_{k=0}^{dp} q_k z^k := D_{n,\Lambda} Q_n^*(z), \quad p_{n,I}(z) = \sum_{k=0}^{dp} p_{n,I,k} z^k := D_{n,\Lambda} P_{n,I}^*(z)$$

satisfy the conditions (22) and (23).

We next use (21) and (23) to get

$$|g_{n,I,\ell}| = \left| \sum_{k=0}^{dp} q_k a_{n,I,\ell-k} \right| \leq (dp+1) e^{\epsilon c_2 p^2 r^n} e^{\epsilon r^n (1+\ell)} \leq e^{\epsilon(c_3 p^2 + \ell) r^n}$$

for all $\ell \geq \tau_I$. This is the first inequality in (24). To get the second inequality in (24) we note that the construction above gives, by using the notations of section 2,

$$C_{n,I}(z) g_{n,I}(z^{r^n}) = D_{n,I}(z) q_n(z^{r^n}) \underline{\Phi}_0^I(z) - C_{n,I}(z) p_{n,I}(z^{r^n}).$$

We use the assumption (8), $\mu(\underline{\Phi}_0^I) \leq c(I)$, and compare the orders on both sides of this equality to get

$$\tau_I \leq c(I)(dp + L^*).$$

The first inequality in (24) and the above estimate for τ_I together with (21) gives an upper bound

$$\|g_{n,I,\tau_I}\| \leq e^{\epsilon(c_5 p^2 + \tau_I) r^n} \leq e^{\epsilon c_6 p^2 r^n}.$$

Since $|\gamma| \geq \|\gamma\|^{-2[K:\mathbb{Q}]}$ for all nonzero $\gamma \in K$ and $g_{n,I,\tau_I} \neq 0$, we get immediately the second estimate in (24). Thus Lemma 2 is proved.

Our next lemma considers the function $T_{n,I}(z)$ defined after (16).

Lemma 3. *If*

$$\epsilon < \frac{1}{4} \log(|\alpha|^{-1}), \tag{26}$$

then

$$T_{n,I}(\alpha) = \underline{\Phi}_0^I(\alpha) + \delta_{n,I}, \quad |\delta_{n,I}| \leq c_7 \, |\alpha|^{r^n/2}. \tag{27}$$

Proof. We have

$$T_{n,I}(z) = \prod_{k=0}^{n-1} \frac{e_{k,I}(z^{r^k})}{f_{k,I}(z^{r^k})} = \underline{\Phi}_0^I(z) \prod_{k=n}^{\infty} \frac{f_{k,I}(z^{r^k})}{e_{k,I}(z^{r^k})}.$$

By using [7, Lemma 1] we get

$$\prod_{k=n}^{\infty} \frac{f_{k,I}(z^{r^k})}{e_{k,I}(z^{r^k})} = \prod_{k=0}^{\infty} \frac{f_{n+k,I}(z^{r^{n+k}})}{e_{n+k,I}(z^{r^{n+k}})} = 1 + \sum_{\ell=1}^{\infty} \sigma_\ell z^{r^n \ell},$$

where, by (26) together with $\ell + 1 \leq 2\ell$,

$$|\overline{\sigma_\ell}| \leq e^{\epsilon r^n (1+\ell)} \geq |\alpha|^{-r^n \ell/2}.$$

Thus we obtain

$$|\delta_{n,I}| \leq |\underline{\Phi}_0^I(\alpha)||\alpha^{r^n/2}| \sum_{\ell=0}^{\infty} |\alpha|^{-r^n \ell/2} \leq c_7 \, |\alpha|^{r^n/2}.$$

This proves Lemma 3.

Proof of Theorem 1. By using the construction of Lemma 2 we get, analogously to (16),

$$q_n(\alpha^{r^n}) \underline{\Phi}_0^I(\alpha) = T_{n,I}(\alpha) P_{n,I}(\alpha^{r^n}) + T_{n,I}(\alpha) g_{n,I}(\alpha^{r^n}).$$

Let $D_n(z)$ be the polynomial defined after (17). Its degree and the degrees of the polynomials $D_n(z)T_{n,I}(z)$ are bounded from above by dL^*r^n, and there exists a common denominator $D^* \le e^{c_8 n}$ for the coefficients of all these polynomials. Let

$$\Gamma = a_{\underline{0}} + \sum_{I\in\Lambda} a_I \underline{\Phi}_0^I(\alpha).$$

Then

$$Q_n\Gamma - P_n = \alpha^{(dp+p+1)r^n} R_n, \tag{28}$$

where

$$\begin{aligned}
Q_n &= D^* D_n(\alpha) q_n(\alpha^{r^n}),\\
P_n &= a_{\underline{0}} D^* D_n(\alpha) q_n(\alpha^{r^n}) + \sum_{I\in\Lambda} a_I D^* D_n(\alpha) T_{n,I}(\alpha) p_{n,I}(\alpha^{r^n}),\\
R_n &= \sum_{I\in\Lambda} a_I D^* D_n(\alpha) T_{n,I}(\alpha) g^*_{n,I}(\alpha^{r^n}), \quad g^*_{n,I}(z) z^{dp+p+1} := g_{n,I}(z).
\end{aligned}$$

In the following we shall use (28) to prove Theorem 1 in five steps.

Step 1. A lower bound for $|P_n|$ in the case $P_n \neq 0$. Note first that $D^*D_n(\alpha)q_n(\alpha^{r^n})$ and $D^*D_n(\alpha)T_{n,I}(\alpha)p_{n,I}(\alpha^{r^n})$ appearing in P_n are polynomials in $\mathcal{O}_K[\alpha]$ of degrees at most $d(p+L^*)r^n$. Moreover, by using (6) and (23), we get an upper bound

$$e^{\epsilon c_9 p^2 r^n} \tag{29}$$

for the absolute values of the conjugates of the coefficients of these polynomials. Thus

$$\begin{aligned}
\prod_{w\neq v} |P_n|_w^{\kappa_w/\kappa} &\le \prod_{w\nmid\infty} \mathrm{M}_w(\alpha)^{d(p+L^*)r^n} \cdot \prod_{w\mid\infty, w\neq v} \mathrm{M}_w(\alpha)^{d(p+L^*)r^n} e^{\epsilon c_{10} p^2 r^n} \mathrm{M}_w(\underline{a})\\
&\le H(\underline{a}) \max(1, |\underline{a}|)^{-\kappa_v/\kappa} e^{\epsilon c_{11} p^2 r^n} H(\alpha)^{d(p+L^*)r^n}.
\end{aligned}$$

Then the product formula implies in the case $P_n \neq 0$

$$|P_n| \ge \max(1, |\underline{a}|) H(\underline{a})^{-\frac{\kappa}{\kappa_v}} e^{-\epsilon c_{12} p^2 r^n} H(\alpha)^{-\frac{\kappa}{\kappa_v} d(p+L^*)r^n}. \tag{30}$$

Step 2. An upper bound for $|R_n|$. By using (23) and (29) we obtain

$$|D^* D_n(\alpha) T_{n,I}(\alpha)| \le e^{\epsilon c_{13} p^2 r^n}.$$

Further, the first inequality in (24) gives

$$\left|g^*_{n,I}(\alpha^{r^n})\right| \le \sum_{\ell=0}^{\infty} e^{\epsilon(c_3 p^2 + dp + p + 1 + \ell)r^n} |\alpha|^{\ell r^n} \le e^{\epsilon c_{14} p^2 r^n} \sum_{\ell=0}^{\infty} (e^{\epsilon} |\alpha|)^{\ell r^n}.$$

Assuming (26) we then have

$$|R_n| \le |\underline{a}|\, e^{\epsilon c_{15} p^2 r^n}.$$

By combining this bound with (28) and (30) we obtain in the case $P_n \neq 0$

$$|Q_n\Gamma| \geq |\underline{a}|\, H(\underline{a})^{-\frac{\kappa}{\kappa_v}} \Omega(p,\epsilon)\big(1 - H(\underline{a})^{\frac{\kappa}{\kappa_v}} e^{\epsilon(c_{12}+c_{15})p^2r^n}\, |\alpha|^{\omega(p)r^n}\big), \tag{31}$$

where

$$\Omega(p,\epsilon) = e^{-\left(\epsilon c_{12}p^2 + \frac{\kappa}{\kappa_v} d(p+L^*)\log H(\alpha)\right)r^n}, \quad \omega(p) = p^* - \lambda(\alpha)d(p+L^*)$$

with $p^* = dp + p + 1$.

In the case $P_n = 0$ we have $Q_n\Gamma = \alpha^{p^*r^n} R_n$, and to study this case we need a lower bound for $|R_n|$. Let q be the least integer such that $g_{n,I,p^*+q} \neq 0$ for some $I \in \Lambda$. By the assumption (8),

$$q \leq c_{16}p.$$

We shall divide the following argument into two parts, in the first part $d = 1$ and in the second $d \geq 2$.

Step 3. A lower bound for $|R_n|$, the case $d = 1$. Let $\Lambda = \{I\}$. Then we have

$$R_n = a_I D^* D_n(\alpha) T_{n,I}(\alpha)\Big(g_{n,I,p^*+q} + \sum_{\ell \geq q+1} g_{n,I,p^*+\ell}\alpha^{r^n\ell}\Big).$$

It follows from the second inequality in (24) that

$$|g_{n,I,p^*+q}| \geq e^{-\epsilon c_4 p^2 r^n}$$

and

$$\left|\sum_{\ell \geq q+1} g_{n,I,p^*+\ell}\alpha^{r^n\ell}\right| \leq e^{\epsilon c_{17}p^2r^n}\, |\alpha|^{(q+1)r^n} \tag{32}$$

if (26) holds. Since the condition (10) gives $D^*D_n(\alpha)T_{n,I}(\alpha) \neq 0$, similarly to Step 1, we then obtain

$$|D^*D_n(\alpha)T_{n,I}(\alpha)| \geq e^{-\epsilon c_{18}p^2r^n} H(\alpha)^{-\frac{\kappa}{\kappa_v}d(p+L^*)r^n}.$$

Moreover, by (9), $|a_I| \geq H(a_I)^{-\frac{\kappa}{\kappa_v}}$. (Theorem 1 holds trivially if $a_I = 0$, so we may assume that $a_I \neq 0$.) The above estimates give

$$|R_n| \geq H(a_I)^{-\frac{\kappa}{\kappa_v}} H(\alpha)^{-\frac{\kappa}{\kappa_v}d(p+L^*)r^n} e^{-\epsilon(c_4+c_{18})p^2r^n}\Big(1 - e^{\epsilon(c_4+c_{17})p^2r^n}\, |\alpha|^{(q+1)r^n}\Big). \tag{33}$$

Step 4. A lower bound for $|R_n|$, the case $d \geq 2$. We assume now that Theorem 1 holds for all Λ with $|\Lambda| \leq d-1$. By this induction hypothesis we may assume in the following that $a_I \neq 0$ for all $I \in \Lambda$. We have

$$R_n = D^*D_n(\alpha) \sum_{I\in\Lambda} a_I T_{n,I}(\alpha) \sum_{\ell \geq q} g_{n,I,p^*+\ell}\alpha^{r^n\ell},$$

for which let us write

$$R_n = D^* D_n(\alpha)\left(\sum_1 + \sum_2\right),$$

where

$$\sum_1 = \sum_{I\in\Lambda} a_I T_{n,I}(\alpha) g_{p^*+q}\alpha^{r^n q}, \quad \sum_2 = \sum_{I\in\Lambda} a_I T_{n,I}(\alpha) \sum_{\ell\geq q+1} g_{n,I,p^*+\ell}\alpha^{r^n \ell}.$$

We first consider $\sum_1$ and use (27) to divide it into two parts

$$\sum_1 = D^* D_n(\alpha)\alpha^{r^n q}\left(\sum_{1,1} + \sum_{1,2}\right),$$

where

$$\sum_{1,1} = \sum_{I\in\Lambda} g_{n,I,p^*+q} a_I \underline{\Phi}_0^I(\alpha), \quad \sum_{1,2} = \sum_{I\in\Lambda} g_{n,I,p^*+q} a_I \delta_{n,I}.$$

By the first inequality in (24) and (27),

$$\left|\sum_{1,2}\right| \leq |\underline{a}|\, e^{\epsilon c_{19} p^2 r^n}\, |\alpha|^{r^n/2} \tag{34}$$

Further, take $J \in \Lambda$ in such a way that $g_{n,J,p^*+q} \neq 0$ and denote $\Lambda^* = \Lambda \setminus \{J\}$. Then

$$\sum_{1,1} = \underline{\Phi}_0^J(\alpha)\Big(g_{n,J,p^*+q} a_J + \sum_{I\in\Lambda^*} g_{n,I,p^*+q} a_I \underline{\Phi}_0^{I-J}(\alpha)\Big).$$

In this expression

$$g_{n,I,\ell} = \sum_{k=0}^{dp} q_k a_{n,I,\ell-k}$$

and so the least common denominator G_n of all $g_{n,I,\ell}$, $I \in \Lambda$, appearing above satisfies, by (21),

$$|G_n| \leq e^{c_{20} p}.$$

Thus the induction hypothesis together with the first inequality in (24) gives an estimate

$$\left|G_n\Big(g_{n,J,p^*+q} a_J + \sum_{I\in\Lambda^*} g_{n,I,p^*+q} a_I \underline{\Phi}_0^{I-J}(\alpha)\Big)\right| \geq \big(e^{\epsilon c_{21} p^2 r^n}\widetilde{h}\big)^{-\phi(\Lambda^*)}$$

with $\widetilde{h} = \max(\widetilde{h}_0, H(\underline{a}))$, where $\phi(\Lambda^*)$ and $\widetilde{h}_0$ are positive constants depending only on α, $\underline{\Phi}_0$ and Λ^*. This implies

$$\left|\sum_{1,1}\right| \geq e^{-\epsilon c_{22} p^2 r^n}\widetilde{h}^{-c_{23}}. \tag{35}$$

To estimate $\sum_2$ we assume (26) and use (27) and (32) to get

$$\left|\sum_2\right| \le |\underline{a}|\, e^{\epsilon c_{24} p^2 r^n}\, |\alpha|^{(q+1)r^n}\,. \tag{36}$$

Finally, by using the above estimates (34), (35)and (36) with the first inequality in (9) and a lower bound for $|D^* D_n(\alpha)| \ge e^{-c_{25} r^n}$ (obtained as in Step 1) we have

$$|R_n| \ge e^{-c_{25} r^n}\, |\alpha|^{r^n q}\, e^{-\epsilon c_{22} p^2 r^n} \widetilde{h}^{-c_{23}} \big(1 - \widetilde{h}^{c_{26}} \Delta_1(p,\epsilon)^{r^n} - \widetilde{h}^{c_{26}} \Delta_2(p,\epsilon)^{r^n}\big) \tag{37}$$

if (26) holds, where

$$\Delta_1(p,\epsilon) = e^{\epsilon(c_{19}+c_{22})p^2}\, |\alpha|^{1/2}, \quad \Delta_2(p,\epsilon) = e^{\epsilon(c_{22}+c_{24})p^2}\, |\alpha|\,.$$

Step 5. End of the proof. By our assumption (11) we can fix the parameter p such that $\omega(p)$ in (31) is positive. In the case $d = 1$ we then choose and fix $\epsilon > 0$ in such a way that (26) and

$$\epsilon < \min\left(\frac{\omega(p)\log(|\alpha|^{-1})}{(c_{12}+c_{15})p^2}, \frac{(q+1)\log(|\alpha|^{-1})}{(c_4+c_{17})p^2}\right) \tag{38}$$

are satisfied. Then

$$e^{\epsilon(c_4+c_{17})p^2 r^n}\, |\alpha|^{(q+1)r^n} < \frac{1}{2} \tag{39}$$

for all $n \ge n(p,\epsilon)$, where $n(p,\epsilon)$ is taken so that all the estimates given above also hold for all $n \ge n(p,\epsilon)$. After that we fix the parameter n to be the least integer satisfying

$$h^{\frac{\kappa}{\kappa_v}} \Delta(p,\epsilon)^{r^n} < \frac{1}{2}, \tag{40}$$

where

$$\Delta(p,\epsilon) = e^{\epsilon(c_{12}+c_{15})p^2}\, |\alpha|^{\omega(p)}$$

and $h = \max(h_0, H(\underline{a}))$ with a positive constant h_0 large enough to imply $n \ge n(p,\epsilon)$. Then we have

$$h^{\frac{\kappa}{\kappa_v}} \Delta(p,\epsilon)^{r^{n-1}} \ge \frac{1}{2},$$

which gives $r^n \le \hat{c}_{27} \log h$ (here $\hat{c}_{27}$ as also $\hat{c}_{28}$ later depends on p and ϵ, which we fixed above). This estimate together with (31), (33), (39) and (40) ensures the truth of Theorem 1 in the case $d = 1$.

We now assume that $d \ge 2$. In the choice of ϵ we in this case replace (38) by

$$\epsilon < \min\left(\frac{\omega(p)\log(|\alpha|^{-1})}{(c_{12}+c_{15})p^2}, \frac{\log(|\alpha|^{-1})}{2(c_{19}+c_{22})p^2}, \frac{\log(|\alpha|^{-1})}{(c_{22}+c_{24})p^2}\right).$$

Let then n be the least integer satisfying

$$\max\big(h^{\frac{\kappa}{\kappa_v}} \Delta(p,\epsilon)^{r^n}, h^{c_{26}} \Delta_1(p,\epsilon)^{r^n}, h^{c_{26}} \Delta_2(p,\epsilon)^{r^n}\big) < \frac{1}{3},$$

where $h = \max(h_0, H(\underline{a}))$ and $h_0 \geq \widetilde{h}_0$ large enough to imply $n \geq n(p, \epsilon)$. As above we have $r^n \leq \hat{c}_{28} \log h$. The use of the estimates (31) and (37) with the above results immediately leads to a desired lower bound for $|\Gamma|$. Theorem 1 now follows by induction.

Proof of the corollary and the examples. The corollary follows imediately from Theorem 1 together with Lemma 1. Also both Example 1 and Example 2 are direct consequences of the corollary. Hence, there remains to ensure the validity of Example 3.

As is shown in [1, Section 4] we have, for $k \geq 1$,

$$1 + \frac{b_{1,k}}{F_{cr^k+d_1}} = \frac{E_{1,k}(\rho^{-cr^k})}{F_{1,k}(\rho^{-cr^k})}, \quad 1 + \frac{b_{2,k}}{L_{cr^k+d_2}} = \frac{E_{2,k}(\rho^{-cr^k})}{F_{2,k}(\rho^{-cr^k})},$$

where $\rho = (1+\sqrt{5})/2$ and

$$\begin{aligned} E_{1,k}(z) &= 1 + \sqrt{5} b_{1,k} \rho^{-d_1} z - \rho^{-2d_1} z^2, \quad & F_{1,k}(z) &= 1 - \rho^{-2d_1} z^2, \\ E_{2,k}(z) &= 1 + b_{2,k} \rho^{-d_2} z - \rho^{-2d_2} z^2, \quad & F_{2,k}(z) &= 1 - \rho^{-2d_2} z^2. \end{aligned}$$

Therefore, if we take

$$\Phi_j(z) = \prod_{k=0}^{\infty} \frac{E_{j,n(k)}(z^{r^{n(k)}})}{F_{j,n(k)}(z^{r^{n(k)}})}, \quad j = 1, 2,$$

then

$$\Phi_1(\rho^{-c}) = \prod_{k=0}^{\infty} \left(1 + \frac{b_{1,n(k)}}{F_{cr^{n(k)}+d_1}}\right), \quad \Phi_2(\rho^{-c}) = \prod_{k=0}^{\infty} \left(1 + \frac{b_{2,n(k)}}{L_{cr^{n(k)}+d_2}}\right),$$

where we have $\lambda(\rho^{-c}) = 1$. We show that $\mu(\Phi_1^s \Phi_2^t) < \infty$ for any $(s,t) \in \mathbb{Z}^2 \backslash \{(0,0)\}$. To this end, by Lemma 1, it is enough to show that

$$E_{1,k}(z)^s F_{2,k}(z)^t \neq E_{2,k}(z)^t F_{1,k}(z)^s, \quad k \geq 0.$$

Assume on the contrary that the equality holds. Then, comparing the zeros and the poles in both sides with multiplicity, we should have

$$s(\alpha_{1,k} + \beta_{1,k}) + t(\rho^{-d_2} - \rho^{-d_2}) = t(\alpha_{2,k} + \beta_{2,k}) + s(\rho^{-d_1} - \rho^{-d_1}),$$

that is, $s(\alpha_{1,k} + \beta_{1,k}) = t(\alpha_{2,k} + \beta_{2,k})$, where $\alpha_{j,k}$ and $\beta_{j,k}$ are the zeros of $E_{j,k}(z)$ for $j = 1, 2$. This implies that $s\rho^{d_1}\sqrt{5} b_{1,k} = t\rho^{d_2} b_{2,k}$. Since $\rho^{d_1-d_2}\sqrt{5}$ is irrational, this holds only when $s = t = 0$. This contradicts our assumption, and hence $\mu(\Phi_1^s \Phi_2^t) < \infty$. Thus Example 3 follows from the corollary.

We finally note that Lemma 1 is not applicable in the case $r = 2$, since $F_{j,k}(z)$ cannot have only positive zeros.

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Masaaki Amou
Department of Mathematics
Gunma University
Tenjin-cho 1-5-1
Kiryu 376-8515, Japan
E-mail: amou@gunma-u.ac.jp

Keijo Väänänen
Department of Mathematics
University of Oulu
P.O. Box 3000
90014 Oulu, Finland
E-mail: keijo.vaananen@oulu.fi