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# UNIFORM CONVEXITY AND ASSOCIATE SPACES

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*Abstract.* We prove that the associate space of a generalized Orlicz space $L^{\varphi(\cdot)}$ is given by the conjugate modular $\varphi^*$ even without the assumption that simple functions belong to the space. Second, we show that every weakly doubling $\Phi$-function is equivalent to a doubling $\Phi$-function. As a consequence, we conclude that $L^{\varphi(\cdot)}$ is uniformly convex if $\varphi$ and $\varphi^*$ are weakly doubling.

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## 1. Intorduction

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940s. A major synthesis of functional analysis in these spaces, based on work, e.g. of Hudzik, Kamińska and Musielak, is given in the monograph [16]. Following ideas of Maeda, Mizuta, Ohno and Shimomura (e.g. [15]), we have studied these spaces from a point-of-view which emphasizes the possibility of choosing the $\Phi$-function generating the norm in the space appropriately [5], [9], [10], [12]. From this perspective, some classical concepts, like convexity of the $\Phi$-function, are too rigid.

Renewed interest in the topic has arisen recently from studies of PDE with nonstandard growth, including the variable exponent case $\varphi(x,t) = t^{p(x)}$ and the double phase case $\varphi(x,t) = t^p + a(x)t^q$. Such problems have been studied e.g. in [2], [3], [4], [8], [17]. For a detailed motivation of our context and additional references we refer to the introduction of [11].

In this note, we tie up some loose ends concerning the basic functional analysis of generalized Orlicz spaces in our monograph [6]. In the book we relied on the assumption that all simple functions belong to our space. This excludes for instance

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the case $\varphi(x,t) := |x|^{-n}t^2$, where $n$ is the dimension. We can now remove this assumption from the following result (cf. [6], Theorem 2.7.4). For simplicity, we consider only the Lebesgue measure on subsets of $\mathbb{R}^n$. See the next sections for definitions.

**Theorem 1.1.** *Let $A \subset \mathbb{R}^n$ be measurable. If $\varphi \in \Phi_w(A)$, then $(L^\varphi)' = L^{\varphi^*}$, i.e. for all measurable $f\colon A \to \mathbb{R}$*

$$\|f\|_{\varphi(\cdot)} \approx \sup_{\|g\|_{\varphi^*(\cdot)} \leqslant 1} \int_A |f(x)g(x)|\,\mathrm{d}x.$$

The proof relies among other things on upgrading the weak $\Phi$-function to a strong $\Phi$-function based on our earlier work. The next result is of the same type, upgrading weak doubling to strong doubling.

**Theorem 1.2.** *Let $A \subset \mathbb{R}^n$ be measurable. If $\varphi \in \Phi_w(A)$ satisfies $\Delta_2^w$ and $\nabla_2^w$, then there exists $\psi \in \Phi_w(A)$ with $\varphi \sim \psi$ satisfying $\Delta_2$ and $\nabla_2$.*

Recall that a vector space $X$ is *uniformly convex* if it has a norm $\|\cdot\|$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$\|x - y\| \geqslant \varepsilon \quad \text{or} \quad \|x + y\| \leqslant 2(1 - \delta)$$

for all unit vectors $x$ and $y$. In the Orlicz case, it is well known that the space $L^\varphi$ is reflexive and uniformly convex if and only if $\varphi$ and $\varphi^*$ are doubling [18], Theorem 2, page 297. Hudzik in [13] showed in 1983 that the same conditions are sufficient for uniform convexity (see also [7], [14]). With the equivalence technique, we are able to give a very simple proof of this result.

**Theorem 1.3.** *Let $A \subset \mathbb{R}^n$ be measurable and $\varphi \in \Phi_w(A)$. If $\varphi$ satisfies $\Delta_2^w$ and $\nabla_2^w$, then $L^{\varphi(\cdot)}$ is uniformly convex and reflexive.*

## 2. $\Phi$-functions

By $A \subset \mathbb{R}^n$ we denote a measurable set. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leqslant Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. By $c$ we denote a generic constant whose value may change between appearances. A function $f$ is *almost increasing* if there exists a constant $L \geqslant 1$ such that $f(s) \leqslant Lf(t)$ for all $s \leqslant t$ (abbreviated $L$-almost increasing). *Almost decreasing* is defined analogously.

**Definition 2.1.** We say that $\varphi\colon A\times[0,\infty)\to[0,\infty]$ is a *weak* $\Phi$*-function*, and write $\varphi\in\Phi_w(A)$, if the following conditions hold:

- ▷ For every $t\in[0,\infty)$ the function $x\mapsto\varphi(x,t)$ is measurable and for every $x\in A$ the function $t\mapsto\varphi(x,t)$ is non-decreasing and left-continuous.
- ▷ $\varphi(x,0)=\lim\limits_{t\to 0^+}\varphi(x,t)=0$ and $\lim\limits_{t\to\infty}\varphi(x,t)=\infty$ for every $x\in A$.
- ▷ The function $t\mapsto\varphi(x,t)/t$ is $L$-almost increasing for $t>0$ uniformly in $A$. "Uniformly" means that $L$ is independent of $x$.

If $\varphi\in\Phi_w(A)$ is convex, then it is called a $\Phi$*-function*, and we write $\varphi\in\Phi(A)$. If $\varphi\in\Phi(A)$ is continuous as a function into the extended real line $[0,\infty]$, then it is a *strong* $\Phi$*-function*, and we write $\varphi\in\Phi_s(A)$.

We say that $\varphi,\psi\in\Phi_w(A)$ are *weakly equivalent*, $\varphi\sim\psi$, if there exist $D>1$ and $h\in L^1(A)$ such that

$$\varphi(x,t)\leqslant\psi(x,Dt)+h(x)\quad\text{and}\quad\psi(x,t)\leqslant\varphi(x,Dt)+h(x).$$

Two functions $\varphi$ and $\psi$ are *equivalent*, $\varphi\simeq\psi$, if the previous conditions hold with $h\equiv 0$. Note that $\varphi\sim\psi$ if and only if $L^{\varphi(\cdot)}=L^{\psi(\cdot)}$. In the case $\varphi,\psi\in\Phi$, this has been proved in [6], Theorem 2.8.1. For the weak $\Phi$-functions the proof is the same.

We define the *doubling condition* $\Delta_2$ and the *weak doubling condition* $\Delta_2^w$ by

$$\varphi(x,2t)\lesssim\varphi(x,t),\quad \varphi(x,2t)\lesssim\varphi(x,t)+h(x),$$

respectively, where $h\in L^1$ and the implicit constant are independent of $x$. If $\varphi\in\Phi_w(A)$, then we define a conjugate $\Phi$-function by

$$\varphi^*(x,t):=\sup_{s\geqslant 0}(st-\varphi(x,s)).$$

We say that $\varphi$ satisfies $\nabla_2$ or $\nabla_2^w$ if $\varphi^*$ satisfies $\Delta_2$ or $\Delta_2^w$, respectively. All these assumptions are invariant under equivalence, $\simeq$, of $\Phi$-functions.

In some situations, it is useful to have a more quantitative version of the $\Delta_2$ and $\nabla_2$ conditions. It can be shown that (aDec) is equivalent to $\Delta_2$ and (aInc) to $\nabla_2$ (cf. [11], Lemma 2.6, and [5], Proposition 3.6), where (aInc) and (aDec) means the following:

(aInc) There exist $\gamma^->1$ and $L\geqslant 1$ such that $t\mapsto\varphi(x,t)/t^{\gamma^-}$ is $L$-almost increasing in $(0,\infty)$.

(aDec) There exist $\gamma^+>1$ and $L\geqslant 1$ such that $t\mapsto\varphi(x,t)/t^{\gamma^+}$ is $L$-almost decreasing in $(0,\infty)$.

Note that the optimal $\gamma^-$ and $\gamma^+$ correspond to the lower and upper Matuszewska-Orlicz indexes, respectively.

Let us start by showing that weak doubling can be upgraded to strong doubling via weak equivalence of $\Phi$-functions. For this we will use the *left-inverse* of a weak $\Phi$-function, defined by the formula

$$\varphi^{-1}(x,\tau) := \inf\{t > 0\colon\ \varphi(x,t) \geqslant \tau\}.$$

We point out that if $\varphi \in \Phi_s(\Omega)$, then by [9], page 4, we have for every $t$ that

$$\varphi(x,\varphi^{-1}(x,t)) = t. \tag{2.1}$$

P r o o f of Theorem 1.2. By [10], Proposition 2.3, we may assume without loss of generality that $\varphi \in \Phi_s(A)$. By assumption,

$$\varphi(x,2t) \leqslant D\varphi(x,t) + h(x), \quad \varphi^*(x,2t) \leqslant D\varphi^*(x,t) + h(x)$$

for some $D > 2$, $h \in L^1$ and all $x \in A$ and $t \geqslant 0$. Using $\varphi = \varphi^{**}$ (see [6], Corollary 2.6.3), and the definition of the conjugate $\Phi$-function, we obtain from the second inequality that

$$\begin{aligned}\varphi(x,2t) &= \sup_{u\geqslant 0}(2tu - \varphi^*(x,u)) \leqslant \sup_{u\geqslant 0}\Big(2tu - \frac{1}{D}(\varphi^*(x,2u) - h(x))\Big)\\ &= \sup_{u\geqslant 0}\Big(2tu - \frac{1}{D}\varphi^*(x,2u)\Big) + \frac{1}{D}h(x) = \frac{1}{D}\sup_{u\geqslant 0}(Dt2u - \varphi^*(x,2u)) + \frac{1}{D}h(x)\\ &= \frac{1}{D}\varphi(x,Dt) + \frac{1}{D}h(x).\end{aligned}$$

Define $t_x := \varphi^{-1}(x,h(x))$ and suppose that $t \geqslant t_x$ so that $h(x) \leqslant \varphi(x,t)$. By convexity, we conclude that $Dh(x) \leqslant D\varphi(x,t) \leqslant \varphi(x,Dt)$. Hence in the case $t \geqslant t_x$ we have

$$\varphi(x,2t) \leqslant (D+1)\varphi(x,t), \quad \varphi(x,2t) \leqslant \frac{D+1}{D^2}\varphi(x,Dt).$$

Let $p := \log_2(D+1)$ and

$$q := \frac{\log(D^2/(D+1))}{\log(D/2)}.$$

Note that $q > 1$ since $D^2/(D+1) > D/2$. Divide the first inequality by $(2t)^p$ and the second one by $(2t)^q$:

$$\begin{aligned}\frac{\varphi(x,2t)}{(2t)^p} &\leqslant \frac{D+1}{2^p}\frac{\varphi(x,t)}{t^p} = \frac{\varphi(x,t)}{t^p},\\ \frac{\varphi(x,2t)}{(2t)^q} &\leqslant \frac{(D+1)D^q}{D^2 2^q}\frac{\varphi(x,Dt)}{(Dt)^q} = \frac{\varphi(x,Dt)}{(Dt)^q}.\end{aligned}$$

Let $s > t \geqslant t_x$. Then there exists $k \in \mathbb{N}$ such that $2^k t < s \leqslant 2^{k+1} t$. Hence

$$\frac{\varphi(x,s)}{s^p} \leqslant \frac{\varphi(x,2^{k+1}t)}{(2^k t)^p} = 2^p \frac{\varphi(x,2^{k+1}t)}{(2^{k+1}t)^p} \leqslant 2^p \frac{\varphi(x,2^k t)}{(2^k t)^p} \leqslant \ldots \leqslant 2^p \frac{\varphi(x,t)}{t^p},$$

so $\varphi$ satisfies (aDec) with $\gamma^+ = p$ for $t \geqslant t_x$. Similarly, we find that $\varphi$ satisfies (aInc) with $\gamma^- = q$ for $t \geqslant t_x$.

Define

$$\psi(x,t) := \begin{cases} \varphi(x,t) & \text{for } t \geqslant t_x, \\ c_x t^2 & \text{otherwise,} \end{cases}$$

where $c_x$ is chosen so that $\psi$ is continuous at $t_x$. Then $\psi$ satisfies (aDec) on $[0, t_x]$ and $[t_x, \infty)$, hence on the whole real axis with $\gamma^+ = \max\{p, 2\}$, similarly for (aInc) with $\gamma^- = \min\{q, 2\}$.

Furthermore, $\varphi(x,t) = \psi(x,t)$ when $t \geqslant t_x$, and so it follows that $|\varphi(x,t) - \psi(x,t)| \leqslant \varphi(x,t_x) = h(x)$, where (2.1) is used for the last step. Since $h \in L^1$, this means that $\varphi \sim \psi$, so $\psi$ is the required function. $\square$

**Remark 2.2.** From the proof of the previous theorem, we see that the two conditions are not interdependent, i.e. if $\varphi \in \Phi_w(A)$ satisfies $\Delta_2^w$, then there exists $\psi \in \Phi_w(A)$ with $\varphi \sim \psi$ satisfying $\Delta_2$; similarly for only $\nabla_2^w$ and $\nabla_2$.

## 3. Associate spaces

We denote by $L^0(A)$ the set of measurable functions in $A$.

**Definition 3.1.** Let $\varphi \in \Phi_w(A)$ and define the *modular* $\varrho_{\varphi(\cdot)}$ for $f \in L^0(A)$ by

$$\varrho_{\varphi(\cdot)}(f) := \int_A \varphi(x, |f(x)|)\, \mathrm{d}x.$$

The *generalized Orlicz space*, also called Musielak-Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(A) := \{f \in L^0(A)\colon \lim_{\lambda \to 0^+} \varrho_{\varphi(\cdot)}(\lambda f) = 0\}$$

equipped with the (Luxemburg) quasinorm

$$\|f\|_{\varphi(\cdot)} := \inf\Big\{\lambda > 0\colon \varrho_{\varphi(\cdot)}\Big(\frac{f}{\lambda}\Big) \leqslant 1\Big\}.$$

Let us start with a lemma which shows that we can approximate the function 1 with a monotonically increasing sequence of functions in the generalized Orlicz space. Note that the next lemma is trivial if $L^\infty \subset L^{\varphi(\cdot)}$, as was assumed in [6] when dealing with associate spaces.

**Lemma 3.2.** *Let $\varphi \in \Phi_w(A)$. There exists positive $h_k \in L^{\varphi(\cdot)}(A)$, $k \in \mathbb{N}$, such that $h_k \nearrow 1$ and $\{h_k = 1\} \nearrow A$.*

P r o o f. For $k \geqslant 1$ we define

$$E_k := \{x\colon\ \varphi(x, 2^{-k}) \leqslant 1\}.$$

Since $\varphi(\cdot, t)$ is assumed to be measurable, $E_k$ is a measurable set. Since $\lim\limits_{t\to 0^+} \varphi(x,t) = 0$, there exists for every $x \in A$ an index $k_x$ such that $x \in E_{k_x}$. And since $\varphi$ is non-decreasing, it follows that $E_k \nearrow A$ as $k \to \infty$. We define

$$h(x) := \sum_{i=0}^{\infty} 2^{-i-1} \chi_{E_i}(x).$$

Then $h(x) \in (0,1]$ for every $x$, and $h$ is measurable. Suppose that $x \in E_{k+1} \setminus E_k$ for some $k \in \mathbb{N}$. Then

$$h(x) = \sum_{i=k+1}^{\infty} 2^{-i-1} = 2^{-(k+1)}.$$

Hence, by the definition of $E_{k+1}$, we find that $\varphi(x, h(x)) \leqslant 1$. Since $A = \bigcup\limits_k E_k$, we have $\varphi(x, h(x)) \leqslant 1$ in $A$. (The function $h$ can alternatively be constructed using the left-inverse of $\varphi$, as in the previous section.)

Let us define $h_k := \min\{kh\chi_{B(0,k)\cap A}, 1\}$. Then

$$\varrho_{\varphi(\cdot)}(k^{-1}h_k) \leqslant \int_{B(0,k)\cap A} \varphi(x,h)\,\mathrm{d}x \leqslant |B(0,k)| < \infty,$$

so that $h_k \in L^{\varphi(\cdot)}(A)$. Since $h > 0$, it follows that $kh\chi_{B(0,k)\cap A} \nearrow \infty$ for every $x$, and so $h_k \nearrow 1$, as required. $\square$

We define the *associate space* by $(L^{\varphi(\cdot)})'(A) := \{f \in L^0(A)\colon\ \|f\|_{(L^{\varphi(\cdot)})'} < \infty\}$, where

$$\|f\|_{(L^{\varphi})'} := \sup_{\|g\|_{\varphi(\cdot)} \leqslant 1} \int_A fg\,\mathrm{d}x.$$

If $g \in (L^{\varphi})'$ and $f \in L^{\varphi}$, then $fg \in L^1$ by the definition of the associate space. In particular, the integral $\int_A fg\,\mathrm{d}x$ is well defined and

$$\left|\int_A fg\,\mathrm{d}x\right| \leqslant \|g\|_{(L^{\varphi})'}\|f\|_{\varphi(\cdot)}.$$

Hölder's inequality holds in generalized Orlicz spaces with constant 2, without restrictions on the $\Phi_w$-function ([6], Lemma 2.6.5):

$$\int_A |f|\,|g|\,\mathrm{d}x \leqslant 2\|f\|_{\varphi(\cdot)}\|g\|_{\varphi^*(\cdot)}. \tag{3.1}$$

Here $\varphi^*$ is the conjugate $\Phi$-function defined in the previous section. Furthermore, we can define a conjugate modular on the dual space by the formula

$$(\varrho_{\varphi(\cdot)})^*(J) := \sup_{f\in L^{\varphi(\cdot)}} (J(f) - \varrho_{\varphi(\cdot)}(f))$$

for $J \in (L^{\varphi(\cdot)})^*$, i.e. $J\colon L^{\varphi(\cdot)} \to \mathbb{R}$ is a bounded linear functional. By $J_f$ we denote the functional $g \mapsto \int fg\,\mathrm{d}x$.

Proof of Theorem 1.1. We follow the outlines of [6], Theorem 2.7.4, but use Lemma 3.2 to get rid of the extraneous assumption that simple functions belong to the space. The inequality $\|f\|_{(L^{\varphi})'} \leqslant 2\|f\|_{\varphi^*(\cdot)}$ follows from (3.1).

Let then $f \in (L^{\varphi})'$ and $\varepsilon > 0$. Let $\{q_1, q_2, \ldots\}$ be an enumeration of non-negative rational numbers with $q_1 = 0$. For $k \in \mathbb{N}$ and $x \in A$ define

$$r_k(x) := \max_{j=1,\ldots,k} q_j|f(x)| - \varphi(x, q_j).$$

The special choice $q_1 = 0$ implies $r_k(x) \geqslant 0$ for all $x \geqslant 0$. Since $\mathbb{Q}$ is dense in $[0,\infty)$ and $\varphi(x,\cdot)$ is left-continuous, $r_k(x) \nearrow \varphi^*(x, |f(x)|)$ for every $x \in A$ as $k \to \infty$.

Since $f$ and $\varphi(\cdot, t)$ are measurable functions, the sets

$$E_{i,k} := \{x \in A\colon\ q_i|f(x)| - \varphi(x, q_i) = \max_{j=1,\ldots,k} (q_j|f(x)| - \varphi(x, q_j))\}$$

are measurable. Let $F_{i,k} := E_{i,k} \setminus (E_{1,k} \cup \ldots \cup E_{i-1,k})$. Define

$$g_k := \sum_{i=1}^{k} q_i \chi_{F_{i,k}}.$$

Then $g_k$ is measurable and bounded and

$$r_k(x) = g_k(x)|f(x)| - \varphi(x, g_k(x))$$

for all $x \in A$.

Let $h_k \in L^{\varphi(\cdot)}(A)$ be as in Lemma 3.2, i.e. $\{h_k = 1\} \nearrow A$ and $0 < h_k \leqslant 1$. Since $g_k$ is bounded, it follows that $w := \operatorname{sgn} f\, h_k g_k \in L^{\varphi(\cdot)}$. Denote $E := \{fw \geqslant \varphi(x,w)\}$.

Since the conjugate modular is defined as a supremum over functions in $L^{\varphi(\cdot)}$, we get a lower bound by using the particular function $w\chi_E$. Thus

$$\begin{aligned}(\varrho_{\varphi(\cdot)})^*(J_f) &\geqslant J_f(w\chi_E) - \varrho_{\varphi(\cdot)}(w\chi_E) = \int_E fw - \varphi(x,w)\,\mathrm{d}x \\ &\geqslant \int_{\{h_k=1\}} g_k|f| - \varphi(x, g_k)\,\mathrm{d}x = \int_A r_k\chi_{\{h_k=1\}}\,\mathrm{d}x.\end{aligned}$$

Since $r_k \chi_{\{h_k=1\}} \nearrow \varphi^*(x,|f|)$, it follows by monotone convergence that $(\varrho_{\varphi(\cdot)})^*(J_f) \geqslant \varrho_{\varphi^*(\cdot)}(f)$. From the definitions of $(\varrho_{\varphi(\cdot)})^*$ and $\varrho_{\varphi^*(\cdot)}$,

$$(\varrho_{\varphi(\cdot)})^*(J_f) = \sup_{g\in L^{\varphi(\cdot)}} \int_A fg - \varphi(x,g)\,\mathrm{d}x \leqslant \int_A \varphi^*(x,f)\,\mathrm{d}x = \varrho_{\varphi^*(\cdot)}(f).$$

Hence $(\varrho_{\varphi(\cdot)})^*(J_f) = \varrho_{\varphi^*(\cdot)}(f)$.

Since $f \mapsto J_f$ is linear, it follows that $(\varrho_{\varphi(\cdot)})^*(\lambda J_f) = \varrho_{\varphi^*(\cdot)}(\lambda f)$ for every $\lambda > 0$ and therefore $\|f\|_{\varphi^*(\cdot)} = \|J_f\|_{(\varrho_{\varphi(\cdot)})^*} \leqslant \|J_f\|_{(L^{\varphi(\cdot)})^*} = \|f\|_{(L^{\varphi(\cdot)})'}$, where the second step follows from [6], Theorem 2.2.10.

Taking into account that $\varphi^{**} \simeq \varphi$, we have shown that $L^{\varphi(\cdot)} = (L^{\varphi^*(\cdot)})'$. By the definition of the associate space norm, this means that

$$\|f\|_{\varphi(\cdot)} \approx \sup_{\|g\|_{\varphi^*(\cdot)}\leqslant 1} \int |f|\,|g|\,\mathrm{d}x$$

for $f \in L^{\varphi(\cdot)}$. In the case $f \in L^0 \setminus L^{\varphi(\cdot)}$, we can approximate $h_k \min\{|f|,k\} \nearrow |f|$ with $h_k$ as before. Since $h_k \min\{|f|,k\} \in L^{\varphi(\cdot)}$, the previous result implies that the formula holds, in the form $\infty = \infty$, when $f \in L^0 \setminus L^{\varphi(\cdot)}$. $\square$

## 4. Uniform convexity

The function $\varphi \in \Phi_w(\mathbb{R}^n)$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varphi\Big(x, \frac{s+t}{2}\Big) \leqslant (1-\delta)\frac{\varphi(x,s)+\varphi(x,t)}{2}$$

for every $x \in \mathbb{R}^n$ whenever $|s-t| \geqslant \varepsilon \max\{|s|,|t|\}$.

**Theorem 4.1.** *The function $\varphi \in \Phi_w(A)$ is equivalent to a uniformly convex $\Phi$-function if and only if it satisfies* (aInc).

P r o o f. Assume first that $\varphi$ satisfies (aInc) with $\gamma^- = p > 1$. By [10], Lemma 2.2, there exists $\psi \in \Phi(A)$ such that $\varphi \simeq \psi$ and $\psi^{1/p}$ is convex for some $p > 1$. The claim follows once we show that $\psi$ is uniformly convex. Let $\varepsilon \in (0,1)$ and $s - t \geqslant \varepsilon s$ with $s > t > 0$. Since $\psi^{1/p}$ is convex,

$$\psi\Big(x, \frac{s+t}{2}\Big)^{1/p} \leqslant \frac{\psi(x,s)^{1/p} + \psi(x,t)^{1/p}}{2}.$$

Since $t \leqslant (1-\varepsilon)s$ and $\psi$ is convex, we find that $\psi(x,t) \leqslant \psi(x,(1-\varepsilon)s) \leqslant (1-\varepsilon)\psi(x,s)$. Therefore $\psi(x,t)^{1/p} \leqslant (1-\varepsilon')\psi(x,s)^{1/p}$ for some $\varepsilon' > 0$. Since $t^p$ is uniformly convex, we obtain that

$$\Big(\frac{\psi(x,s)^{1/p}+\psi(x,t)^{1/p}}{2}\Big)^p \leqslant (1-\delta)\frac{\psi(x,s)+\psi(x,t)}{2}.$$

Combined with the previous estimate, this shows that $\psi$ is uniformly convex.

Assume now conversely that $\varphi \simeq \psi$ and $\psi$ is uniformly convex. Choose $\varepsilon = \frac{1}{2}$ and $t = 0$ in the definition of uniform convexity:

$$\psi(x, s/2) \leqslant \tfrac{1}{2}(1-\delta)\psi(x,s).$$

Divide this equation by $(s/2)^p$, where $p$ is chosen so that $2^{p-1}(1-\delta) = 1$:

$$\frac{\psi(x,s/2)}{(s/2)^p} \leqslant 2^{p-1}(1-\delta)\frac{\psi(x,s)}{s^p} = \frac{\psi(x,s)}{s^p}.$$

The previous inequality holds for every $s > 0$. If $0 < t < s$, then we can choose $k \in \mathbb{N}$ such that $2^k t \leqslant s < 2^{k+1}t$. Then by the previous inequality and monotonicity of $\psi$,

$$\frac{\psi(x,t)}{t^p} \leqslant \frac{\psi(x,2t)}{(2t)^p} \leqslant \ldots \leqslant \frac{\psi(x,2^k t)}{(2^k t)^p} \leqslant 2^p \frac{\psi(x,s)}{s^p}.$$

Hence, $\psi$ satisfies (aInc) with $\gamma^- = p$. Since this property is invariant under equivalence, it holds for $\varphi$ as well. □

We can now prove the uniform convexity of the space.

*Proof* of Theorem 1.3. By Theorem 1.2, $\Delta_2^w$ and $\nabla_2^w$ imply $\Delta_2$ and $\nabla_2$. If $\varphi$ satisfies (aInc), then it follows from Theorem 4.1 that it is equivalent to a uniformly convex $\Phi$-function $\psi$. By (aDec), also $\psi$ is doubling. Hence by [16], Theorem 11.6 (see also [6], Theorem 2.4.14), $L^{\psi(\cdot)}$ is uniformly convex. Since $\varphi \simeq \psi$, $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$, and hence we have proved $L^{\varphi(\cdot)}$ is uniformly convex. Furthermore, every uniformly convex Banach space is reflexive [1], Chapter 1. □

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